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The Bundle Structure of Spherical Non-Commutative Tori

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Abstract. Let A_{cd} be a cd-homogeneous C^*-algebra over $\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^{r+2}$ of which no non-trivial matrix algebra can be factored out. The spherical non-commutative torus $\mathbb{S}^{cd}_\rho$ is defined by twisting $C^*(\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2})$ in $A_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by a totally skew multiplier ρ on $\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2}$. It is shown that $\mathbb{S}^{cd}_\rho \otimes M_{p^\infty}$ is isomorphic to $C(\prod^e S^2 \times \prod^s S^1) \otimes C^*(\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2}, \rho) \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p.

1. Introduction

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$, which is the universal object for unitary ω-representations of G. $C^*(\mathbb{Z}^m, \omega)$ is said to be a *non-commutative torus of rank m* and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group*, and the multiplier ω is called *totally skew* if the symmetry group S_ω is trivial. And A_ω is called *completely irrational* if ω is totally skew. See [1, 8, 11]. It was shown in [1] that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G, \omega)$ is a simple C^*-algebra.

An important problem, in the bundle theory of geometry, is to compute the set $[M, BPU(cd)]$ of homotopy classes of continuous maps of a compact CW-complex M into the classifying space $BPU(cd)$ of the Lie group $PU(cd)$. The set $[M, BPU(cd)]$ is in bijective correspondence with the set of equivalence classes of principal $PU(cd)$-bundles over M, which is in bijective correspondence with the set of cd-homogeneous C^*-algebras over M. That is, each cd-homogeneous C^*-algebra A over M is isomorphic to the C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η with base space M, fibres $M_{cd}(\mathbb{C})$, and structure group $\mathrm{Aut}(M_{cd}(\mathbb{C})) \cong PU(cd)$. See [10] for details. So each cd-homogeneous

C^*-algebra over $\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^{r+2}$ is realized as the C^*-algebra $\Gamma(\zeta)$ of sections of a locally trivial C^*-algebra bundle ζ over $\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^{r+2}$ with fibres $M_{cd}(\mathbb{C})$. Thus the spherical non-commutative torus $\mathbb{S}_\rho^{cd}$, defined in Section 2, is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod^e S^2 \times \prod^s S^1$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$, where P_ρ^d is defined in Section 2.

We are going to show that $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod^e S^2 \times \prod^s S^1) \otimes C^*(\widehat{\mathbb{T}^{r+2} \times \mathbb{Z}^{m-2}}, \rho) \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p. And it is shown that $\mathbb{S}_\rho^{cd} \otimes \mathcal{O}_{2u}$ is isomorphic to $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes \mathcal{O}_{2u}$ if and only if cd and $2u-1$ are relatively prime, and that $\mathbb{S}_\rho^{cd} \otimes \mathcal{O}_\infty$ is not isomorphic to $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes \mathcal{O}_\infty$ if $cd > 1$, where $\mathcal{O}_u$ and $\mathcal{O}_\infty$ denote the Cuntz algebra and the generalized Cuntz algebra, respectively.

2. Homogeneous C^*-Algebras Over $\prod^e S^2 \times \prod^{s+r+2} S^1$

$[S^2, BPU(cd)] = [S^1, PU(cd)] \cong \mathbb{Z}_{cd}$, which is a cyclic group. So each group has a generator, and there is a unitary $U(z) \in PU(cd)$ such that the generating cd-homogeneous C^*-algebra over S^2 can be realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^2 with fibres $M_{cd}(\mathbb{C})$ characterized by the unitary $U(z) \in PU(cd)$ over S^1. If $(cd, k) = p$ $(p > 1)$, then consider the cd-homogeneous C^*-algebra over S^2 corresponding to each $k \in \mathbb{Z}_{cd}$ as the tensor product of $M_p(\mathbb{C})$ with a $\frac{cd}{p}$-homogeneous C^*-algebra over S^2, which is given by $U(z)^{k/p} \in PU(\frac{cd}{p})$. Consider $U(z)^k$ as $U(z)^{k/p} \otimes I_p \in PU(cd)$, where I_p denotes the $p \times p$ identity matrix. Then each cd-homogeneous C^*-algebra $B_{k/cd}$ over S^2 can be realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^2 with fibres $M_{cd}(\mathbb{C})$ characterized by the unitary $U(z)^k \in PU(cd)$ over S^1 for some $k \in \mathbb{Z}_{cd}$.

Krauss and Lawson [10] proved that each cd-homogeneous C^*-algebra over S^2 is isomorphic to one of the C^*-subalgebras $B_{k/cd} = C_{g_k}(e_+^2 \amalg e_-^2, M_{cd}(\mathbb{C}))$, $k \in \mathbb{Z}_{cd}$, given as follows: if $f \in B_{k/cd}$, then the following condition is satisfied

$$f_+(z) = U(z)^k f_-(z) U(z)^{-k}$$

for all $z \in S^1$, where $U(z) \in PU(cd) = \mathrm{Inn}(M_{cd}(\mathbb{C}))$ is the unitary given above, and e_+^2 (resp. e_-^2) is the 2-dimensional northern (resp. southern) hemisphere.

Since there is a map of degree 1 from S^2 to $S^1 \times S^1$, there are cd-homogeneous C^*-algebras over $S^1 \times S^1$ induced from cd-homogeneous C^*-algebras over S^2. So every cd-homogeneous C^*-algebra over $S^1 \times S^1$ is isomorphic to one of the C^*-subalgebras $A_{k/cd}$, $k \in \mathbb{Z}_{cd}$, of $C(S^1 \times [0,1], M_{cd}(\mathbb{C}))$, given as follows: if $f \in A_{k/cd}$, then the following condition is satisfied

$$f(z, 1) = U(z)^k f(z, 0) U(z)^{-k}$$

for all $z \in S^1$, where $U(z) \in PU(cd)$ is the unitary given above. See [3].

Lemma 2.1. *Let $B_{k/cd}$ be a cd-homogeneous C^*-algebra over S^2 of which no non-trivial matrix algebra can be factored out. Then $[1_{B_{k/cd}}] \in K_0(B_{k/cd}) \cong \mathbb{Z}^2$ is primitive.*

Proof. It was shown in [3, Lemma 3.1] that $B_{k/cd}$ is stably isomorphic to $C(S^2)\otimes M_{cd}(\mathbb{C})$. So $K_0(B_{k/cd}) \cong K_0(C(S^2)) \cong \mathbb{Z}\oplus\mathbb{Z}$. Since

$$[S^2, BPU(cd)] \cong [S^1, PU(cd)] \cong [S^1\times S^1, BPU(cd)] \cong \mathbb{Z}_{cd},$$

$B_{k/cd}$ corresponds to $A_{k/cd}$ with respect to the conditions on sections over the boundaries S^1 of $e_+^2 \amalg e_-^2$ and $S^1\times[0,1]$. The proof of Elliott's theorem given in [6, Theorem 2.2] implies that the canonical embedding of each factor $C(\mathbb{T}^1)$ of $C(\mathbb{T}^2)$ into $A_{k/cd}$ induces an isomorphism of $K_0(C(\mathbb{T}^2))$ into $K_0(A_{\frac{k}{cd}})$ such that the class $[1_{C(\mathbb{T}^2)}]$ of the unit $1_{C(\mathbb{T}^2)}$ maps to the class $[1_{A_{k/cd}}]$ of the unit $1_{A_{k/cd}}$. The canonical embedding of $C(S^1)$ into $A_{k/cd}$ which induces the isomorphism of $K_0(C(S^1\times S^1))$ into $K_0(A_{k/cd})$ corresponds to the embedding ϕ of $C(S^1)$ into $B_{k/cd}$. The canonical embedding ϕ of $C(S^1)$ into $B_{k/cd}$ induces an isomorphism μ of $K_0(C(S^2))$ into $K_0(B_{\frac{k}{cd}})$, where $S^1 = \partial e_{\pm}^2$. The unit $1_{C(S^1)}$ maps to the unit $1_{C(S^2)}$ under the canonical embedding ψ of $C(S^1)$ into $C(S^2)$. $[1_{C(S^1)}]\in K_0(C(S^1)) \cong \mathbb{Z}$ maps to $[1_{C(S^2)}]\in K_0(C(S^2))\cong\mathbb{Z}^2$, primitive in $K_0(C(S^2))$ (see [9]). In the commutative diagram

$$\begin{array}{ccc} K_0(C(S^1)) & \xrightarrow{\psi_*} & K_0(C(S^2)) \\ \Big\downarrow{\scriptstyle (\text{identity})_*} & & \Big\downarrow{\scriptstyle \mu(\cong)} \\ K_0(C(S^1)) & \xrightarrow{\phi_*} & K_0(B_{k/cd}), \end{array}$$

$\mu([1_{C(S^2)}]) = \phi_*\circ(\text{identity})_*\circ\psi_*^{-1}([1_{C(S^2)}]) = [1_{B_{k/cd}}]$. So $[1_{B_{k/cd}}]$ is the image of the primitive element $[1_{C(S^2)}]\in K_0(C(S^2))$ under the isomorphism μ. Hence $[1_{B_{k/cd}}]\in K_0(B_{k/cd})$ is primitive.

Therefore, $[1_{B_{k/cd}}]\in K_0(B_{k/cd})\cong\mathbb{Z}^2$ is primitive. ■

The proof given in Lemma 2.1 implies that the canonical embedding of $C(S^1)$ into $B_{k/cd}$ induces an isomorphism of $K_0(C(S^2))$ into $K_0(B_{k/cd})$ such that the class $[1_{C(S^2)}]$ of the unit $1_{C(S^2)}$ maps to the class $[1_{B_{k/cd}}]$ of the unit $1_{B_{k/cd}}$.

If $s+r$ is odd, one can make the integer even by tensoring with $C(S^1)$. So one can assume that $s+r$ is even.

In [3, Theorem 2.5], the authors constructed cd-homogeneous C^*-subalgebras $E^{a_1,\dots,a_e}_{b_1,\dots,b_{(s+r+2)/2}}$ $(a_1,\dots,a_e,b_1,\dots,b_{(s+r+2)/2}\in\mathbb{Z})$ over $\prod^e S^2\times\prod^{s+r+2}S^1$ of $C\Big(\prod^e(e_+^2\amalg e_-^2)\times\prod^{(s+r+2)/2}(S^1\times[0,1]), M_{cd}(\mathbb{C})\Big)$, and constructed all cd-homogeneous C^*-algebras over $\prod^e S^2\times\prod^{s+r+2}S^1$.

Theorem 2.2. *Let A_{cd} be a cd-homogeneous C^*-algebra over $\prod^e S^2\times\prod^{s+r+2}S^1$, of which any non-trivial matrix algebra cannot be factored. Then $K_0(A_{cd})\cong K_1(A_{cd})\cong\mathbb{Z}^{2^{e+s+r+1}}$, and $[1_{A_{cd}}]\in K_0(A_{cd})$ is primitive.*

Proof. It was shown in [3, Lemma 3.1] that A_{cd} is stably isomorphic to $C(\prod^e S^2\times\prod^{s+r+2}S^1)\otimes M_{cd}(\mathbb{C})$. By Künneth's theorem [2, Theorem 23.1.3]

$$K_0(A_{cd}) \cong K_0(C(\prod^{e} S^2 \times \prod^{s+r+2} S^1))$$
$$\cong K_0(C(\prod^{e} S^2)) \otimes K_0(C(\prod^{s+r+2} S^1)) \oplus K_1(C(\prod^{e} S^2)) \otimes K_1(C(\prod^{s+r+2} S^1))$$
$$\cong \mathbb{Z}^{2^e} \otimes \mathbb{Z}^{2^{s+r+1}} \oplus \{0\} \otimes \mathbb{Z}^{2^{s+r+1}} \cong \mathbb{Z}^{2^{e+s+r+1}}.$$

Similarly, one obtains that $K_1(A_{cd}) \cong \mathbb{Z}^{2^{e+s+r+1}}$.

It is enough to show that $[1_{A_{cd}}] \in K_0(A_{cd})$ is primitive. First of all, we show that $[1_{E^{a_1,\dots,a_e}_{b_1,\dots,b_{(s+r+2)/2}}}] \in K_0(E^{a_1,\dots,a_e}_{b_1,\dots,b_{(s+r+2)/2}})$ is primitive. By the same reasoning as in the proof given in Lemma 2.1, the canonical embedding ϕ of $C(\prod^e S^1 \times \prod^{(s+r+2)/2} S^1)$ into $E^{a_1,\dots,a_e}_{b_1,\dots,b_{(s+r+2)/2}}$ induces an isomorphism μ of $K_0(C(\prod^e S^2 \times \prod^{s+r+2} S^1))$ into $K_0(E^{a_1,\dots,a_e}_{b_1,\dots,b_{(s+r+2)/2}})$. The unit $1_{C(\prod^e S^1 \times \prod^{(s+r+2)/2} S^1)}$ maps to the unit $1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}$ under the canonical embedding ψ of $C(\prod^e S^1 \times \prod^{(s+r+2)/2} S^1)$ into $C(\prod^e S^2 \times \prod^{s+r+2} S^1)$.

$$[1_{C(\prod^e S^1 \times \prod^{(s+r+2)/2} S^1)}] \in K_0(C(\prod^{e} S^1 \times \prod^{(s+r+2)/2} S^1))$$

maps to

$$[1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}] \in K_0(C(\prod^{e} S^2 \times \prod^{s+r+2} S^1)),$$

primitive in $K_0(C(\prod^e S^2 \times \prod^{s+r+2} S^1))$ (see [9]). In the commutative diagram

$$\begin{array}{ccc} K_0(C(\prod^{e} S^1 \times \prod^{(s+r+2)/2} S^1)) & \xrightarrow{\psi_*} & K_0(C(\prod^{e} S^2 \times \prod^{s+r+2} S^1)) \\ \Big\downarrow{\scriptstyle (\text{identity})_*} & & \Big\downarrow{\scriptstyle \mu(\cong)} \\ K_0(C(\prod^{e} S^1 \times \prod^{(s+r+2)/2} S^1)) & \xrightarrow{\phi_*} & K_0(E^{a_1,\dots,a_e}_{b_1,\dots,b_{(s+r+2)/2}}), \end{array}$$

$$\mu([1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}])$$
$$= \phi_* \circ (\text{identity})_* \circ \psi_*^{-1}([1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}]) = [1_{E^{a_1,\dots,a_e}_{b_1,\dots,b_{(s+r+2)/2}}}].$$

So $[1_{E^{a_1,\dots,a_e}_{b_1,\dots,b_{(s+r+2)/2}}}]$ is the image of the primitive element

$$[1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}] \in K_0(C(\prod^{e} S^2 \times \prod^{s+r+2} S^1))$$

under the isomorphism μ. Hence $[1_{E^{a_1,\dots,a_e}_{b_1,\dots,b_{(s+r+2)/2}}}] \in K_0(E^{a_1,\dots,a_e}_{b_1,\dots,b_{(s+r+2)/2}})$ is primitive.

Next, assume that $A_{cd} \cong C_1 \otimes C_2 \otimes \ldots \otimes C_q$, where C_i are of the type above. Then

$$K_0(A_{cd}) \cong K_0(C_1) \otimes K_0(C_2) \otimes \ldots \otimes K_0(C_q) \oplus \ldots$$

and $[1_{A_{cd}}]$ is the image of the primitive element $[1_{C_1}] \otimes [1_{C_2}] \otimes \ldots \otimes [1_{C_q}] \in K_0(C_1) \otimes K_0(C_2) \otimes \ldots \otimes K_0(C_q)$ under the isomorphism. So $[1_{A_{cd}}] \in K_0(A_{cd})$ is primitive.

Now assume that A_{cd} is a general *cd*-homogeneous C^*-algebra over $\prod^e S^2 \times \prod^{s+r+2} S^1$. The proof given above implies that the canonical embedding of each factor $C(S^1)$ of $C(\prod^e S^1 \times \prod^{s+r+2} S^1)$ into A_{cd} induces an isomorphism of $K_0(C(\prod^e S^2 \times \prod^{s+r+2} S^1))$ into $K_0(A_{cd})$ such that the class $[1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}]$ of the unit $1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}$ maps to the class $[1_{A_{cd}}]$ of the unit $1_{A_{cd}}$. By the same reasoning as in the proof given above, the canonical embedding ϕ of each factor $C(S^1)$ of $C(\prod^e S^1 \times \prod^{s+r+2} S^1)$ into A_{cd} induces an isomorphism μ of

$$K_0(C(\prod^e S^2 \times \prod^{s+r+2} S^1)) \to K_0(A_{cd}).$$

The unit $1_{C(\prod^e S^1 \times \prod^{s+r+2} S^1)}$ maps to the unit $1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}$ under the canonical embedding ψ of $C(\prod^e S^1 \times \prod^{s+r+2} S^1)$ into $C(\prod^e S^2 \times \prod^{s+r+2} S^1)$.

$$[1_{C(\prod^e S^1 \times \prod^{s+r+2} S^1)}] \in K_0(C(\prod^e S^1 \times \prod^{s+r+2} S^1))$$

maps to

$$[1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1}] \in K_0(C(\prod^e S^2 \times \prod^{s+r+2} S^1)),$$

primitive in $K_0(C(\prod^e S^2 \times \prod^{s+r+2} S^1))$. In the commutative diagram

$$\begin{array}{ccc} K_0(C(\prod^e S^1 \times \prod^{s+r+2} S^1)) & \xrightarrow{\psi_*} & K_0(C(\prod^e S^2 \times \prod^{s+r+2} S^1)) \\ \text{(identity)}_* \downarrow & & \downarrow \mu(\cong) \\ K_0(C(\prod^e S^1 \times \prod^{s+r+2} S^1)) & \xrightarrow{(\otimes^{e+s+r+2}\phi)_*} & K_0(A_{cd}), \end{array}$$

$$\mu([1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}]) = (\otimes^{e+s+r+2}\phi)_* \circ (\text{identity})_* \circ \psi_*^{-1}([1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}]) = [1_{A_{cd}}],$$

i.e., μ must be the canonical extension of $(\text{identity})_*$:

$$K_0(C(\prod^e S^1 \times \prod^{s+r+2} S^1)) \to K_0(C(\prod^e S^1 \times \prod^{s+r+2} S^1)).$$

So $[1_{A_{cd}}]$ is the image of the primitive element

$$[1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}] \in K_0(C(\prod^e S^2 \times \prod^{s+r+2} S^1))$$

under the isomorphism μ. Hence $[1_{A_{cd}}] \in K_0(A_{cd})$ is primitive.

Therefore, $K_0(A_{cd}) \cong K_1(A_{cd}) \cong \mathbb{Z}^{2^{e+s+r+1}}$, and $[1_{A_{cd}}] \in K_0(A_{cd})$ is primitive. ■

The proof given in Theorem 2.2 implies that the canonical embedding of each factor $C(S^1)$ of $C(\prod^e S^1 \times \prod^{s+r+2} S^1)$ into A_{cd} induces an isomorphism of $K_0(C(\prod^e S^2 \times \prod^{s+r+2} S^1))$ into $K_0(A_{cd})$ such that $[1_{C(\prod^e S^2 \times \prod^{s+r+2} S^1)}]$ maps to $[1_{A_{cd}}]$.

3. Spherical Non-Commutative Tori

The non-commutative torus A_ω of rank m is obtained by an iteration of $m-1$ crossed products by actions of $\mathbb{Z}$, the first action on $C(\mathbb{T}^1)$ (see [6]). When A_ω is not simple, by a change of basis, A_ω is obtained by an iteration of $m-2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{l/d}$. Since the fibre $M_d(\mathbb{C})$ of $A_{l/d}$ is a factor of the fibre of A_ω, A_ω can be obtained by an iteration of $m-2$ crossed products by actions of $\mathbb{Z}$, the first action on $A_{l/d}$, where the actions of $\mathbb{Z}$ on the fibre $M_d(\mathbb{C})$ of $A_{l/d}$ are trivial. So one can assume that A_ω is given by twisting $C^*(d\mathbb{Z} \times d\mathbb{Z} \times \mathbb{Z}^{m-2})$ in $A_{l/d} \otimes C^*(\mathbb{Z}^{m-2})$ by the restriction of the multiplier ω to $d\mathbb{Z} \times d\mathbb{Z} \times \mathbb{Z}^{m-2}$, where $\widehat{d\mathbb{Z}} \times \widehat{d\mathbb{Z}}$ is the primitive ideal space of $A_{l/d}$ and $C^*(d\mathbb{Z} \times d\mathbb{Z}, \text{res of } \omega) = C^*(d\mathbb{Z} \times d\mathbb{Z})$.

Definition 3.1 [3, Definition 1.1]. *Let A_{cd} be a cd-homogeneous C^*-algebra over $\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2$ whose cd-homogeneous C^*-subalgebra restricted to the subspace $\mathbb{T}^r \times \mathbb{T}^2$ of $\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2$ is realized as $C(\mathbb{T}^r) \otimes A_{l/d} \otimes M_c(\mathbb{C})$ for $A_{l/d}$ a rational rotation algebra. The C^*-algebra which is given by twisting $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2})$ in $A_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by a totally skew multiplier ρ on $\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}$ is said to be a spherical non-commutative torus of rank $e+s+r+m$, and denoted by $\mathbb{S}^{cd}_\rho$, where $C^*(\widehat{\mathbb{T}^2}, \text{ of } \rho) = C^*(\widehat{\mathbb{T}^2})$ and $\mathbb{T}^2$ is the primitive ideal space of $A_{l/d}$.*

Then the fibre of $\mathbb{S}^d_\rho$, denoted by P^d_ρ, can be obtained by an iteration of $r+m-2$ crossed products by actions α_i of $\mathbb{Z}$, the first action on the rational rotation algebra $A_{l/d}$, where the actions α_i on the fibre $M_d(\mathbb{C})$ of $A_{l/d}$ are trivial.

$$A_\rho = C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}, \rho) \cong C^*(d\mathbb{Z} \times d\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_{r+m}} \mathbb{Z}.$$

Thus the spherical non-commutative torus $\mathbb{S}^{cd}_\rho$ is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod^e S^2 \times \prod^s S^1$ with fibres $P^d_\rho \otimes M_c(\mathbb{C})$.

We are going to show that $[1_{\mathbb{S}^{cd}_\rho}] \in K_0(\mathbb{S}^{cd}_\rho)$ is primitive.

Theorem 3.2. *Let $\mathbb{S}^{cd}_\rho$ be a spherical non-commutative torus of rank $e+s+r+m$ defined above. Assume that no non-trivial matrix algebra can be factored out of A_{cd}. Then $K_0(\mathbb{S}^{cd}_\rho) \cong K_1(\mathbb{S}^{cd}_\rho) \cong \mathbb{Z}^{2^{e+s+r+m-1}}$, and $[1_{\mathbb{S}^{cd}_\rho}] \in K_0(\mathbb{S}^{cd}_\rho)$ is primitive.*

Proof. It was shown in [3, Theorem 3.4] that the spherical non-commutative torus $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$, where A_ρ is a non-commutative torus of rank $r+m$. But by Künneth's theorem and by Elliott's theorem [6, Theorem 2.2] and Theorem 2.2

$$\begin{aligned} &K_0(C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho) \\ &\cong K_0(C(\prod^e S^2 \times \prod^s S^1)) \otimes K_0(A_\rho) \oplus K_1(C(\prod^e S^2 \times \prod^s S^1)) \otimes K_1(A_\rho) \\ &\cong \mathbb{Z}^{2^{e+s-1}} \otimes \mathbb{Z}^{2^{r+m-1}} \oplus \mathbb{Z}^{2^{e+s-1}} \otimes \mathbb{Z}^{2^{r+m-1}} \cong \mathbb{Z}^{2^{e+s+r+m-1}} \end{aligned}$$

Similarly, one obtains that $K_1(C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho) \cong \mathbb{Z}^{2^{e+s+r+m-1}}$. Hence

$$K_0(\mathbb{S}_\rho^{cd}) \cong K_0(C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho) \cong \mathbb{Z}^{2^{e+s+r+m-1}},$$

$$K_1(\mathbb{S}_\rho^{cd}) \cong K_1(C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho) \cong \mathbb{Z}^{2^{e+s+r+m-1}}.$$

So it is enough to show that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive. The proof is by induction on m. Assume that $m = 2$. By the Elliott theorem [6, Theorem 2.2], the canonical embedding of each factor $C(\mathbb{T}^1)$ of $C(\mathbb{T}^r \times \mathbb{T}^2)$ into A_ρ induces an isomorphism of $K_0(C(\mathbb{T}^r \times \mathbb{T}^2))$ into $K_0(A_\rho)$ such that the class $[1_{C(\mathbb{T}^r \times \mathbb{T}^2)}]$ of the unit $1_{C(\mathbb{T}^r \times \mathbb{T}^2)}$ maps to the class $[1_{A_\rho}]$ of the unit 1_{A_ρ}. By the same reasoning as the proof given in Theorem 2.2, the canonical embedding ϕ of each factor $C(S^1)$ or $C(\mathbb{T}^1)$ of $C(\prod^e S^1 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)$ into $\mathbb{S}_\rho^{cd}$ induces an isomorphism μ of

$$K_0(C(\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)) \to K_0(\mathbb{S}_\rho^{cd}).$$

The unit $1_{C(\prod^e S^1 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)}$ maps to the unit $1_{C(\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)}$ under the canonical embedding ψ of $C(\prod^e S^1 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)$ into $C(\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)$.

$$[1_{C(\prod^e S^1 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)}] \in K_0(C(\prod^e S^1 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2))$$

maps to

$$[1_{C(\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)q}] \in K_0(C(\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)),$$

primitive in $K_0(C(\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2))$. In the commutative diagram

$$\begin{array}{ccc} K_0(C(S)) & \xrightarrow{\psi_*} & K_0(C(\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)) \\ \Big\downarrow{\scriptstyle (\text{identity})_*} & & \Big\downarrow{\scriptstyle \mu(\cong)} \\ K_0(C(S)) & \xrightarrow{(\otimes^{e+s+r+2}\phi)_*} & K_0(\mathbb{S}_\rho^{cd}), \end{array}$$

where $S = \prod^e S^1 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2$.

$$\begin{aligned}&\mu([1_{C(\prod^e S^2\times\prod^s S^1\times\mathbb{T}^r\times\mathbb{T}^2)}])\\&= (\otimes^{e+s+r+2}\phi)_* \circ (\text{identity})_* \circ \psi_*^{-1}([1_{C(\prod^e S^2\times\prod^s S^1\times\mathbb{T}^r\times\mathbb{T}^2)}]) = [1_{\mathbb{S}_\rho^{cd}}],\end{aligned}$$

i.e., μ must be the canonical extension of $(\text{identity})_*$:

$$K_0(C(\prod^e S^1 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)) \to K_0(C(\prod^e S^1 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2)).$$

So $[1_{\mathbb{S}_\rho^{cd}}]$ is the image of the primitive element

$$[1_{C(\prod^e S^2\times\prod^s S^1\times\mathbb{T}^r\times\mathbb{T}^2)}] \in K_0(C(\prod^e S^2 \times \prod^s S^1 \times \mathbb{T}^r \times \mathbb{T}^2))$$

under the isomorphism μ. Hence $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive.

Next, assume that the result is true for all spherical non-commutative tori with $m = i-1$. Write $\mathbb{S}_i = C^*(\mathbb{S}_{i-1}, u_i)$, where $\mathbb{S}_i = C^*(\mathbb{S}_\rho^{cd}, u_3, \dots, u_i)$, where $\mathbb{S}_\rho^{cd}$ is the case above, $m = 2$. Then the inductive hypothesis applies to $\mathbb{S}_{i-1}$. Also, we can think of $\mathbb{S}_i$ as the crossed product of $\mathbb{S}_{i-1}$ by an action α of $\mathbb{Z}$, where the generator of $\mathbb{Z}$ corresponds to u_i, which acts on $C^*(v_1, \dots, v_r, u_1^d, u_2^d, u_3, \dots, u_{i-1})$ by conjugation (sending u_j to $u_i u_j u_i^{-1} = e^{2\pi i\theta_{ji}} u_j, j \neq 1, 2$, sending u_j^d to $u_i u_j^d u_i^{-1} = e^{2\pi i d\theta_{ji}} u_j^d, j = 1, 2$, and sending v_j to $u_i v_j u_i^{-1} = e^{2\pi i\beta_{ji}} v_j$), and which acts trivially on $C(\prod^e S^2 \times \prod^s S^1) \otimes M_{cd}(\mathbb{C})$. Here $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2}, \text{res of } \rho) \cong C^*(v_1, v_2, \dots, v_r, u_1^d, u_2^d)$. Note that this action is homotopic to the trivial action, since we can homotope θ_{ji} and β_{ji} to 0. Hence $\mathbb{Z}$ acts trivially on the K-theory of $\mathbb{S}_{i-1}$. The Pimsner-Voiculescu exact sequence for a crossed product gives an exact sequence

$$K_0(\mathbb{S}_{i-1}) \xrightarrow{1-\alpha_*} K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \to K_1(\mathbb{S}_{i-1}) \xrightarrow{1-\alpha_*} K_1(\mathbb{S}_{i-1})$$

and similarly for K_1, where the map Φ is induced by inclusion. Since $\alpha_* = 1$ and since the K-groups of $\mathbb{S}_{i-1}$ are free abelian, this reduces a split short exact sequence

$$\{0\} \to K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \to K_1(\mathbb{S}_{i-1}) \to \{0\}$$

and similarly for K_1. So $K_0(\mathbb{S}_i)$ and $K_1(\mathbb{S}_i)$ are free abelian of rank $2 \cdot 2^{e+s+r+i-2} = 2^{e+s+r+i-1}$. Furthermore, since the inclusion $\mathbb{S}_{i-1} \to \mathbb{S}_i$ sends $1_{\mathbb{S}_{i-1}}$ to $1_{\mathbb{S}_i}$, $[1_{\mathbb{S}_i}]$ is the image of $[1_{\mathbb{S}_{i-1}}]$, which is primitive in $K_0(\mathbb{S}_{i-1})$ by inductive hypothesis. Hence the image is primitive, since the Pimsner-Voiculescu exact sequence is a split short exact sequence of torsion-free groups.

Therefore, $K_0(\mathbb{S}_\rho^{cd}) \cong K_1(\mathbb{S}_\rho^{cd}) \cong \mathbb{Z}^{2^{e+s+r+m-1}}$, and $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive. ■

Corollary 3.3. *Let q be a positive integer, and $\mathbb{S}_\rho^{cd}$ a spherical non-commutative torus given above. Assume that no non-trivial matrix algebra can be factored out* *of A_{cd}. Then $\mathbb{S}^{cd}_\rho \otimes M_q(\mathbb{C})$ is not isomorphic to $A \otimes M_{pq}(\mathbb{C})$ for any C^*-algebra A and any integer p greater than 1. In particular, no non-trivial matrix algebra can be factored out of $\mathbb{S}^{cd}_\rho$, P^{cd}_ρ and A_ρ.*

Proof. Assume that $\mathbb{S}^{cd}_\rho \otimes M_q(\mathbb{C})$ is isomorphic to $A \otimes M_{pq}(\mathbb{C})$. Then the unit $1_{\mathbb{S}^{cd}_\rho} \otimes I_q$ maps to the unit $1_A \otimes I_{pq}$. So

$$[1_{\mathbb{S}^{cd}_\rho} \otimes I_q] = [1_A \otimes I_{pq}].$$

Thus there is a projection $e \in \mathbb{S}^{cd}_\rho$ such that $q[1_{\mathbb{S}^{cd}_\rho}] = (pq)[e]$. But $K_0(\mathbb{S}^{cd}_\rho)$ is torsion-free, so $[1_{\mathbb{S}^{cd}_\rho}] = p[e]$. This contradicts Theorem 3.2 if $p > 1$.

Therefore, $\mathbb{S}^{cd}_\rho \otimes M_q(\mathbb{C})$ is not isomorphic to $A \otimes M_{pq}(\mathbb{C})$. ■

4. Tensor Products of Spherical Non-Commutative Tori with UHF-Algebras and Cuntz Algebras

In this section, we are going to assume that no non-trivial matrix algebra can be factored out of A_{cd}. Using the fact that $[1_{\mathbb{S}^{cd}_\rho}] \in K_0(\mathbb{S}^{cd}_\rho)$ is primitive, we are going to show that the tensor product of the spherical non-commutative torus $\mathbb{S}^{cd}_\rho$ with a UHF-algebra M_{p^∞} of type p^∞ is isomorphic to $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p.

Theorem 4.1. *Let $\mathbb{S}^{cd}_\rho$ be a spherical non-commutative torus defined as before. Then $\mathbb{S}^{cd}_\rho \otimes M_{p^\infty}$ is isomorphic to $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p.*

Proof. Assume that the set of prime factors of cd is a subset of the set of prime factors of p. To show that $\mathbb{S}^{cd}_\rho \otimes M_{p^\infty}$ is isomorphic to $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$, it is enough to show that $\mathbb{S}^{cd}_\rho \otimes M_{(cd)^\infty}$ is isomorphic to $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{(cd)^\infty}$. But there exist the C^*-algebra homomorphisms which are the canonical inclusions $\mathbb{S}^{cd}_\rho \otimes M_{(cd)^g}(\mathbb{C}) \hookrightarrow C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{(cd)^g}(\mathbb{C})$ and the $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho$-module maps $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{(cd)^g}(\mathbb{C}) \hookrightarrow \mathbb{S}^{cd}_\rho \otimes M_{(cd)^g}(\mathbb{C})$:

$$\mathbb{S}^{cd}_\rho \hookrightarrow C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \hookrightarrow \mathbb{S}^{cd}_\rho \otimes M_{cd}(\mathbb{C})$$
$$\hookrightarrow C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{(cd)^2}(\mathbb{C}) \hookrightarrow \cdots .$$

The inductive limit of the odd terms

$$\cdots \to \mathbb{S}^{cd}_\rho \otimes M_{(cd)^g}(\mathbb{C}) \to \mathbb{S}^{cd}_\rho \otimes M_{(cd)^{g+1}}(\mathbb{C}) \to \cdots$$

is $\mathbb{S}^{cd}_\rho \otimes M_{(cd)^\infty}$, and the inductive limit of the even terms

$$\to C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{(cd)^g}(\mathbb{C}) \to C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{(cd)^{g+1}}(\mathbb{C}) \to$$

is $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{(cd)^\infty}$. Thus by the Elliott theorem [7, Theorem 2.1], $\mathbb{S}_\rho^{cd} \otimes M_{(cd)^\infty}$ is isomorphic to $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{(cd)^\infty}$.

Conversely, assume that

$$\mathbb{S}_\rho^{cd} \otimes M_{p^\infty} \cong C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}.$$

Then the unit $1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}$. So

$$[1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}] = [1_{C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}]$$
$$[1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}] = [1_{\mathbb{S}_\rho^{cd}}] \otimes [1_{M_{p^\infty}}]$$
$$[1_{C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}]$$
$$= cd([1_{C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho}] \otimes [1_{M_{p^\infty}}]).$$

Under the assumption that the unit $1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}$ maps to the unit

$$1_{C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd},$$

if there is a prime factor q of cd such that $q \nmid p$, then $[1_{M_{p^\infty}}] \neq q[e_\infty]$ for e_∞ a projection in M_{p^∞}. So there is a projection $e \in \mathbb{S}_\rho^{cd}$ such that $[1_{\mathbb{S}_\rho^{cd}}] = q[e]$. This contradicts Theorem 3.2. Thus the set of prime factors of cd is a subset of the set of prime factors of p.

Therefore, $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p. ■

In particular, $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ has the trivial bundle structure if the set of prime factors of cd is a subset of the set of prime factors of p.

Let us study the tensor products of spherical non-commutative tori with (even) Cuntz algebras.

The Cuntz algebra $\mathcal{O}_u, 2 \leq u < \infty$, is the universal C^*-algebra generated by u isometries $s_1, \ldots, s_u$, i.e., $s_j^* s_j = 1$ for all j, with the relation $s_1 s_1^* + \cdots + s_u s_u^* = 1$. Cuntz [4, 5] proved that $\mathcal{O}_u$ is simple and the K-theory of $\mathcal{O}_u$ is $K_0(\mathcal{O}_u) = \mathbb{Z}/(u-1)\mathbb{Z}$ and $K_1(\mathcal{O}_u) = 0$. He proved that $K_0(\mathcal{O}_u)$ is generated by the class of the unit.

Proposition 4.2. *Let $\mathbb{S}_\rho^{cd}$ be a spherical non-commutative torus with fibres $P_\rho^d \otimes M_c(\mathbb{C})$ for cd a positive integer greater than 1. Let u be a positive integer such that cd and $u-1$ are not relatively prime. Then $\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}$ is not isomorphic to $\mathcal{O}_u \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$.*

Proof. Let p be a prime such that $p \mid cd$ and $p \mid u-1$. Suppose that $\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}$ is isomorphic to $\mathcal{O}_u \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. Then the unit $1_{\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}}$ maps to the unit $1_{\mathcal{O}_u \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho} \otimes I_{cd}$. So

$$[1_{\mathcal{O}_u \otimes \mathbb{S}_\rho^{cd}}] = [1_{\mathcal{O}_u \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho} \otimes I_{cd}] = cd[1_{\mathcal{O}_u \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho}].$$

Hence there is a projection e in $\mathcal{O}_u \otimes \mathbb{S}^{cd}_\rho$ such that $[1_{\mathcal{O}_u \otimes \mathbb{S}^{cd}_\rho}] = cd[e]$. But $[1_{\mathcal{O}_u \otimes \mathbb{S}^{cd}_\rho}] = [1_{\mathcal{O}_u}] \otimes [1_{\mathbb{S}^{cd}_\rho}]$ and $[1_{\mathcal{O}_u}]$ is a generator of $K_0(\mathcal{O}_u) \cong \mathbb{Z}/(u-1)\mathbb{Z}$ (see [5]). But $p \mid u-1$. $[1_{\mathcal{O}_u}] \neq p[e_*]$ for e_* a projection in $\mathcal{O}_u$. So $[1_{\mathbb{S}^{cd}_\rho}] = p[e']$ for e' a projection in $\mathbb{S}^{cd}_\rho$. This contradicts Theorem 3.2. Hence cd and $u-1$ are relatively prime.

Therefore, $\mathcal{O}_u \otimes \mathbb{S}^{cd}_\rho$ is not isomorphic to $\mathcal{O}_u \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if cd and $u-1$ are not relatively prime. ■

The following result is useful to understand the bundle structure of $\mathcal{O}_u \otimes \mathbb{S}^{cd}_\rho$.

Proposition 4.3 [12, Theorem 7.2]. *Let A and B be unital simple inductive limits of even Cuntz algebras. If $\alpha : K_0(A) \to K_0(B)$ is an isomorphism of abelian groups satisfying $\alpha([1_A]) = [1_B]$, then there is an isomorphism $\phi : A \to B$ which induces α.*

Corollary 4.4.

(1) *Let p be an odd integer such that p and $2u-1$ are relatively prime. Then $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{(2u-1)p+1} \otimes M_{p^\infty}$. That is, $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty}$.*

(2) *$\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(2u)^\infty}$.*

Theorem 4.5. *Let $\mathbb{S}^{cd}_\rho$ be a spherical non-commutative torus with fibres $P^d_\rho \otimes M_c(\mathbb{C})$. Then $\mathcal{O}_{2u} \otimes \mathbb{S}^{cd}_\rho$ is isomorphic to $\mathcal{O}_{2u} \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if and only if cd and $2u-1$ are relatively prime.*

Proof. Assume that cd and $2u-1$ are relatively prime. Let $cd = p2^v$ for some odd integer p. Then p and $2u-1$ are relatively prime. Then by Corollary 4.4 $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty}$, and $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(2u)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(2u)^\infty} \otimes M_{(2^v)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(2^v)^\infty}$. So $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty} \otimes M_{(2^v)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(cd)^\infty}$. Thus by Theorem 4.1 $\mathcal{O}_{2u} \otimes \mathbb{S}^{cd}_\rho$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(cd)^\infty} \otimes \mathbb{S}^{cd}_\rho$, which in turn is isomorphic to $\mathcal{O}_{2u} \otimes M_{(cd)^\infty} \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. Thus $\mathcal{O}_{2u} \otimes \mathbb{S}^{cd}_\rho$ is isomorphic to $\mathcal{O}_{2u} \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$.

The converse was proved in Proposition 4.2.

Therefore, $\mathcal{O}_{2u} \otimes \mathbb{S}^{cd}_\rho$ is isomorphic to $\mathcal{O}_{2u} \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if and only if cd and $2u-1$ are relatively prime. ■

Cuntz [5] computed the K-theory of the generalized Cuntz algebra $\mathcal{O}_\infty$, generated by a sequence of isometries with mutually orthogonal ranges, $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$. He proved that $K_0(\mathcal{O}_\infty)$ is generated by the class of the unit.

Proposition 4.6. *Let $\mathbb{S}^{cd}_\rho$ be a spherical non-commutative torus with fibres $P^d_\rho \otimes M_c(\mathbb{C})$. Then $\mathcal{O}_\infty \otimes \mathbb{S}^{cd}_\rho$ is not isomorphic to $\mathcal{O}_\infty \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if $cd > 1$.*

Proof. Suppose $\mathcal{O}_\infty \otimes \mathbb{S}^{cd}_\rho$ is isomorphic to $\mathcal{O}_\infty \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$.

The unit $1_{\mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}}$ maps to the unit $1_{\mathcal{O}_\infty \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho} \otimes I_{cd}$. By the same trick as in the proof of Proposition 4.2, one can show that $[1_{\mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}}] = cd[e]$ for a projection $e \in \mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}$. $[1_{\mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}}] = [1_{\mathcal{O}_\infty}] \otimes [1_{\mathbb{S}_\rho^{cd}}]$ and $[1_{\mathcal{O}_\infty}]$ is a primitive element of $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ (see [5]). So $[1_{\mathbb{S}_\rho^{cd}}] = cd[e']$ for a projection $e' \in \mathbb{S}_\rho^{cd}$. This contradicts Theorem 3.2.

Therefore, $\mathcal{O}_\infty \otimes \mathbb{S}_\rho^{cd}$ is not isomorphic to $\mathcal{O}_\infty \otimes C(\prod^e S^2 \times \prod^s S^1) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. ■

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