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Calibrated Foliations and Invariant Metrics

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Abstract. In this paper we study calibrated foliations on a smooth manifold X where there is an action of a compact Lie group G. We extend the results of R. Harvey and H. B. Lawson Jr. in [5] for calibrated foliations in the generalized case.

1. Introduction

The interest in theory of calibration and its various applications have been increasing in recent years. The idea of Calibration Theory is contained in Huyghens algorithm for the family of light rays in a non-homogeneous environment (see [15]). Dao Trong Thi used covariant constant forms as calibration forms to examine globally minimal currents and surfaces in Riemannian manifolds. In particular, covariant constant forms have been used to study the minimality of such important classes of surfaces in compact Lie groups as totally geodesic submanifolds and primitive cycles, constructed by Pontryagin (see [9, 11, 12]). R. Harvey and H. B. Lawson Jr. were the first to use the terminology "*Calibration*" and they have conducted a systematic investigation of theory of calibration (see [3]). In particular, they have studied and used calibrations for studying foliations. In this paper, the theory of calibration is used for the study of invariant foliations on Riemannian manifolds. We consider here a problem, which seems particularly well suitable for studying the internal dynamics of a foliation.

Problem. *Given an action of a compact Lie group G on a smooth manifold X, and let $\mathcal{F}$ be an oriented, G-invariant, k-dimensional foliation of class C^1 on the manifold X.*

Can one find a closed k-form ω and a G-invariant Riemannian metric with respect to which ω becomes a calibration for $\mathcal{F}$?

This paper has two aims, first to establish sufficient conditions for an affirmative answer of the Problem, and then to investigate the relationship of tightness with the existence of invariant metrics and the exponential growth of leaves.

Note. When G is trivial, the problem was answered by R. Harvey and H. B. Lawson Jr. in [5].

This paper is organized as follows. Section 2 is preliminary. In Sec. 3, we establish the relationship of homologically tight and geometrically tight foliations with the existence of invariant metrics. In Sec. 4, we apply our results obtained in Sec. 3 to the investigation of calibrated foliations on symmetric spaces. In the last section, we establish the relationship of the existence of invariant metrics with the exponential growth of leaves of a tight foliation.

2. Preliminaries

In this section we briefly recall some of the important ideas and results of Geometric Measure Theory (more detail see [1, 14]).

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We denote by $\Lambda_{k,n}\mathbb{R}^n$ and $\Lambda^{k,n}\mathbb{R}^n$ the space of k-vectors and k-covectors respectively. The direct sums

$$\Lambda_{*,n} = \bigoplus_{k\geq 0} \Lambda_{k,n}\mathbb{R}^n$$

and

$$\Lambda^{*,n} = \bigoplus_{k\geq 0} \Lambda^{k,n}\mathbb{R}^n$$

form respectively the contravariant and covariant Grassmann algebras of $\mathbb{R}^n$ with the operation of exterior multiplication $\wedge$. The inner product $(.\,,.)$ of $\mathbb{R}^n$ and the corresponding norm $|\,.\,|$ in $\mathbb{R}^n$ induce the inner product and norm in $\Lambda_{*,n}$ and $\Lambda^{*,n}$, which will be also denoted by $(.\,,.)$ and $|\,.\,|$. Below, we will identify $\Lambda_{*,n}$ and $\Lambda^{*,n}$ by means of the scalar product.

Definition 2.1. *The comass of the k-covector ω is defined by*

$$\|\omega\|^* = \sup\{\omega(\xi);\ \xi \text{ is a simple } k\text{-vector and } |\xi| = 1\}$$

and the mass of the k-vector ξ is defined by

$$\|\xi\| = \sup\{\omega(\xi) : \omega \in \Lambda^{k,n}\mathbb{R}^n \text{ and } \|\omega\|^* \leq 1\}.$$

Let X be a Riemannian manifold. We denote the real vector space of differential k-forms of class ∞ on X by E^kX, and we equip it with the topology of compact convergence of all partial derivatives. Thus, the convergence of a sequence of k-forms in E^kX means uniform convergence, on each compact subset $K \subset X$, of each partial derivative of any order. The direct sum

$$E^*X = \bigoplus_k E^k X$$

is a graded differential algebra with exterior multiplication $\wedge$ and exterior differentiation d.

Definition 2.2. *The support* $\operatorname{spt}\varphi$ *of a k-form $\varphi \in E^k X$ is defined as the closure of the set*

$$\{x \in X : \varphi_x \neq 0\}.$$

Definition 2.3. *A k-current S (with compact support) on a Riemannian manifold X is a real continuous linear functional on $E^k X$.*

Definition 2.4. *The support* $\operatorname{spt} S$ *of a k-current S is the smallest closed set K such that $S(\varphi) = 0$ for all $\varphi \in E^k X$ with $\operatorname{spt}\varphi \subset X \setminus K$, and the mass of S is defined by*

$$M(S) = \sup\{S(\varphi) : \varphi \in E^k X,\ \|\varphi_x\|^* \leq 1\ \forall x \in X\}.$$

We see that the set $\operatorname{spt} S$ is always compact. We denote by $E_k X$ the space of k-currents (with compact support and) with finite mass on X, equipped with the weak topology which is given by the seminorms

$$S \to \sup_{\varphi \in A} |S(\varphi)|,$$

where A is an arbitrary finite set of k-forms in $E^k X$.

The direct sum

$$E_* X = \bigoplus_k E_k X$$

is a chain complex with boundary operator ∂ defined by the formula

$$\partial S(\varphi) = S(d\varphi).$$

Clearly, d and ∂ are continuous.

Definition 2.5. *A current S is called closed if $\partial S = 0$ and exact if S is a boundary, i.e., $S = \partial T$, where T is some current.*

Definition 2.6. *The topological dual space $\mathfrak{M}_k(X) \subset E_k X$ is the space of k-dimensional currents represented by integration. A current S belongs to $\mathfrak{M}_*(X)$ if S has compact support and has measure coefficients in any local coordinate system.*

To any *k-current* $S \in \mathfrak{M}_k(X)$, there corresponds a local Radon measure $\|S\|$ (called the *variational measure* of S) on X such that for an arbitrary real nonnegative continuous function f on X,

$$\int_X f d\|S\| = \sup\{S(\varphi) : \varphi \in E^k X,\ \|\varphi_x\|^* \leq f(x)\ \forall x \in X\}.$$

Note that when $S \in \mathfrak{M}_k(X)$, $M(S) = \int_X d\|S\|$. Furthermore, if S is a current defined by integration over a compact oriented k-dimensional submanifold with boundary $L^k \subset X$, then $\|S\|$ is just Hausdorff k-measure restricted to L^k. Namely, $\int_X f d\|S\| = \int_L f dvol$. In particular, $M(S) = \mathrm{vol}(L^k)$.

From Radon-Nikodym theorem, with any current $S \in \mathfrak{M}_k(X)$ one associates a measurable field of tangent k-vectors $\overrightarrow{S}_x$, defined $\|S\|$-a.e. and for arbitrary k-form $\varphi \in E^k(X)$ such that

$$S(\varphi) = \int \varphi(\overrightarrow{S}_x) d\|S\|(x).$$

(More briefly we denote $S = \overrightarrow{S}\|S\|$.) Here the vector $\overrightarrow{S}_x \in \Lambda^k T_x X$ and $\|\overrightarrow{S}_x\| = 1$ almost everywhere in the sense of the measure $\|S\|$. If S is given by integration over a submanifold L^k as above, then $\overrightarrow{S}_x$ is the unit simple k-vector representing the oriented tangent space to L^k at x.

It was shown in [1, 2] that certain compact families of currents can be used to define the homology of a Riemannian manifold X. We denote by $\mathfrak{R}_k X$ the set of all rectifiable k-currents. This is defined as the closure in the M-topology on $\mathfrak{M}_k(X)$ of the integral Lipschitz k-chains.

Further, we put

$$N_k X = \{S \in E_k X : M(S) + M(\partial S) < \infty\},$$

$$I_k X = \{S \in \mathfrak{R}_k X : \partial S \in \mathfrak{R}_{k-1} X\};$$

the direct sums

$$N_* X = \bigoplus_k N_k X$$

and

$$I_* X = \bigoplus_k I_k X$$

form, respectively, chain complexes of normal and integral currents with boundary operator ∂. It was proved by Federer and Fleming in [2] that there are natural isomorphisms

$$H_k(N_* X) \cong H_k(X; \mathbb{R})$$

and

$$H_k(I_* X) \cong H_k(X; \mathbb{Z}).$$

It is known that Geometric Measure Theory is fit for the inverstigation of foliations. Suppose the $\mathcal{F}$ is an oriented k-dimensional foliation of class C^r $(r \geq 1)$ on a manifold X. Given a Riemannian metric on X and let $P \in \Gamma(\Lambda^k TX)$ be the field of unit k-vectors tangent to the foliation $\mathcal{F}$.

Definition 2.7. *A current $S \in \mathfrak{M}_k(X)$ such that*

$$\overrightarrow{S}_x = P(x) \quad \text{for } \|S\|\text{-a.e. } x$$

is called a foliation current for $\mathcal{F}$. If in addition $dS = 0$ then S is called a d-closed (or closed) foliation current. A foliation current S which is rectifiable is called a foliation chain, or, in the case $dS = 0$ a foliation cycle.

3. Tightness and the Existence of Invariant Metrics

Definition 3.1. *Suppose a Lie group G acts on a Riemannian manifold X. We say that a foliation $\mathcal{F}$ of manifold X is a G-invariant foliation if the map*

$$\Pi_g : X \longrightarrow X$$

takes leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$ for any $g \in G$.

Definition 3.2. *Let a Lie group G act on a Riemannian manifold X and P is a k-plane field on X. The k-plane field P is called invariant under action of Lie group G, or simply a G-invariant plane field if $P(gx) = g_*P(x)$ for every $x \in X$ and $g \in G$.*

Definition 3.3. *Suppose $\mathcal{F}$ is an oriented k-dimensional foliation on a Riemannian manifold X, and let P denote the unit oriented k-plane field determined by $\mathcal{F}$. Then a calibration for $\mathcal{F}$ is a d-closed k-form ω of comass one such that*

$$\omega(P) \equiv 1$$

on X.

We note that there is a more general concept of a calibration that has been introduced and studied in detail by R.Harvey and H.B.Lawson, Jr. (see [3]).

Our first result is the following.

Proposition 3.4. *Suppose we are given an action of a compact Lie group G on a Riemannian manifold X. Let $\mathcal{F}$ be an oriented, G-invariant, calibrated, k-dimensional foliation of class C^1 on X defined by an oriented, k-plane field P. If there exists a number $N \in \mathbb{R}^+$ such that on each principal direction $P(x)$ of each leaf F of the foliation $\mathcal{F}$, one of the following conditions is satisfied:*
1. $|g_*P(x)|_{s^*} \leq N$ for all points $x \in X$ and for all $g \in G$,
2. $|g_*P(x)|_{s^*} \geq N$ for all points $x \in X$ and for all $g \in G$,
where s^* is an inner product in $\Lambda_k X_x$ which is induced from a Riemannian metric on X in which the foliation $\mathcal{F}$ can be calibrated.
Then there exists a closed k-form $\tilde{\omega}'$ and a G-invariant metric with respect to which $\tilde{\omega}'$ has comass one and $\tilde{\omega}'$ restricts to be the Riemannian volume form on the leaves.

Proof. 1. Assume that $\mathcal{F}$ is calibrated with calibration ω in a Riemannian metric s. By definition of the calibration, we have

(a) $\quad \omega(\xi) \leq |\xi|_{s^*}$ for all simple k-vectors ξ

and

(b) $\quad \omega_x(P(x)) = |P(x)|_{s^*}$ for all points $x \in F$.

Two conditions (a) and (b) are equivalent to the following conditions

(a′) $\omega(\xi) \leq |\xi|_{s^*}$ for all simple k-vectors ξ

and

(b′) $\omega_x(P(x)) = 1$ for $|P(x)|_{s^*} = 1$ and for all points $x \in F$.

Since the group G is a compact Lie group, there exists on G, as it was known, a unique bilateral invariant Haar measure such that the measure of the whole group G is equal to unity. Now we will use the method of integral average to construct the desired metric.

For each ω, we put

$$\widetilde{\omega}(\xi) = \int_G g^*\omega(\xi)dg = \int_G \omega(g_*\xi)dg$$

and we construct $\widetilde{s}$ as follows

$$\widetilde{s}(u, v) = \int_G g^* s(u, v)dg \text{ for all } u,\, v \in X_x.$$

$\widetilde{s}$ induces, as it is known, $\widetilde{s}^*$ on $\Lambda_k X_x$ defined by the following equality

$$\widetilde{s}^*(\xi, \eta) = \int_G g^* s^*(\xi, \eta)dg \text{ for all } \xi,\, \eta \in \Lambda_k X_x.$$

Lemma 3.5.
(i) *$\widetilde{\omega}$ is a closed k-form.*
(ii) *If ω is a G-invariant form, then $\widetilde{\omega} = \omega$.*

Proof. (i) Since ω is a closed k-form and the action of g^* commutes with operator d, we have

$$d\widetilde{\omega}(\xi) = \int_G dg^*\omega(\xi)dg = \int_G g^* d\omega(\xi)dg = 0$$

for any $\xi \in \Lambda_k X_x$, which implies that $d\widetilde{\omega} = 0$, that is, $\widetilde{\omega}$ is a closed k-form.

(ii) If ω is a G-invariant form, then $g^*\omega = \omega$ for any $g \in G$. Consequently, $\widetilde{\omega} = \omega$. ∎

Lemma 3.6. *$\widetilde{s}$ is a G-invariant metric.*

Proof. It is obvious that $\widetilde{s}$ is a Riemannian metric. Also, for each $g \in G$ we have

$$g^*\widetilde{s}(u, v) = \widetilde{s}(g_*(u, v)) = \int_G h^* s(g_*(u, v))dh = \int_G g^* h^* s(u, v)dh$$

$$= \int_G (hg)^* s(u, v)dhg = \widetilde{s}(u, v)$$

for all $u,\, v \in X_x$, that is, $g^*\widetilde{s} = \widetilde{s}$. Thus, $\widetilde{s}$ is G-invariant. This proves the lemma. ∎

Now we construct $\widetilde{s}'$ as follows

$$\widetilde{s}\,'(u,v) = \frac{1}{N^2}\int_G g^* s(u,v)dg = \frac{1}{N^2}\widetilde{s}(u,v) \text{ for all } u,\, v \in X_x,$$

and we construct $\widetilde{\omega}\,'$ as follows

$$\widetilde{\omega}\,'(\xi) = \frac{1}{N}\int_G g^*\omega(\xi)dg = \frac{1}{N}\widetilde{\omega}(\xi).$$

By the construction of $\widetilde{\omega}\,'$ and $\widetilde{s}\,'$, and from Lemmas 3.5 and 3.6 we conclude that $\widetilde{\omega}\,'$ is a closed k-form and $\widetilde{s}\,'$ is a G-invariant metric. Indeed, we see from Lemma 3.6 and the construction of $\widetilde{s}\,'$, that $\widetilde{s}\,'$ is a bilinear, symmetric, positive-definite form.

Now we only need prove the G-invariant property of $\widetilde{s}\,'$.

We consider

$$g^*\widetilde{s}\,'(u,v) = \widetilde{s}\,'(g_*(u,v)) \text{ for all } u,\, v \in X_x \text{ and for any } g \in G.$$

By definition of $\widetilde{s}\,'$,

$$\widetilde{s}\,'(g_*(u,v)) = \frac{1}{N^2}\widetilde{s}(g_*(u,v)) \text{ for all } u,\, v \in X_x \text{ and for any } g \in G.$$

From Lemma 3.6, we have

$$\frac{1}{N^2}\widetilde{s}(g_*(u,v)) = \frac{1}{N^2}\widetilde{s}(u,v) \text{ for all } u,\, v \in X_x \text{ and for any } g \in G,$$

and by the construction of $\widetilde{s}\,'$,

$$\frac{1}{N^2}\widetilde{s}(u,v) = \widetilde{s}\,'(u,v) \text{ for all } u,\, v \in X_x.$$

Hence $g_*\widetilde{s}\,'(u,v) = \widetilde{s}\,'(u,v)$ for all $u, v \in X_x$ and for all $g \in G$, as we wanted to prove.

In the following, we will show that $\widetilde{s}\,'$ is the desired metric.

Since $\mathcal{F}$ is a G-invariant foliation and from conditions (a) and (b), we have conditions (a") and (b") as follows

(a") $\omega(g_*\xi) \leq |g_*\xi|_{s^*}$ for all simple k-vectors ξ and for all $g \in G$;

(b") $\omega_{gx}(g_*(P(x))) = |g_*P(x)|_{s^*}$ for all points $x \in F$ and for all $g \in G$.

From condition (a$''$), we have

$$\frac{1}{N}\int_G \omega(g_*\xi)dg \leq \frac{1}{N}\int_G |g_*\xi|_{s^*}dg. \tag{3.1}$$

Using Hölder's inequality for integral, we obtain

$$\frac{1}{N}\int_G |g_*\xi|_{s^*}dg \leq \Big(\int_G |g_*\xi|^2_{s^*}dg\Big)^{1/2}\Big(\int_G \frac{1}{N^2}dg\Big)^{1/2}. \tag{3.2}$$

But

$$\Big(\int_G |g_*\xi|_{s^*}^2 dg\Big)^{1/2}\Big(\int_G \frac{1}{N^2}dg\Big)^{1/2} = \frac{1}{N}\Big(\int_G |g_*\xi|_{s^*}^2 dg\Big)^{1/2} = |\xi|_{\widetilde{s}',*} \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$\frac{1}{N}\int_G |g_*\xi|_{s^*} dg \le |\xi|_{\widetilde{s}',*}. \tag{3.4}$$

From (3.1) and (3.4) it follows that

$$\frac{1}{N}\int_G \omega(g_*\xi)dg \le |\xi|_{\widetilde{s}',*}. \tag{3.5}$$

Since $\widetilde{\omega}'(\xi) = \frac{1}{N}\int_G g^*\omega(\xi)dg = \frac{1}{N}\int_G \omega(g_*\xi)dg$, we deduce from (3.5) that

$$\widetilde{\omega}'(\xi) \le |\xi|_{\widetilde{s}',*}. \tag{3.6}$$

Assuming now that $|P(x)|_{\widetilde{s}',*} = 1$ we must show that

$$\widetilde{\omega}'_x(P(x)) = 1.$$

Indeed , by condition (b")

$$\frac{1}{N}\omega_{gx}(g_*P(x)) = \frac{1}{N}|g_*(P(x))|_{s^*}, \tag{3.7}$$

and by the assumption of Proposition 3.4, $|g_*(P(x))|_{s^*} \le N$ for all points $x \in X$ and for all $g \in G$, we have

$$\frac{1}{N}|g_*(P(x))|_{s^*} \ge \frac{1}{N^2}|g_*(P(x))|_{s^*}^2. \tag{3.8}$$

for all points $x \in X$ and for all $g \in G$.

From (3.7) and (3.8) it follows that

$$\frac{1}{N}\int_G \omega_{gx}(g_*(P(x)))dg \ge \frac{1}{N^2}\int_G |g_*(P(x))|_{s^*}^2 dg \tag{3.9}$$

for all points $x \in X$ and for all $g \in G$.

Since $\widetilde{\omega}'_x(P(x)) = \frac{1}{N}\int_G g^*\omega_x(P(x))dg = \frac{1}{N}\int_G \omega_{gx}(g_*(P(x)))dg$ and from the construction of $\widetilde{s}'$, we deduce from (3.9) that

$$\widetilde{\omega}'_x(P(x)) \ge |P(x)|_{\widetilde{s}',*}^2. \tag{3.10}$$

By assumption $|P(x)|_{\widetilde{s}',*} = 1$, and from (3.10)

$$\widetilde{\omega}'_x(P(x)) \ge 1.$$

On the other hand, in the previous proof we have

$$\widetilde{\omega}'_x(P(x)) \le |P(x)|_{\widetilde{s}',*} = 1.$$

Hence, we obtain

$$\widetilde{\omega}'_x(P(x)) = 1 \tag{3.11}$$

as claimed.

Since $\widetilde{s}'$ is a G-invariant metric we conclude from (3.6) and (3.11) that $\widetilde{s}'$ is the desired metric.

2. We proceed as above in this case, we construct $\widetilde{s}'$ and $\widetilde{\omega}'$ by the method of integral average mentioned above, i.e.,

$$\widetilde{\omega}'(\xi) = \frac{1}{N}\widetilde{\omega}(\xi) = \frac{1}{N}\int_G g^*\omega(\xi)dg,$$

and

$$\widetilde{s}'(u,v) = \frac{1}{N^2}\widetilde{s}(u,v) = \frac{1}{N^2}\int_G g^*s(u,v)dg$$

for all u, $v \in X_x$.

Hence, as before, we conclude that $\widetilde{\omega}'$ is a closed k-form and $\widetilde{s}'$ is a G-invariant metric.

Analogous to the case 1 of Proposition 3.4, we use Hölder's inequality for integral and we also obtain

$$\widetilde{\omega}'(\xi) \leq |\xi|_{\widetilde{s}'^*} \tag{3.12}$$

for all simple k-vectors ξ.

Now suppose that $|P(x)|_{\widetilde{s}'^*} = 1$, we must show that

$$\widetilde{\omega}'_x(P(x)) = 1.$$

Indeed, by assumption of Proposition 3.4 $|g_*(P(x))|_{s^*} \geq N$ for all points $x \in X$ and for all $g \in G$, we have

$$\frac{1}{N}|g_*(P(x))|_{s^*} \geq 1 \tag{3.13}$$

for all points $x \in X$ and for all $g \in G$.

By condition (b")

$$\frac{1}{N}\int_G \omega_{gx}(g_*(P(x)))dg = \frac{1}{N}\int_G |g_*(P(x))|_{s^*}dg.$$

From (3.13) it follows that

$$\frac{1}{N}\int_G |g_*(P(x))|_{s^*}dg \geq \int_G 1\ dg = 1.$$

Hence

$$\frac{1}{N}\int_G \omega_{gx}(g_*(P(x)))dg \geq 1. \tag{3.14}$$

Since $\widetilde{\omega}'_x(P(x)) = \frac{1}{N}\int_G g^*\omega_x(P(x))dg = \frac{1}{N}\int_G \omega_{gx}(g_*(P(x)))dg$, we deduce from (3.14) that

$$\widetilde{\omega}'_x(P(x)) \geq 1.$$

On the other hand, in the above proof we have

$$\widetilde{\omega}'_x(P(x)) \leq |P(x)|_{\widetilde{s}'^*} = 1,$$

hence we obtain

$$\widetilde{\omega}_x'(P(x)) = 1 \tag{3.15}$$

as claimed.

Since $\widetilde{s}'$ is a G-invariant metric we conclude from (3.12) and (3.15) that $\widetilde{s}'$ is the desired metric.

The proof of Proposition 3.4 is complete. ∎

Remark 1. If X is a compact Riemannian manifold, then there always exists a closed k-form $\widetilde{\omega}'$ and a G-invariant metric on X with respect to which $\widetilde{\omega}'$ has comass one and $\widetilde{\omega}'$ restricts to be Riemannian volume form on the leaves of $\mathcal{F}$.

Indeed, since the map $f : G \times X \longrightarrow \mathbb{R}$ defined by

$$f(g,x) = |g_* P(x)|_{s^*}$$

is a continuous function of g and x, we conclude from the above assumption that there always exists a number N which satisfies conditions of Proposition 3.4. Thus, there exists a closed k-form $\widetilde{\omega}'$ and a G-invariant metric on X with respect to which $\widetilde{\omega}'$ has comass one and $\widetilde{\omega}'$ restricts to be the Riemannian volume form on the leaves of $\mathcal{F}$.

Definition 3.7. *A k-dimensional current S (with compact support) on a Riemannian manifold X is said to be homologically mass minimizing if the mass of current S, $M(S) < \infty$ and $M(S) \le M(S')$ for every k-current S' homologous to S.*

Definition 3.8. *An oriented k-dimensional foliation $\mathcal{F}$ on a Riemannian manifold X is said to be geometrically tight if every foliation current is homologically mass minimizing.*

Note that if $\mathcal{F}$ is geometrically tight then every leaf in $\mathcal{F}$ is a minimal submanifold of X.

Definition 3.9. *An oriented k-dimensional foliation $\mathcal{F}$ on a manifold X is said to be homologically tight if no (non-zero) closed foliation current is homologous to zero as a compactly supported de Rham current in X.*

We note that the calibrated foliation is always geometrically tight.

In Proposition 3.4, we have established sufficient conditions for an affirmative answer to the Problem. However, the arguments presented there actually prove much more, namely:

Theorem 3.10. *Suppose we are given an action of a compact Lie group G on a Riemannian manifold X. Let $\mathcal{F}$ be an oriented, G-invariant, k-dimensional foliation of class C^1 on X defined by an oriented, k-plane field P. Consider the following conditions.*

(1) $\mathcal{F}$ is homologically tight.

(2) $\mathcal{F}$ is a calibrated foliation and there exists a number $N \in \mathbb{R}^+$ such that on each principal direction $P(x)$ of each leaf F of the foliation $\mathcal{F}$, one of the following conditions is satisfied:
(a) $|g_* P(x)|_{s^*} \leq N$ for all points $x \in X$ and for all $g \in G$,
(b) $|g_* P(x)|_{s^*} \geq N$ for all points $x \in X$ and for all $g \in G$,
where s^* is an inner product in $\Lambda_k X_x$, which is induced from a Riemannian metric on X in which the foliation $\mathcal{F}$ can be calibrated.

(3) There exists a closed k-form $\widetilde{\omega}'$ and a G-invariant metric with respect to which $\widetilde{\omega}'$ has comass one and $\widetilde{\omega}'$ restricts to be the Riemannian volume form on the leaves of $\mathcal{F}$.

(4) There exists a G-invariant metric on X in which $\mathcal{F}$ is geometrically tight.

Then (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1); and if X is compact the conditions are equivalent.

Corollary 3.11. *Suppose we are given an action of a compact Lie group on a Riemannian manifold X. Let $\mathcal{F}$ be an oriented, calibrated, k-dimensional foliation of class C^1 on X defined by an oriented and G-invariant plane field P. Then there exists a closed k-form $\widetilde{\omega}$ and a G-invariant metric on X with respect to which $\widetilde{\omega}$ has comass one and restricts to be the Riemannian volume form on the leaves.*

Proof. We can prove Corollary 3.11 directly from Theorem 3.10. Here $N = 1$. We see, therefore, that $\widetilde{\omega}$ is the desired k-form and $\widetilde{s}$ is the desired metric. ■

4. Calibrated Foliations on Symmetric Spaces

Let X be a compact Riemannian symmetric space. It is known that X admits a representation in the form G/H, where $G = I_0(X)$ is a connected group of isometries of X, and H is a stationary subgroup. Suppose that σ is an involutory automorphism of G, whose set of fixed points coincides with H. It is known that, for a compact symmetric space X, every G-invariant generalized k-form is a closed differential form and is identified with the H-invariant k-covectors determined by it at an arbitrary fixed point. Further, there is a unique G-invariant form in every cohomology class such that the group of G-invariant forms is isomorphic to the cohomology group.

Definition 4.1. *A foliation $\mathcal{F}$ of a Riemannian manifold X is called a totally geodesic foliation if each leaf of the foliation $\mathcal{F}$ is a totally geodesic submanifold of the Riemannian manifold X.*

Theorem 4.2. *Let $X = G/H$ be a compact Riemannian symmetric space, $\mathcal{F}$ is a totally geodesic foliation, G-invariant, on X. Then there exists a G-invariant metric on X in which the foliation $\mathcal{F}$ can be calibrated.*

Proof. This theorem is a direct consequence of Theorem 3.10 and Remark 1. ■

Now we apply Theorem 4.2 to the investigation of compact Lie groups.

Let G be a connected compact Lie group. It is known that G admits a representation in the form G/H, where the group $G \times G$ acts on G by the formula

$$(g_1, g_2)h = g_1 . h . g_2^{-1} = L_{g_1} . R_{g_2^{-1}} h \tag{4.1}$$

for any $h \in G$.

The stationary subgroup H is a group of the inner automorphisms of G, that is, H acts on G by the following formula

$$\mathrm{Int}_g h = ghg^{-1} \text{ for any } h \in G. \tag{4.2}$$

Lemma 4.3. (see [13]) *Let X be a Riemannian globally symmetric space and p_0 any point in X. If $G = I_0(X)$, and H is the subgroup of G which leaves p_0 fixed, then H is a compact subgroup of the connected group G and G/H is analytically diffeomorphic to X under the mapping $gH \mapsto g.p_0$, $g \in G$. Identifying as usual the tangent space X_{p_0} with the subspace $\mathfrak{p}$ of the Lie algebra of $I(X)$, let $\mathfrak{s}$ be a Lie triple system contained in $\mathfrak{p}$. Put $S = \mathrm{Exp}\,\mathfrak{s}$. Then S has a natural differentiable structure in which it is a totally geodesic submanifold of X satisfying $S_{p_0} = \mathfrak{s}$.*

On the other hand, if S is a totally geodesic submanifold of X and $p_0 \in S$, then the subspace $\mathfrak{s} = S_{p_0}$ of $\mathfrak{p}$ is a Lie triple system.

Now we will consider a foliation of a connected compact Lie group G.

Suppose that K is a closed Lie subgroup of G. Then the Lie algebra $\mathfrak{k}$ of K is a subalgebra of $\mathfrak{g}$, the Lie algebra of G. We may identify $\mathfrak{g} = G_e$ and $\mathfrak{k} = K_e$, where e is the unit element of Lie group G. From Lemma 4.3, we conclude that K is a totally geodesic submanifold of G. Assuming now that $\dim K = k$ we consider the k-plane field P on G determined by the formula

$$P(g) = L_{g*}\vec{K}_e \text{ for any } g \in G, \tag{4.3}$$

where $\vec{K}_e$ is the simple unit k-vector associated to K_e.

It is easy to prove that the k-plane field P on G defined by the formula (4.3) is completely integrable.

We conclude from the Frobenius theorem that there exists a C^1 foliation $\mathcal{F}$ of dimension k on G such that $T_g(\mathcal{F}) = P(g)$ for every $g \in G$. Further, the foliation $\mathcal{F}$ is unique.

Remark 2. We see that each leaf F of the foliation $\mathcal{F}$, corresponding with the k-plane field P defined by the formula (4.3) is a coset gK, $g \in G$.

Theorem 4.4. *Suppose $\mathcal{F}$ is a k-dimensional foliation of Lie group G, corresponding with the k-plane field P defined by the formula*

$$P(g) = L_{g*}\vec{K}_e \text{ for any } g \in G,$$

where $\vec{K}_e$ is the simple unit k-vector associated to K_e, and K is a closed Lie subgroup of G, e is the unit element of Lie group G. Then there exists a G-invariant metric on G in which the foliation $\mathcal{F}$ can be calibrated.

Proof. The proof of Theorem 4.4 follows immediately from Theorem 4.2 and Remark 2. ■

5. The Exponential Growth of Leaves of Invariant Foliations

In this section, we investigate the relationship of tightness with the existence of G-invariant metrics and the exponential growth of leaves.

Theorem 5.1. *Let a compact Lie group G act on a compact manifold X and let $\mathcal{F}$ be a tight, G-invariant foliation of dimension k on the manifold X whose k^{th} homology group (over $\mathbb{R}$) is zero. Then there exists a G-invariant metric on X in which the growth of each leaf of $\mathcal{F}$ must be exponential.*

Proof. Assume that $\mathcal{F}$ is calibrated with calibration ω. By Theorem 3.10 and Remark 1, there exists a closed k-form $\tilde{\omega}'$ and a G-invariant metric $\tilde{s}'$ with respect to which $\tilde{\omega}'$ becomes a calibration for $\mathcal{F}$. In the following, we will show that $\tilde{s}'$ is the desired metric.

Let $f(r)$ denote the volume of a geodesic ball B_r of radius r in the leaf F (centered at some point $a \in F$). Then $f'(r)$ is the volume or mass of the boundary of this ball. By the construction of $\tilde{\omega}'$, and since $H_k(X;\mathbb{R}) = 0$, there exists a form ψ with $d\psi = \omega$, consequently

$$f(r) = \int_{B_r} \tilde{\omega}' = \frac{1}{N}\int_{B_r}\int_G g^*\omega dg = \frac{1}{N}\int_{B_r}\int_G g^* d\psi dg. \tag{5.1}$$

Since the action of g^* commutes with operator d, we deduce from (5.1) that

$$f(r) = \frac{1}{N}\int_{B_r}\int_G dg^*\psi dg. \tag{5.2}$$

From Fubini's theorem, we have

$$\frac{1}{N}\int_{B_r}\int_G dg^*\psi dg = \frac{1}{N}\int_G\int_{B_r} dg^*\psi dg. \tag{5.3}$$

From (5.2) and (5.3) it follows that

$$f(r) = \frac{1}{N}\int_G\int_{B_r} dg^*\psi dg. \tag{5.4}$$

Using Stokes' theorem, we obtain

$$\frac{1}{N}\int_G\int_{B_r} dg^*\psi dg = \frac{1}{N}\int_G\int_{\partial B_r} g^*\psi dg. \tag{5.5}$$

From (5.4) and (5.5) it follows that

$$f(r) = \frac{1}{N}\int_G\int_{\partial B_r} g^*\psi dg.$$

Since we have used the method of integral average to construct the metric $\tilde{s}'$ and the form $\tilde{\omega}'$, we have $\int_G dg = 1$, consequently

$$f(r) = \frac{1}{N}\int_G \int_{\partial B_r} g^*\psi dg \leq \frac{1}{N}\int_G c_1 f'(r)dg = c_2 f'(r),$$

which integrates to $f(r) \geq le^{\frac{r}{c_2}}$, where c_1 is the supremum of comasses of the forms $g^*\psi$ with $\forall g \in G$, and $c_1 > 0$.
The proof of Theorem 5.1 is complete. ■

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