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Solution Sensitivity of a Generalized Variational Inequality

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Abstract. Based on the theory of maximal monotone operators, we obtain some results on continuity and Hölder continuity of the locally unique solution to a parametric generalized variational inequality in a reflexive Banach space. Local boundedness and strict monotonicity of the involved set-valued mapping are assumed. Our results extend some results of A. Domokos and N. D. Yen.

1. Introduction

It is well known that the necessary optimality condition to a convex programming problem can be written as a generalized variational inequality. Namely, if X is a locally convex topological vector space with the dual X^*, $K \subset X$ a nonempty closed convex subset and $\varphi : X \to R \cup \{+\infty\}$ a convex function, then x is a solution of the optimization problem

$$\varphi(x) \longrightarrow \inf, \quad x \in K, \tag{1.1}$$

if and only if x satisfies the following inclusion

$$0 \in \partial\varphi(x) + N_K(x). \tag{1.2}$$

Here

$$\partial\varphi(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \ \forall y \in X\}$$

denotes the subdifferential of f at x, and

$$N_K(x) = \begin{cases} \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0 \ \forall y \in K\} & \text{if } x \in K \\ \emptyset & \text{if } x \notin K \end{cases}$$

denotes the normal cone to K at x. The *generalized variational inequality* (GVI, for short) defined by a set-valued operator $F : X \to 2^{X^*}$ and a closed convex set K in X is the problem of finding $x \in X$ satisfying the inclusion

$$0 \in F(x) + N_K(x). \tag{1.3}$$

For $F(x) = \partial\varphi(x)$, where φ is a convex function defined on X, (1.3) becomes (1.2). Therefore, convex programming problems can be studied by using the GVI model (1.3). If $F(x) = \{f(x)\}$, where $f : X \to X^*$ is a single-valued operator, then (1.3) has the form

$$0 \in f(x) + N_K(x), \tag{1.4}$$

which is called a *variational inequality* (VI, for short).

Conditions for the solution existence of variational inequalities and generalized variational inequalities have been considered in many papers (see, for instance, [2] and references therein).

In the last two decades, several authors have studied the solution sensitivity of parametric variational inequalities (see, for instance, [4, 6, 12, 13], and references therein). By using the metric projection method of Dafermos [4], Yen [12] obtained a theorem on the Hölder continuity of the solution to a parametric variational inequality in Hilbert spaces. Recently, Domokos [6] has extended that theorem to the case of VIs in reflexive Banach spaces. However, the proof of the main result in [6] had an inaccuracy, which will be corrected in the proof of Theorem 3.1 below. (See also the discussion in Sec. 4 of this paper). Domokos' method is based on the theory of maximal monotone operators [8, 16].

In this paper, by using the method of Domokos [6], we shall obtain some results on solution sensitivity of parametric generalized variational inequalities in reflexive Banach spaces. Our results generalize the corresponding results of [9] and [6]. As a by-product, we obtain some facts on solution sensitivity of parametric convex programming problems in reflexive Banach spaces.

From now on X denotes a reflexive Banach space with the dual X^*, (Λ, d) and (M, d) are metric spaces; $x_0 \in X$, $\lambda_0 \in \Lambda$ and $\mu_0 \in M$ are some given points. Let $F : X \times M \to 2^{X^*}$ and $K : \Lambda \to 2^X$ be two given set-valued mappings. It is always assumed that $K(\cdot)$ has nonempty closed convex values. The problem of finding $x \doteq x(\mu, \lambda)$ satisfying the inclusion

$$0 \in F(x, \mu) + N_{K(\lambda)}(x), \tag{1.5}$$

where $(\mu, \lambda) \in M \times \Lambda$ is a pair of parameters, is called a *parametric GVI.*

The paper is organized as follows. In Sec. 2 we recall some definitions and facts concerning maximal monotone operators, which will be used in subsequent sections. In Sec. 3 we will show that under some suitable conditions on $F(\cdot)$ and $K(\cdot)$, the unique solution $x = x(\mu, \lambda)$ of (1.5) is continuous with respect to (μ, λ). Moreover, if F is Lipschitz continuous and strongly monotone then $x = x(\mu, \lambda)$ will be a Hölder continuous function. In Sec. 4 we will study some special cases of (1.5) where $F(\cdot)$ is a single-valued map. Sec. 5 is devoted to an application of our result to the solution sensitivity of problem (1.1).

2. Auxiliaries

For a given point a in a metric space and a real number $\rho > 0$, denote by $B(a,\rho)$ (resp. $\overline{B}(a,\rho)$) the open (resp. closed) ball centered at a with the radius ρ. For a set-valued map $G : X \to 2^{X^*}$, the set $\operatorname{dom} G := \{x \in X : G(x) \neq \emptyset\}$ and $\operatorname{gr} G := \{(x, x^*) \in X \times X^* : x^* \in G(x)\}$ is called, respectively, the effective domain and the graph of G.

Definition 2.1. *G is called lower semicontinuous in the sense of Hausdorff (H-l.s.c) at $x_0 \in X$ if, for any $\epsilon > 0$, there exists a neighborhood U of x_0 in X such that $G(x_0) \subset G(x) + \epsilon B_{X^*}$ for every $x \in U$, where $B_{X^*} := \{x^* \in X^* : \|x^*\| < 1\}$.*

Definition 2.2. [8, p. 28] *G is said to be demicontinuous at $x_0 \in X$ if, for any weakly* open subset $V \subset X^*$ satisfying $G(x_0) \subset V$, there exists a neighborhood U of x_0 in X such that $G(x) \subset V$ for every $x \in U$. G is said to be hemicontinuous at $x_0 \in X$ if, for every $v \in X$, $\bar{t} \in [0,1]$ and every weakly* open subset $V \subset X^*$ satisfying $G(\bar{t}x_0 + (1-\bar{t})v) \subset V$, there exists $\delta > 0$ such that $G(tx_0 + (1-t)v) \subset V$ for every $t \in [0,1]$ with $|t - \bar{t}| < \delta$.*

Definition 2.3. [16, p. 852] *$G : X \to 2^{X^*}$ is said to be monotone if for any $(x_1, x_1^*), (x_2, x_2^*) \in \operatorname{gr} G$ one has*

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \geq 0.$$

G is maximal monotone if it is monotone and there exists no monotone map $G' : X \to 2^{X^*}$ such that $\operatorname{gr} G$ is a proper subset of $\operatorname{gr} G'$.

Combining the following two properties one has a criterion for checking whether the operator G is maximal monotone.

Lemma 2.1. [8, Lemma 2.5, p. 32] *Let $G : X \to 2^{X^*}$ be a monotone and hemicontinuous mapping. If $U \subset \operatorname{dom} G$ is such that $G(x)$ is a closed convex set for every $x \in U$ then G is demicontinuous at every $x_0 \in U$.*

Lemma 2.2. [8, Lemma 2.13, p. 34] *Let $G : X \to 2^{X^*}$ be a monotone and demicontinuous mapping. If $G(x)$ is a nonempty closed convex set for every $x \in X$ then G is maximal monotone.*

The following proposition gives a criterion for checking maximal monotonicity of a sum of two maximal monotone operators.

Lemma 2.3. (Rockafellar 1970; see [13, Theorem 32.I]) *If $G_1, G_2 : X \to 2^{X^*}$ are maximal monotone mappings satisfying* $\operatorname{int}(\operatorname{dom} G_1) \cap \operatorname{dom} G_2 \neq \emptyset$, *where* $\operatorname{int} D$ *denotes the interior of a set $D \subset X$, then the sum $G_1 + G_2 : X \to 2^{X^*}$ defined by setting $(G_1 + G_2)(x) = G_1(x) + G_2(x)$, is also maximal monotone.*

The following fact is fundamental in the theory of maximal monotone operators.

Lemma 2.4. [16, Corollary 32.35] *If $G : X \to 2^{X^*}$ is maximal monotone and* $\operatorname{dom} G$ *is bounded, then G is surjective, i.e.* $\cup_{x \in X} G(x) = X^*$.

In the sequel, we shall need the notions of strict monotonicity and uniform monotonicity (with respect to a gauge function ω) of a set-valued mapping $G : X \to 2^{X^*}$. For the case of single-valued mappings, these notions can be seen in [16, pp. 500-501].

Definition 2.4. $G : X \to 2^{X^*}$ *is said to be strictly monotone if for any* (x_1, x_1^*), $(x_2, x_2^*) \in \operatorname{gr} G$, $x_1 \neq x_2$, *one has* $\langle x_2^* - x_1^*, x_2 - x_1 \rangle > 0$.

Obviously, if $F : X \to 2^{X^*}$ is a strictly monotone operator then the problem (1.3) has at most one solution. Indeed, suppose that $x_1, x_2 \in K$ are two solutions of (1.3). Then there exist $x_1{}^* \in F(x_1)$, $x_2{}^* \in F(x_2)$ such that

$$\langle x_1{}^*, x_2 - x_1 \rangle \geq 0, \quad \langle x_2{}^*, x_1 - x_2 \rangle \geq 0.$$

Therefore $\langle x_2{}^* - x_1{}^*, x_2 - x_1 \rangle \leq 0$. By the monotonicity of F, $\langle x_2{}^* - x_1{}^*, x_2 - x_1 \rangle \geq 0$. Hence $\langle x_2{}^* - x_1{}^*, x_2 - x_1 \rangle = 0$. Then, by the assumed strict monotonicity of F, $x_1 = x_2$.

Definition 2.5. [1] *Let ω be a non-decreasing real function defined on the set* $R_+ = \{t \in R : t \geq 0\}$ *such that $\omega(t) > 0$ for every $t > 0$. $G(.)$ is said to be ω-uniformly monotone if, for any* (x_1, x_1^*), $(x_2, x_2^*) \in \operatorname{gr} G$, *one has*

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \geq \omega(\|x_2 - x_1\|)\|x_2 - x_1\|. \tag{2.1}$$

For $\omega(t) = \alpha t$, $\alpha > 0$, (2.1) becomes

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \geq \alpha \|x_2 - x_1\|^2. \tag{2.2}$$

In this case $G(\cdot)$ is said to be a strongly monotone operator.

3. Continuity Properties of the Solution to a Parametric GVI

Consider a parametric generalized variational inequality in the form (1.5), where $F(x, \mu), K(\lambda), M, \Lambda$ are defined as in Sec. 1. Suppose $(x_0, \mu_0, \lambda_0) \in X \times M \times \Lambda$ is such that

$$0 \in F(x_0, \mu_0) + N_{K(\lambda_0)}(x_0). \tag{3.1}$$

Our first result on the solution sensitivity of the problem (1.5) can be stated as follows.

Theorem 3.1. *Suppose that the following conditions are fulfilled:*

(a_1) *For every $\mu \in M$, $F(\cdot, \mu)$ is a maximal monotone operator.*

(a_2) *There exists a neighborhood U of x_0 such that, for every $\epsilon > 0$ there exists $\delta > 0$ with the property that if $(x_1, x_1^*), (x_2, x_2^*) \in \operatorname{gr} F(\cdot, \mu) \cap (U \times X^*)$ for some $\mu \in M$, and $\|x_2 - x_1\| > \epsilon$, then $\langle x_2^* - x_1^*, x_2 - x_1 \rangle > \delta$.*

(a_3) *There exist a neigborhood U' of x_0, a neighborhood W of μ_0 and a constant $\gamma > 0$ such that $F(x, \mu) \neq \emptyset$ for every $(x, \mu) \in U' \times W$,*

$$\sup\{\|x^*\| : x^* \in F(x,\mu), x \in U', \ \mu \in W\} < \gamma, \tag{3.2}$$

and for every $x \in U$, $F(x,\cdot)$ is lower semicontinuous in the sense of Hausdorff at every $\mu \in W$

(a_4) *There exist a real function $\beta : R_+ \to R_+$ with $\lim_{t\to 0} \beta(t) = 0$, a neighborhood U'' of x_0 and a neighborhood V of λ_0 such that*

$$K(\lambda') \cap U'' \subset K(\lambda) + \beta(d(\lambda',\lambda))\overline{B}_X \tag{3.3}$$

for all $\lambda', \lambda \in V$, where $\overline{B}_X$ denotes the closed unit ball in X.

Then there exist a neighborhood $\widetilde{W}$ of μ_0, a neighborhood $\widetilde{V}$ of λ_0 such that, for every $(\mu,\lambda) \in \widetilde{W} \times \widetilde{V}$, there exists a unique solution $x = x(\mu,\lambda) \in U$ of the following generalized variational inequality

$$0 \in F(x,\mu) + N_{K(\lambda)}(x). \tag{3.4}$$

Besides, $x(\mu_0,\lambda_0) = x_0$ and the function $(\mu,\lambda) \mapsto x(\mu,\lambda)$ is continuous on $\widetilde{W} \times \widetilde{V}$.

The following remarks allow us to have a closer look at the assumptions $(a_1) - (a_4)$.

Remark 1. If there exists a constant $\alpha > 0$ such that, for any $\mu \in M$ and $(x_1,x_1^*), (x_2,x_2^*) \in \mathrm{gr}F(\cdot,\mu)$, inequality (2.2) holds, then (a_2) is fulfilled. The proof is obvious. Note also that if there exists a nondecreasing real function $\omega : R_+ \to R_+$, $\omega(t) > 0$, for every $t > 0$ such that for every $\mu \in M$ and for any $(x_1,x_1^*), (x_2,x_2^*) \in \mathrm{gr}\, F(\cdot,\mu)$ inequality (2.1) holds true, then (a_2) is fulfilled.

Remark 2. If (a_1), (a_2) are valid then, for every $\mu \in M$, the restriction of the map $F(\cdot,\mu)$ on U is strictly monotone. Indeed, by (a_1), $F(\cdot,\mu)$ is monotone. Suppose that for some $(x_1,x_1^*), (x_2,x_2^*) \in \mathrm{gr}F(\cdot,\mu) \cap (U \times X^*)$, $x_1 \neq x_2$, one has $\langle x_2^* - x_1^*, x_2 - x_1\rangle = 0$. Let $\epsilon = \|x_2 - x_1\|/2$. For any $\delta > 0$, one has $\|x_2 - x_1\| > \epsilon$ but $\langle x_2^* - x_1^*, x_2 - x_1\rangle = 0 < \delta$. This contradicts (a_2).

Remark 3. If $F(x,\mu) = \{f(x,\mu)\}$, where $f : X \times M \to X^*$ is a continuous single-valued map, then (a_3) is fulfilled.

Remark 4. Suppose that Λ is a subset of a normed space and $\beta(t) = kt$, where $k > 0$ is a constant. Then (3.2) becomes

$$K(\lambda') \cap U'' \subset K(\lambda) + k\|\lambda' - \lambda\|\overline{B}_X$$

for all $\lambda', \lambda \in V$. In this case, one says that $K(\cdot)$ is pseudo-Lipschitz at (λ_0, x_0). In the terminology of [11], the map $K(\cdot)$ has the Aubin property at λ_0 for x_0. We have seen that if $K(\cdot)$ has the Aubin property at λ_0 for x_0 then (a_4) is fulfilled. Sufficient conditions for the Aubin property of implicit set-valued mappings can be found, for instance, in [5, 11, 14].

Remark 5. If for every $\mu \in M$ the map $F(x,\mu)$ has nonempty closed convex values and it is monotone and hemicontinuous on X, then (a_1) holds. For the proof, it suffices to apply Lemmas 2.1 and 2.2.

Remark 6. Theorem 1 of [6] corresponds to Theorem 4.1 below, which is a special case of Theorem 3.1 above where F is single-valued. A detailed comparison of our Theorem 4.1 with Theorem 1 from [6] will be given in Sec. 4.

The notion of uniformly monotone operators (w.r.t. a certain gauge function ω) has proved to be very useful in functional analysis (see [16]). In [10] and [7], it is shown that one can characterize the uniform convexity of Banach spaces by using uniformly monotone operators. Note that the class of strongly monotone operators is too small and not suitable for obtaining such characterizations.

It is not difficult to give examples of ω-uniformly monotone operators which are not strongly monotone.

Example 1. [16, pp. 502-503] Consider the set-valued map $F : R \to 2^R$ defined by the formula $F(u) = \{|u|^{p-2}u\}$ for all $u \in R$, where $p > 2$ is a fixed constant. Since there exists $c > 0$ such that

$$\langle |u|^{p-2}u - |v|^{p-2}v, u - v\rangle \geq c|u - v|^p$$

for all $u, v \in R$, then $F(\cdot)$ is a ω-uniformly monotone operator, where $\omega(t) := ct^{p-1}$. Meanwhile, it can be shown that $F(\cdot)$ is not a strongly monotone operator.

Example 2. Let $\varphi : R \to R, \varphi(x) = x^4$. Obviously, φ is a convex function and $\partial\varphi(x) = \{\varphi'(x)\}$. For all $x, y \in R$ we have

$$\begin{aligned}\langle \varphi'(y) - \varphi'(x), y - x\rangle &= (4y^3 - 4x^3)(y - x)\\ &= 4(y^2 + xy + x^2)(x - y)^2\\ &\geq (y - x)^4.\end{aligned}$$

So $F(\cdot) := \partial\varphi(\cdot)$ is ω-uniformly monotone, where $\omega(t) = t^3$. Note that it is not a strongly monotone operator.

Example 3. Let $X = L^p([0, 1])$, $p > 2$, be the Banach space of all measurable functions defined on $[0, 1]$, for which $\int_0^1 |x(s)|^p ds < +\infty$. By definition, $\|x\| = (\int_0^1 |x(s)|^p ds)^{1/p}$. Let $\phi(x) = \|x\|^p/p$ for all $x \in X$. Let $F : X \to 2^{X^*}$ be the set-valued mapping defined by the formula $F(\cdot) = \partial\phi(\cdot)$. In fact, $F(\cdot)$ is the normalized duality mapping on X (see [7]). By Corollary 2.1 from [9],

$$\|tx + (1 - t)y\|^p \leq t\|x\|^p + (1 - t)\|y\|^p - \frac{1}{p2^p}c(t)\|x - y\|^p$$

for all $x, y \in X$ and $t \in [0, 1]$, where $c(t) := t(1 - t)^p + t^p(1 - t)$. Using this fact and arguing similarly as in the proof of Proposition 5.1 below, we get

$$\langle x^* - y^*, x - y\rangle \geq \alpha\|x - y\|^p$$

for all $x, y \in X$, $x^* \in F(x)$, $y^* \in F(y)$, where $\alpha = 2/p^2 2^p$. This implies that F is a ω-uniformly monotone operator with $\omega(t) := \alpha t^{p-1}$. However, F is not strongly monotone. Indeed, suppose to the contrary that there exists $\beta > 0$ such that

$$\langle x^* - y^*, x - y\rangle \geq \beta\|x - y\|^2$$

for all $x, y \in X$, $x^* \in F(x), y^* \in F(y)$. For $x := 2y, y \neq 0$, we have

$$\langle x^* - y^*, y\rangle \geq \beta\|y\|^2$$

for all $y \in X$, $x^* \in F(2y)$, $y^* \in F(y)$. By Theorem 2.1 from [7],

$$F(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}.$$

Consequently,

$$\beta\|y\|^2 \leq \langle x^* - y^*, y\rangle = 2^{p-1}\|y\|^p - \|y\|^p = (2^{p-1} - 1)\|y\|^p.$$

Therefore

$$\beta \leq (2^{p-1} - 1)\|y\|^{p-2}.$$

Since this inequality does not hold for $y \in X$ with $\|y\|$ sufficiently small, we have arrived at a contradiction.

Proof of Theorem 3.1. (This proof is based on some ideas of [6]). Let the assumptions (a_1) - (a_4) be satisfied. By (a_3) and (a_4), there exist positive constants $s, \overline{\delta}$, such that $B(x_0, s) \subset U \cap U' \cap U''$, $B(\lambda_0, \overline{\delta}) \subset V$, and

$$\beta d(\lambda, \lambda_0) < s, \text{ for all } \lambda \in B(\lambda_0, \overline{\delta}). \tag{3.5}$$

It follows from (3.2) and (3.3) that

$$\sup\{\|x^*\| : x^* \in F(x, \mu), x \in \overline{B}(x_0, s), \mu \in W\} \leq \gamma, \tag{3.6}$$

$$K(\lambda') \cap \overline{B}(x_0, s) \subset K(\lambda) + \beta d(\lambda', \lambda)\overline{B}_X, \tag{3.7}$$

for all λ', $\lambda \in B(\lambda_0, \overline{\delta})$. Putting $\lambda' = \lambda_0$ in (3.7), we see that for each $\lambda \in B(\lambda_0, \overline{\delta})$ there exists $z_\lambda \in K(\lambda)$ satisfying

$$\|z_\lambda - x_0\| \leq \beta d(\lambda, \lambda_0) < s.$$

Hence $K(\lambda) \cap B(x_0, s) \neq \emptyset$ for all $\lambda \in B(\lambda_0, \overline{\delta})$. Fixing any pair $(\mu, \lambda) \in W \times B(\lambda_0, \overline{\delta})$ we consider the inclusion

$$0 \in F(x, \mu) + N_{K(\lambda)\cap\overline{B}(x_0,s)}(x). \tag{3.8}$$

Since $K(\lambda) \cap \overline{B}(x_0, s)$ is a closed convex set, the normal cone operator

$$x \mapsto N_{K(\lambda)\cap\overline{B}(x_0,s)}(x), \tag{3.9}$$

is maximal monotone (see [16, p. 859]). By (a_1), $F(\cdot, \mu)$ is also a maximal monotone. According to (a_3) and the choice of s, we have $\overline{B}(x_0, s) \subset \operatorname{int} F(\cdot, \mu)$. Since the effective domain of the operator in (3.9) is a bounded nonempty set $K(\lambda) \cap \overline{B}(x_0, s)$, by Lemma 2.3 the set-valued map

$$x \mapsto F(x, \mu) + N_{K(\lambda)\cap\overline{B}(x_0,s)}(x) \tag{3.10}$$

is a maximal monotone operator with a bounded effective domain. Lemma 2.4 shows that there must exist a vector $x = x(\mu, \lambda)$ satisfying the inclusion (3.8). Since $F(\cdot, \mu)$ is a strictly monotone operator (see Remark 3.2), such vector

$x = x(\mu,\lambda)$ is uniquely defined. By (3.8), there exists some $z^*_{\mu,\lambda} \in F(x(\mu,\lambda),\mu)$ such that

$$\langle z^*_{\mu,\lambda}, z - x(\mu,\lambda)\rangle \geq 0 \text{ for all } z \in K(\lambda) \cap \overline{B}(x_0, s)(x).$$

In particular,

$$\langle z^*_{\mu,\lambda}, z_\lambda - x(\mu,\lambda)\rangle \geq 0. \tag{3.11}$$

By (3.7), since $x(\mu,\lambda) \in K(\lambda) \cap \overline{B}(x_0, s)$ then there is $z_0 \in K(\lambda_0)$ satisfying

$$\|x(\mu,\lambda) - z_0\| \leq d(\lambda, \lambda_0).$$

By (3.1) there exists $x_0^* \in F(x_0, \mu_0)$ such that

$$\langle x_0^*, z - x_0\rangle \geq 0 \text{ for all } z \in K(\lambda_0).$$

In particular,

$$\langle x_0^*, z_0 - x_0\rangle \geq 0. \tag{3.12}$$

As $F(\cdot,\mu)$ is a maximal monotone operator, its values must be convex weakly* closed sets in X^* (see [16, Proposition 32.6]). Thus $F(x_0,\mu)$, $\mu \in M$, are convex weakly* closed subsets in X^*. In addition, by (a_3), $F(x_0,\mu) \neq \emptyset$. Since X is a reflexive Banach space, X^* is also a reflexive Banach space, hence there is $y^*_\mu \in F(x_0,\mu)$ satisfying

$$d(x_0^*, F(x_0,\mu)) := \inf_{z^* \in F(x_0,\mu)} \|x_0^* - z^*\| = \|x_0^* - y^*_\mu\|. \tag{3.13}$$

Using (3.11), (3.12), and the monotonicity property of $F(\cdot,\mu)$, we get

$$\begin{aligned}
0 &\leq \langle z^*_{\mu,\lambda} - y^*_\mu, x(\mu,\lambda) - x_0\rangle \\
&\leq \langle z^*_{\mu,\lambda} - y^*_\mu, x(\mu,\lambda) - x_0\rangle + \langle x_0^*, z_0 - x_0\rangle + \langle z^*_{\mu,\lambda}, z_\lambda - x(\mu,\lambda)\rangle \\
&= \langle z^*_{\mu,\lambda}, z_\lambda - x_0\rangle + \langle x_0^*, z_0 - x_0\rangle - \langle y^*_\mu, x(\mu,\lambda) - x_0\rangle \\
&= \langle z^*_{\mu,\lambda}, z_\lambda - x_0\rangle + \langle x_0^* - y^*_\mu, z_0 - x_0\rangle - \langle y^*_\mu, x(\mu,\lambda) - x_0\rangle \\
&\leq \|z^*_{\mu,\lambda}\|\|z_\lambda - x_0\| + \|x_0^* - y^*_\mu\|\|z_0 - x_0\| + \|y^*_\mu\|\|x(\mu,\lambda) - x_0\|.
\end{aligned}$$

By (3.6),

$$\|z^*_{\mu,\lambda}\| \leq \gamma, \|y^*_\mu\| \leq \gamma,$$

$$\|x(\mu,\lambda) - z_0\| \leq \beta d(\lambda,\lambda_0), \|z_\lambda - x_0\| \leq \beta d(\lambda,\lambda_0),$$

$$\|z_0 - x_0\| \leq \|z_0 - x(\mu,\lambda)\| + \|x(\mu,\lambda) - x_0\| \leq \beta d(\lambda,\lambda_0) + 2s < 2s.$$

Therefore

$$0 \leq \langle z^*_{\mu,\lambda} - y^*_\mu, x(\mu,\lambda) - x_0\rangle \leq 2\gamma\beta d(\lambda,\lambda_0)) + 2s\|x_0^* - y^*_\mu\|. \tag{3.14}$$

We claim that $\|x_0^* - y^*_\mu\| \to 0$ as $\mu \to \mu_0$. Indeed, by (a_3) the set-valued map $F(x_0,\cdot)$ is H-l.s.c. at μ_0. Then for any $\epsilon > 0$ there exists $\delta' > 0$ such that $F(x_0,\mu_0) \subset F(x_0,\mu) + \epsilon B_{X^*}$ for all $\mu \in B(\mu_0,\delta')$. As $x_0^* \in F(x_0,\mu_0)$, $d(x_0^*, F(x_0,\mu)) < \epsilon$. By (3.13), $\|x_0^* - y_\mu^*\| < \epsilon$ for every $\mu \in B(\mu_0,\delta')$. Thus we have proved that $\|x_0^* - y_\mu^*\| \to 0$ as $\mu \to \mu_0$. We get from (3.14) that

$$\langle z_{\mu,\lambda}^* - y_\mu^*, x(\mu,\lambda) - x_0\rangle \to 0 \quad \text{as } (\mu,\lambda) \to (\mu_0,\lambda_0). \tag{3.15}$$

Now we can take the advantage of the assumption (a_2). Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that the property stated in (a_2) is valid. By (3.15), there exists $\theta > 0$ such that $\langle z_{\mu,\lambda}^* - y_\mu^*, x(\mu,\lambda) - x_0\rangle \leq \delta$ for any pair (μ,λ) satisfying $d(\mu,\mu_0) < \theta, d(\lambda,\lambda_0) < \theta$. As $(x(\mu,\lambda), z_{\mu,\lambda}^*), (y_\mu^*, x_0) \in \operatorname{gr} F(\cdot,\mu)$, the property in (a_2) implies that

$$\|x(\mu,\lambda) - x_0\| \leq \epsilon,$$

for any pair (μ,λ) satisfying $d(\mu,\mu_0) < \theta, d(\lambda,\lambda_0) < \theta$. This shows that $x(\mu_0,\lambda_0) = x_0$ and $x(\mu,\lambda) \to 0$ as $(\mu,\lambda) \to (\mu_0,\lambda_0)$. As a consequence, there exists an open neighborhood $\widetilde{W}$ of μ_0, an open neighborhood $\widetilde{V}$ of λ_0 such that $\widetilde{W} \subset W$, $\widetilde{V} \subset B(\lambda_0,\overline{\delta})$ and $x(\mu,\lambda) \in B(x_0,s)$ for any pair $(\mu,\lambda) \in \widetilde{W} \times \widetilde{V}$. Then, for any pair $(\mu,\lambda) \in \widetilde{W} \times \widetilde{V}$, since $x = x(\mu,\lambda)$ satisfies (3.8) and

$$N_{K(\lambda)\cap\overline{B}(x_0,s)}x(\mu,\lambda) = N_{K(\lambda)}x(\mu,\lambda),$$

we have

$$0 \in F(x(\mu,\lambda)) + N_{K(\lambda)}(x(\mu,\lambda)).$$

This shows that $x = x(\mu,\lambda)$ is a solution to the GVI problem (1.4).

We now prove that the function $(\mu,\lambda) \mapsto x(\mu,\lambda)$ is continuous on $\widetilde{W} \times \widetilde{V}$. Let $(\overline{\mu},\overline{\lambda}) \in \widetilde{W} \times \widetilde{V}$ be given arbitrarily. Putting $\overline{x} = x(\overline{\mu},\overline{\lambda})$, we get

$$0 \in F(\overline{x},\overline{\mu}) + N_{K(\overline{\lambda})}(\overline{x}).$$

Now, instead of the triple (x_0,μ_0,λ_0) we shall deal with the triple $(\overline{x},\overline{\mu},\overline{\lambda})$.

Observe that the assumptions $(a_1),(a_2)$ do not depend on the choice of (x_0,μ_0,λ_0). As concerning (a_3) and (a_4), we note that W and V, respectively, are also neighborhoods of $\overline{\mu}$ and $\overline{\lambda}$; while U, U' and U'' are neighborhoods of $\overline{x}$. Therefore, the assumptions $(a_1)-(a_4)$, where (x_0,μ_0,λ_0) is replaced by $(\overline{x},\overline{\mu},\overline{\lambda})$, remain valid. Then, by the result established in the preceding part of this proof, there exist open neighborhoods $\overline{W} \subset \widetilde{W}$ and $\overline{V} \subset \widetilde{V}$ of $\overline{\mu}$ and $\overline{\lambda}$, respectively, such that for each $(\mu,\lambda) \in \overline{V} \times \overline{W}$ one can find a unique vector $u = u(\mu,\lambda)$ satisfying (3.4), so that $u(\mu,\lambda) \to \overline{x}$ as $(\mu,\lambda) \to (\overline{\mu},\overline{\lambda})$ and $u(\overline{\mu},\overline{\lambda}) = \overline{x}$. For every pair $(\mu,\lambda) \in M \times \Lambda$, since (3.4) has at most one solution then we get $u(\mu,\lambda) = x(\mu,\lambda)$ for every $(\mu,\lambda) \in \overline{W} \times \overline{V}$. The desired continuity of the function $x(\mu,\lambda)$ at $(\overline{\mu},\overline{\lambda})$ follows from the above-mentioned continuity of the function $u(\mu,\lambda)$ at $(\overline{\mu},\overline{\lambda})$. The proof is complete. ■

We now show that under a strengthened version of $(a_1)-(a_4)$, the solution map $x = x(\mu,\lambda)$ of the parametric problem (3.4) possesses a finer continuity property than that described in the conclusion of Theorem 3.1.

Theorem 3.2. *Suppose that (a_1) and the following conditions are fulfilled:*

(a_2') *There exist a neighborhood U of x_0 and a constant $\alpha > 0$ such that, if $(x_1, x_1^*), (x_2, x_2^*) \in \operatorname{gr} F(\cdot, \mu) \cap (U \times X^*)$ for some $\mu \in M$ then*

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \geq \alpha \|x_2 - x_1\|^2; \tag{3.16}$$

(a_3') *There exist a neighborhood U' of x_0, a neighborhood W of μ_0, and a constant $l > 0$ such that $F(x, \mu) \neq \emptyset$ for every $(x, \mu) \in U' \times W$ and*

$$h(F(x_1, \mu_1), F(x_2, \mu_2)) \leq l(\|x_1 - x_2\| + d(\mu_1, \mu_2)) \tag{3.17}$$

for all $(x_1, \mu_1), (x_2, \mu_2) \in U' \times W$, where

$$h(A, B) := \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\}$$

denotes the Hausdorff distance between two subsets $A, B \subset X^*$;

(a_4') *There exist a neighborhood U'' of μ_0, a neighborhood V of λ_0, and a constant $k > 0$ such that*

$$K(\lambda') \cap U'' \subset K(\lambda) + kd(\lambda', \lambda)\overline{B}_X \tag{3.18}$$

for all $\lambda, \lambda' \in V$.

Then there exist a neighborhood $\widetilde{W}$ of μ_0, a neighborhood $\widetilde{V}$ of λ_0, and constants $k_1, k_2 > 0$ such that, for any $(\mu, \lambda) \in \widetilde{W} \times \widetilde{V}$, there exists a unique solution $x = x(\mu, \lambda) \in U$ of problem (3.4). Besides, $x(\mu_0, \lambda_0) = x_0$ and

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \leq k_1 d(\mu', \mu) + k_2 d(\lambda', \lambda)^{\frac{1}{2}}$$

for all $(\mu', \lambda'), (\mu, \lambda) \in \widetilde{W} \times \widetilde{V}$.

Proof. Firstly, we shall show that the assumptions (a_1) - (a_4) of Theorem 3.1 are fulfilled. It is obvious that (a_2') implies (a_2) and (a_4') implies (a_4). From (a_3') it follows that $x_0 \in \operatorname{int}(\operatorname{dom} F(\cdot, \mu_0))$. Since $F(\cdot, \mu_0)$ is maximal monotone then $F(x_0, \mu_0)$ must be a bounded set (see [16, Proposition 32.33]). By (a_3') and the boundedness of the set $F(x_0, \mu_0)$, there exists a constant $\gamma > 0$ such that requirements stated in (a_3) are fulfilled. Therefore the proof of Theorem 3.1 is applicable to the case we are considering.

By Theorem 3.1, there exist neighborhoods $\widetilde{W}$, $\widetilde{V}$ and a unique continuous function $x(\mu, \lambda)$ defined on $\widetilde{W} \times \widetilde{V}$ such that, for each $(\mu, \lambda) \in \widetilde{W} \times \widetilde{V}$, $x = x(\mu, \lambda)$ solves the inclusion (3.4).

Let s and δ be chosen as in the proof of Theorem 3.1. Let neighborhoods $\widetilde{W}$, $\widetilde{V}$ and function $x = x(\mu, \lambda)$ be the ones obtained in that proof. Let $(\mu, \lambda), (\mu', \lambda') \in \widetilde{W} \times \widetilde{V}$ be given arbitrarily. Since $x(\mu, \lambda) \in K(\lambda) \cap B(x_0, s) \subset K(\lambda) \cap U''$, by (3.18) there is some $z \in K(\lambda')$ such that

$$\|x(\mu, \lambda) - z\| \leq kd(\lambda, \lambda'). \tag{3.19}$$

Similarly, since $x(\mu, \lambda') \in K(\lambda') \cap B(x_0, s) \subset K(\lambda') \cap U''$ then by (3.18) we must find some $y \in K(\lambda)$ such that

$$\|x(\mu, \lambda') - y\| \leq kd(\lambda, \lambda'). \tag{3.20}$$

As $x(\mu, \lambda)$ (resp. $x(\mu, \lambda')$) solves the inclusion

$$0 \in F(x,\mu) + N_{K(\lambda)}(x)$$

(resp., $0 \in F(x,\mu) + N_{K(\lambda')}(x)$), there exists some $y^* \in F(x(\mu,\lambda),\mu)$ (resp., $z^* \in F(x(\mu,\lambda'),\mu)$) such that

$$\langle y^*, y - x(\mu,\lambda)\rangle \geq 0, \;\; \langle z^*, z - x(\mu,\lambda')\rangle \geq 0. \tag{3.21}$$

By (3.16) and (3.19) - (3.21),

$$\begin{aligned}
\alpha\|x(\mu,\lambda') - x(\mu,\lambda)\|^2 &\leq \langle z^* - y^*, x(\mu,\lambda') - x(\mu,\lambda)\rangle \\
&\leq \langle z^* - y^*, x(\mu,\lambda') - x(\mu,\lambda)\rangle + \langle y^*, y - x(\mu,\lambda)\rangle \\
&\quad + \langle z^*, z - x(\mu,\lambda')\rangle \\
&= \langle z^*, z - x(\mu,\lambda)\rangle + \langle y^*, y - x(\mu,\lambda')\rangle \\
&\leq \|z^*\|\|z - x(\mu,\lambda)\| + \|y^*\|\|y - x(\mu,\lambda')\| \\
&\leq 2\gamma k d(\lambda,\lambda').
\end{aligned}$$

Therefore

$$\|x(\mu,\lambda') - x(\mu,\lambda)\| \leq \left\{\frac{2\gamma k}{\alpha} d(\lambda,\lambda')\right\}^{\frac{1}{2}}. \tag{3.22}$$

Now, since $x(\mu,\lambda')$ (resp., $x(\mu',\lambda')$) solves the inclusion

$$0 \in F(x,\mu) + N_{K(\lambda')}(x)$$

(resp., $0 \in F(x,\mu') + N_{K(\lambda')}(x)$), there exists $u^* \in F(x(\mu,\lambda'),\mu)$ (resp., $v^* \in F(x(\mu',\lambda'),\mu')$) such that

$$\langle u^*, x(\mu',\lambda') - x(\mu,\lambda')\rangle \geq 0, \;\; \langle v^*, x(\mu,\lambda') - x(\mu',\lambda')\rangle \geq 0. \tag{3.23}$$

By (3.17), there is some $w^* \in F(x(\mu',\lambda'),\mu)$ satisfying

$$\|v^* - w^*\| \leq d(\mu',\mu). \tag{3.24}$$

By (a_2') and (3.23),

$$\begin{aligned}
\alpha\|x(\mu',\lambda') - x(\mu,\lambda')\|^2 &\leq \langle w^* - u^*, x(\mu',\lambda') - x(\mu,\lambda')\rangle \\
&\leq \langle w^* - u^*, x(\mu',\lambda') - x(\mu,\lambda')\rangle \\
&\quad + \langle u^*, x(\mu',\lambda') - x(\mu,\lambda')\rangle \\
&\quad + \langle v^*, x(\mu,\lambda') - x(\mu',\lambda')\rangle \\
&= \langle w^* - v^*, x(\mu'\lambda') - x(\mu,\lambda')\rangle. \\
&\leq \|w^* - v^*\|\|x(\mu',\lambda') - x(\mu,\lambda')\|.
\end{aligned}$$

Therefore, using (3.24) we have

$$\alpha\|x(\mu',\lambda') - x(\mu,\lambda')\| \leq l d(\mu',\mu). \tag{3.25}$$

As

$$\|x(\mu',\lambda') - x(\mu,\lambda)\| \leq \|x(\mu',\lambda') - x(\mu,\lambda')\| + \|x(\mu,\lambda') - x(\mu,\lambda)\|,$$

by (3.22) and (3.25) we obtain

$$\|x(\mu',\lambda') - x(\mu,\lambda)\| \le \frac{l}{\alpha} d(\mu',\mu) + \left\{\frac{2\gamma k}{\alpha} d(\lambda',\lambda)\right\}^{\frac{1}{2}}.$$

Putting $k_1 = l/\alpha$, $k_2 = (2\gamma k/\alpha)^{1/2}$, we get (3.18). The proof is complete. ∎

4. Special Cases

In this section we study the parametric problem (1.5) in the case $F(x,\mu) = \{f(x,\mu)\}$, where $f : X \times M \to X^*$ is a single-valued map. Let $(x_0, \mu_0, \lambda_0) \in X \times M \times \Lambda$ be such that

$$0 \in f(x_0, \mu_0) + N_{K(\lambda_0)}(x_0).$$

From Theorem 3.1 we can deduce the following result.

Theorem 4.1. *Suppose that* (a_4) *and the following conditions are satisfied:*
(b_1) *For every* $\mu \in M$, $f(\cdot,\mu)$ *is a hemicontinuous monotone operator.*
(b_2) *There exists a neighborhood* U *of* x_0 *such that for every* $\epsilon > 0$, *there exists* $\delta > 0$ *with the property that if* $x_1, x_2 \in U$, $\mu \in M$, *and* $\|x_2 - x_1\| > \epsilon$, *then* $\langle f(x_2,\mu) - f(x_1,\mu), x_2 - x_1\rangle > \delta$.
(b_3) *There exist a neighborhood* U' *of* x_0, *a neighborhood* W *of* μ_0 *and a constant* $\gamma > 0$ *such that*

$$\sup\{\|f(x,\mu)\| : x \in U', \mu \in W\} \,\square\, \gamma$$

and, for every $x \in U$, $f(x,\cdot)$ *is continuous on* W.
Then there exist a neighborhood $\widetilde{W}$ *of* μ_0, *a neighbourhood* $\widetilde{V}$ *of* λ_0 *such that, for every* $(\mu,\lambda) \in \widetilde{W} \times \widetilde{V}$, *there exists a unique solution* $x = x(\mu,\lambda) \in U$ *of the following parametric variational inequality*

$$0 \in f(x,\mu) + N_{K(\lambda)}(x). \tag{4.1}$$

Besides, $x(\mu_0,\lambda_0) = x_0$ *and the function* $(\mu,\lambda) \mapsto x(\mu,\lambda)$ *is continuous on* $\widetilde{W} \times \widetilde{V}$.

Proof. For the proof, it suffices to note that (b_1) implies (a_1) (see Lemma 2.1 and Lemma 2.2) and apply Theorem 3.1. ∎

The following theorem was obtained in [6].

Theorem 4.2. *Suppose that* (a_4) *and the following conditions are satisfied:*
(b_1') *There exist neighborhood* U *of* x_0, *a neighborhood* W *of* μ_0 *such that* f *is continuous on* $U \times W$.
(b_2') *The mapping* $f(\cdot,\mu)$ *are strictly-monotone for all* $\mu \in W$, *and*

$$\langle f(y,\mu) - f(x,\mu), y - x\rangle \to 0 \Rightarrow y \to x$$

uniformly with respect to $\mu \in W$.

Then there exist a neighborhood $\widetilde{W}$ of μ_0, a neighborhood $\widetilde{V}$ of λ_0 such that, for every $(\mu, \lambda) \in \widetilde{W} \times \widetilde{V}$, there exists a unique solution $x = x(\mu, \lambda) \in U$ of the following parametric variational inequality

$$0 \in f(x, \mu) + N_{K(\lambda)}(x). \tag{4.1}$$

Besides, $x(\mu_0, \lambda_0) = x_0$ and the function $(\mu, \lambda) \mapsto x(\mu, \lambda)$ is continuous on $\widetilde{W} \times \widetilde{V}$.

Condition (b_2') is rather difficult to understand. Its exact meaning is explained in condition (b_2) of our Theorem 4.1. Note that Theorem 4.1 extends Theorem 4.2. Indeed, if one assumes that $f(x, \mu)$ is continuous on a neighborhood of (x_0, μ_0) then it is obvious that condition (b_3) in Theorem 4.1 is satisfied. As it has been said in Introduction, the proof in [6] has some inaccuracy. Namely, in our notation, to prove that the function $x(\mu, \lambda)$ is continuous at $(\overline{\mu}, \overline{\lambda}) \in \widetilde{W} \times \widetilde{V}$ the author proves that $x(\cdot, \overline{\lambda})$ (reps., $x(\overline{\mu}, \cdot)$) is continuous at $\overline{\mu}$ (resp., $\overline{\lambda}$) then he concludes that $x(\mu, \lambda)$ is continuous at $(\overline{\mu}, \overline{\lambda})$. This argument is, of course, incorrect! As an example, consider the function

$$\varphi(u, v) = \begin{cases} \dfrac{2uv}{u^2 + v^2} & \text{if } u^2 + v^2 \neq 0 \\ 0 & \text{if } u = v = 0 \end{cases}$$

of two real variables u and v (see [3, p. 464-465]). Observe that $\varphi(\cdot, v)$ is continuous for each v fixed, and $\varphi(u, \cdot)$ is continuous for each u fixed. Since $\varphi(0, 0) = 0$ and $\varphi(u, v) = 1$ if $u = v$ and $u \neq 0$, φ is discontinuous at $(0, 0)$. In the proof of Theorem 3.1, by introducing some modifications to the proof scheme proposed in [6] we have shown that the above-mentioned inaccuracy can be eliminated, and the result of [6] can be extended to the case of generalized variational inequalities.

The following result follows directly from Theorem 3.2.

Theorem 4.3. *Suppose that $(b_1), (a_4')$ and the following assumptions are satisfied:*

(b_2'') There exist a neighborhood U of x_0 and a constant $\alpha > 0$ such that for every $(x_1, \mu), (x_2, \mu) \in U \times M$, it holds

$$\langle f(x_2, \mu) - f(x_1, \mu), x_2 - x_1 \rangle \geq \alpha \|x_2 - x_1\|^2.$$

(b_3'') There exist a neighborhood U' of x_0, a neighborhood W of μ_0, and a constant $l > 0$ such that

$$\|f(x_1, \mu_1) - f(x_2, \mu_2)\| \leq l(\|x_1 - x_2\| + d(\mu_1, \mu_2)),$$

for all $(x_1, \mu_1), (x_2, \mu_2) \in U' \times W$.

Then there exist a neighbourhood $\widetilde{W}$ of μ_0, a neighbourhood $\widetilde{V}$ of λ_0, and constants $k_1, k_2 > 0$ such that for any $(\mu, \lambda) \in \widetilde{W} \times \widetilde{V}$, there exists a unique solution $x = x(\mu, \lambda) \in U$ of the parametric problem (4.1). Besides, $x(\mu_0, \lambda_0) = x_0$ and

$$\|x(\mu',\lambda') - x(\mu,\lambda)\| \le k_1 d(\mu',\mu) + k_2 d(\lambda',\lambda)^{\frac{1}{2}}$$

for all $(\mu',\lambda'), (\mu,\lambda) \in \widetilde{W} \times \widetilde{V}$.

For $X = H$, where H is a real Hilbert space, Theorem 4.3 recovers Theorem 3.1 of [12] except for one thing: We have to impose the additional condition (b_1). Therefore, Theorem 4.2 extends Theorem 3.1 of [12] to the case of variational inequalities in reflexive Banach spaces. The proof in [12] allows one to avoid using the extra condition (b_1), and get the conclusion of Theorem 4.3 by using only the assumptions (a_4), (b_2) and (b_3). Note that, in order to follow the Domokos scheme [6], one has to assume that the operators $f(\cdot,\mu)$, $\mu \in M$, are maximal monotone. This is the reason why we cannot omit the assumption (b_1) in Theorems 4.1 and 4.3.

5. Applications

In this section we want to explain how the obtained results can be applied to study the solution sensitivity of parametric convex programming problems.

Definition 5.1. *A function* $\varphi : X \mapsto R \cup \{+\infty\}$ *is said to be strongly convex if there exists a constant* $\rho > 0$ *such that*

$$\varphi(tx_1 + (1-t)x_2) \le t\varphi(x_1) + (1-t)\varphi(x_2) - \rho t(1-t)\|x_1 - x_2\|^2,$$

for all $t \in [0,1]$ *and* $x_1, x_2 \in X$.

There exists a tight relation between strong convexity of a function and strong monotonicity of the subdifferential map.

Proposition 5.1. *If* $\varphi(\cdot)$ *is strongly convex then there exists a constant* $\alpha > 0$ *such that for all* $x_1, x_2 \in X$, $x_1^* \in \partial\varphi(x_1)$, $x_2^* \in \partial\varphi(x_2)$ *one has*

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \ge \alpha \|x_2 - x_1\|^2. \tag{5.1}$$

Proof. (This simple proof is presented here for the completeness of presentation). For every $x, y \in X$, $x^* \in \partial\varphi(x)$ we have

$$\varphi(x + t(y-x)) - \varphi(x) \le t(\varphi(y) - \varphi(x)) - \rho t(1-t)\|y-x\|^2.$$

Therefore

$$t\langle x^*, y-x\rangle \le t(\varphi(y) - \varphi(x)) - \rho t(1-t)\|y-x\|^2,$$

and hence

$$\langle x^*, y-x\rangle \le \varphi(y) - \varphi(x) - \rho(1-t)\|y-x\|^2 \tag{5.2}$$

for all $x, y \in X$ and $x^* \in \partial\varphi(x)$. Suppose that $x_1, x_2 \in X$, $x_1^* \in \partial\varphi(x_1)$, $x_2^* \in \varphi(x_2)$. From (5.2) we have

$$\langle x_1^*, x_2 - x_1\rangle \le \varphi(x_2) - \varphi(x_1) - \rho(1-t)\|x_2 - x_1\|^2,$$
$$\langle x_2^*, x_1 - x_2\rangle \le \varphi(x_1) - \varphi(x_2) - \rho(1-t)\|x_1 - x_2\|^2.$$

Adding these inequalities yields

$$\langle x_2^* - x_1^*, x_2 - x_1\rangle \ge 2\rho(1-t)\|x_2 - x_1\|^2.$$

Letting $t \to 0$ and putting $\alpha = 2\rho$ we obtain (5.1). The proof is complete. ∎

Note that if $\varphi(\cdot, \mu)$ is not assumed to be Fréchet differentiable on X then the correspondence $x \mapsto \partial\varphi(x)$ defines a set-valued map.

Now we can formulate two results on solution sensitivity of convex programming problems which work for the case of parametric strongly convex programming problems.

Let $\varphi : X \times M \to R \cup \{+\infty\}$ be a real function such that, for every $\mu \in M$, $\varphi(\cdot, \mu)$ is a convex function. Denote by $\partial\varphi(\cdot, \mu)$ the subdifferential of $\varphi(\cdot, \mu)$. Consider the problem

$$\varphi(x, \mu) \longrightarrow \min, \quad x \in K(\lambda). \tag{5.3}$$

Here $(\mu, \lambda) \in M \times \Lambda$ is a pair of parameters. Let x_0 be a solution of (5.3) where $(\mu, \lambda) = (\mu_0, \lambda_0) \in M \times \Lambda$ is a given pair.

Theorem 5.1. *Suppose that (a_4) and the following assumptions are fulfilled:*

(c_1) *For every $\mu \in M$, $\varphi(\cdot, \mu)$ is a lower semicontinuous function on X;*

(c_2) *There exists a neighborhood U of x_0 such that, for every $\epsilon > 0$ there exists $\delta > 0$ with the property that, for any $x_1^* \in \partial_x\varphi(x_1, \mu)$, $x_2^* \in \partial_x\varphi(x_2, \mu)$, $x_1, x_2 \in U$ and $\mu \in M$, if $\|x_2 - x_1\| > \epsilon$ then $\langle x_2^* - x_1^*, x_2 - x_1\rangle > \delta$;*

(c_3) *There exist a neigborhood U' of x_0, a neighborhood W of μ_0 and a constant $\gamma > 0$ such that $\partial_x\varphi(x, \mu) \ne \emptyset$ for every $(x, \mu) \in U' \times W$,*

$$\sup\{\|x^*\| : x^* \in \partial_x\varphi(x, \mu), x \in U', \mu \in W\} < \gamma,$$

and, for any $x \in U$, $\partial_x\varphi(x, \cdot)$ is lower semicontinuous in the sense of Hausdorff at every $\mu \in W$;

Then there exist a neighborhood $\widetilde{W}$ of μ_0, a neighborhood $\widetilde{V}$ of λ_0 such that, for every $(\mu, \lambda) \in \widetilde{W} \times \widetilde{V}$, there exists a unique solution $x = x(\mu, \lambda) \in U$ of the optimization problem (5.3). Besides, $x(\mu_0, \lambda_0) = x_0$ and the function $(\mu, \lambda) \mapsto x(\mu, \lambda)$ is continuous on $\widetilde{W} \times \widetilde{V}$.

Proof. It suffices to note that x is a solution of (5.3) if and only if x solves the following variational inequality:

$$0 \in \partial_x\varphi(x, \mu) + N_{K(\lambda)}(x).$$

Put $F(x, \mu) = \partial_x\varphi(x, \mu)$. By a result of Rockafellar (see [16, Proposition 32.17, p. 860]), for every $\mu \in M$, $F(\cdot, \mu)$ is a maximal monotone operator. Hence we can apply Theorem 3.1 to get the desired conclusions.

Theorem 5.2. *Let (a_4') and the following conditions be fulfilled:*

(c_2') *There exist a neighborhood U of x_0 and a constant $\alpha > 0$ such that, if $x_1^* \in \partial_x\varphi(x_1,\mu)$, $x_2^* \in \partial_x\varphi(x_2,\mu)$ for some $\mu \in M$, where $x_1, x_2 \in U$, then*
$$\langle x_2^* - x_1^*, x_2 - x_1\rangle \geq \alpha\|x_2 - x_1\|^2;$$

(c_3') *There exist an open neighborhood U' of x_0, a neighborhood W of μ_0, and a constant $l > 0$ such that, for every $(x,\mu) \in U' \times W$, $\varphi(\cdot,\mu)$ has the Fréchet derivative $\varphi_x'(x,\mu)$, and*
$$\|\varphi_x'(x_1,\mu_1) - \varphi_x'(x_2,\mu_2)\| \leq l(\|x_1 - x_2\| + d(\mu_1,\mu_2))$$
for all $(x_1,\mu_1), (x_2,\mu_2) \in U' \times W$.

Then there exist a neighborhood $\widetilde{W}$ of μ_0, a neighborhood $\widetilde{V}$ of λ_0, and constants $k_1, k_2 > 0$ such that, for any $(\mu,\lambda) \in \widetilde{W} \times \widetilde{V}$, there exists a unique solution $x = x(\mu,\lambda) \in U$ of the parametric problem (5.3). Besides, $x(\mu_0,\lambda_0) = x_0$ and
$$\|x(\mu',\lambda') - x(\mu,\lambda)\| \leq k_1 d(\mu',\mu) + k_2 d(\lambda',\lambda)^{\frac{1}{2}}$$
for all $(\mu',\lambda'), (\mu,\lambda) \in \widetilde{W} \times \widetilde{V}$.

Proof. This theorem follows from Theorem 3.2 in the same manner as Theorem 5.1 follows from Theorem 3.1. ∎

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