Vietnam Journal of Mathematics **29**:1 (2001) 47–51

Vietnam Journal
of
MATHEMATICS

Common Fixed Points for Condensing and Compact Mappings*

Tran Thi Lan Anh

Institute of Mathematics, P. O. Box 631, Bo Ho, Hanoi, Vietnam

Received November 11, 1999
Revised January 5, 2000

Abstract. We prove common fixed point theorems for a pair of condensing (respectively compact) commuting self-mappings on metric spaces. Examples and applications are also discussed.

1. Introduction

In [1] we established common fixed point theorems for a pair of commuting self-mappings f_1, f_2 on a complete metric space (X, d) which satisfy the so-called g-quasi-contractive condition for a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with the following properties:

($g1$) g is a non-decreasing function;
($g2$) g is right-continuous;
($g3$) $\forall t > 0 \quad g(t) < t$;
($g4$) $\exists \lim\limits_{t \to \infty} g(t)/t < 1$;

and also a metric condition of Fisher-Sessa type or Fisher-Iseki type. A natural question arising here is to what extent one can relax the conditions above for the function g, say, if g satisfies only property ($g3$) for all $x, y \in X$ such that $f_1 x \neq f_2 y$? In this case for the existence of a common fixed point it is desirable to impose conditions of topological nature to the mappings, or to the total space, such as compact or condensing properties.

The aim of the note is to give some results on common fixed points for

* This work was partially supported by the National Natural Science Basic Research Foundation of Vietnam.

compact and condensing mappings which generalize the case of one mapping extensively studied by Janos, Leader, Shih-Yeh, Liu and others (cf. [2-5]).

2. Results

Before stating our main theorems we need the following definitions.

Definition 2.1. *A self-mapping f of a metric space (X,d) is said to be compact if there exists a compact subset K of X such that $f(X) \subset K$.*

Denote by $\gamma(A)$ the Kuratowski measure of non-compactness of a subset A of a bounded metric space (X,d).

Definition 2.2. *A mapping $f\colon X \to X$ is said to be condensing if f is continuous and for any non-empty non-totally bounded subset A of X, we have $\gamma(f(A)) < \gamma(A)$.*

Throughout the note we use the following notations: f_1, f_2 are commuting self-mappings of X, $A \subset X$ a subset of X;

$$\mathcal{O}_{f_i}(x) := \{f_i^n x\colon \quad n = 0,1,2,\dots\}, \quad i = 1,2;$$

$$\mathcal{O}_{f_i}(A) := \{f_i^n x\colon \quad n = 0,1,2,\dots, \quad \forall x \in A\}, \quad i = 1,2;$$

$$\mathcal{O}(x,\infty) := \{f_1^m f_2^n x\colon \quad m,n = 0,1,2,\dots\}$$

Theorem 2.3. *Let (X,d) be a bounded complete metric space; f_i, $i = 1,2$ commuting self-mappings of X satisfying the following conditions:*

(i) $\forall x,y$ *with* $f_1 x \neq f_2 y$

$$d(f_1 x, f_2 y) < \delta\big(\overline{\mathcal{O}_{f_1}(x)} \cup \overline{\mathcal{O}_{f_2}(y)}\big) \tag{1}$$

where $\delta(A) := \sup\{d(x,y) : \ x,y \in A\}$ *for a subset* $A \subset X$.

(ii) f_1, f_2 *are condensing.*

Then there exists a unique common fixed point in X for f_1, f_2.

Proof. Let $\mathcal{O}(x,\infty) := \{f_1^m f_2^n x : \ m,n = 0,1,2,\dots\}$ be the orbit defined as above; x a point of X. Since

$$\gamma(\mathcal{O}_{f_1}(x)) = \max\big\{\gamma(x), \gamma(\mathcal{O}_{f_1}(f_1 x))\big\} = \gamma\big(f_1(\mathcal{O}_{f_1}(x))\big)$$

and f_1 is condensing one concludes that $\mathcal{O}_{f_1}(x)$ is precompact. Similarly for $\mathcal{O}(x,\infty)$

$$\gamma(\mathcal{O}(x,\infty)) = \max\big\{\gamma(\mathcal{O}_{f_1}(x)), \gamma(\mathcal{O}_{f_2}(\mathcal{O}_{f_1}(x)))\big\}.$$

in view of the commutativity of f_1, f_2 and the above. Therefore *(ii)* implies that $\mathcal{O}(x,\infty)$ is totally bounded, hence precompact (because of the completeness of

X). Now putting $D_0 = \overline{\mathcal{O}(x,\infty)}$, so that D_0 is compact. Clearly $f_i(D_0) \subset D_0$, $i = 1,2$. Next let us consider the following sets

$$D_1 := \bigcap_{n=1}^{\infty} f_1^n(D_0),$$

$$D_2 := \bigcap_{n=1}^{\infty} f_2^n(D_1).$$

Thus D_i, $i = 1,2$, are f_i-invariant, and by the finite intersection property, non-empty compact subsets. Suppose $u \in D_1$, there exist $x_n \in f_1^{n-1}(D_0)$ such that $f_1(x_n) = u$ for $n = 1,2,\ldots$. From the compactness of D_0 and by proceeding to a subsequence, if necessary, one may assume that x_n converges to some point $v \in D_0$. Since $\{x_{n+1}, x_{n+2}, \ldots\} \subset f_1^n(D_0)$ and $f_1^n(D_0)$ it follows that $v \in f_1^n(D_0)$ for $n = 1,2,\ldots$. So $v \in D_1$ and $f_1(v) = u$. This shows that $f_1(D_1) = D_1$.

Furthermore by the definition of D_2, the condition (ii) of the theorem and the above $f_1(D_2) = D_2$. Next we shall prove $f_2(D_2) = D_2$. Because D_2 is f_2-invariant it suffices to establish $D_2 \subset f_2(D_2)$. Take $\overline{u} \in D_2$, there exist $x_n \in f_2^{n-1}(D_1)$ such that $f_2(x_n) = \overline{u}$ for $n = 1,2,\ldots$. As above one can assume that x_n converges to some point $\overline{v} \in D_2$, and in a similar way it can be shown that $\overline{v} \in D_2$ and $f_2(\overline{v}) = \overline{u}$. Thus $f_i(D_2) = D_2$, $i = 1,2$.

Now we claim that D_2 is a singleton, say $\{z\}$, and hence z is a common fixed point in X for f_1, f_2. If not, then $\delta(D_2) > 0$. Since D_2 is compact, there exist $a \neq b \in D_2$ such that $\delta(D_2) = d(a,b)$. From the above clearly $a = f_1(a_1)$, $b = f_2(b_1)$ for certain points $a_1, b_1 \in D_2$. Hence

$$\overline{\mathcal{O}_{f_1}(a_1)} \cup \overline{\mathcal{O}_{f_2}(b_1)} \subset \overline{D}_2 = D_2$$

and by (2.4)

$$0 < \delta(D_2) = d(f_1a_1, f_2b_1) < \delta\big(\overline{\mathcal{O}_{f_1}(a_1)} \cup \overline{\mathcal{O}_{f_2}(b_1)}\big) \le \delta(D_2)$$

which is a contradiction. The uniqueness is obvious.

Corollary 2.5. *Let f_i, $i = 1,2$, be commuting self-mappings of a bounded complete metric space (X,d). Suppose that f_1, f_2 are condensing and satisfy*

$$d(f_1x, f_2y) < \delta\big(\{x, y, f_1x, f_2y\}\big) \tag{2}$$

for all x, y with $f_1x \neq f_2y$. Then f_1, f_2 have a unique common fixed point in X.

Remark 1. In the above theorem for the condensing property and metric condition (1) it is sufficient to require for some iterates of f_1, f_2.

Theorem 2.6. *Let f_i, $i = 1,2$, be commuting self-mappings of a metric space (X,d) satisfying the following conditions:*

(i) *$\forall x, y$ with $f_1x \neq f_2y$ one has* (1),

(ii) *f_i, $i = 1,2$, are compact and continuous mappings.*

Then f_1, f_2 have a unique common fixed point in X.

Proof. From (ii) there exist compact subsets $K_1, K_2 \subset X$ such that $f_i(X) \subset K_i,\ i = 1, 2$. Putting $K = K_1 \cup K_2$ one has

$$X \supset K_1 \supset f_1(X) \supset f_1(K) \supset f_1^2(X) \supset f_1^2(K) \supset \ldots \supset f_1^n(X) \supset f_1^n(K) \supset \ldots$$

so

$$D_1 := \bigcap_{n=0}^{\infty} f_1^n(K)$$

is a non-empty compact subset by the finite intersection property. Clearly D_1 is f_i-invariant, $i = 1, 2$. In fact it can be shown also that $f_1(D_1) = D_1$. Thus putting

$$D_2 := \bigcap_{n=0}^{\infty} f_2^n(D_1)$$

one can see that D_2 is a non-empty compact subset and $f_i(D_2) = D_2,\ i = 1, 2$. As in the proof of Theorem 2.3 D_2 is a singleton, say $\{z\}$. Hence z is a unique common fixed point for f_1, f_2.

Corollary 2.7. *Let $f_i,\ i = 1, 2$, be commuting continuous self-mappings of a metric space (X, d). Suppose that $f_1,\ f_2$ are compact and satisfy condition (2) for all x, y with $f_1x \neq f_2y$. Then f_1, f_2 have a unique common fixed point in X.*

Remark 2. 1) It should be noted that (cf. Remark 1) the same conclusions hold true if one replaces $f_i,\ i = 1, 2$, by their iterates in the formulation of Theorem 2.6 for the compactness property of mappings and metric condition (1).

2) As a special case if $f_1 = f_2$ one obtains well-known results previously due to Janos, Leader, Shih-Yeh, Liu and others [2 - 5].

3. Examples

3.1. The following example shows that the condensing condition in 2.3, 2.5 is essential. Let $X = \mathbb{N}$ be the set of positive integers which is complete with metric $d(n, n) = 0,\ d(n, m) = d(m, n) = a + 1/(n + 1)$ if $n < m$, where a is a fixed positive number. Consider $f_1(n) = f_2(n) = n + 1$, which have no fixed points in X. One checks easily that f_1, f_2 satisfy conditions of 2.3, 2.5, except the condensing property, since $\gamma(X) = \gamma(X \backslash \{1\}) = a$. One can show also that the compactness of f_1, f_2 in 2.6 - 2.7 can not be omitted, for a counter-example, let $X = [1, +\infty)$ with usual metric and $f_i(x) = a_i x$, where $a_i > 1,\ i = 1, 2$.

3.2. The motivation of this part seems going back to Euler. We work over non-negative real numbers. Let $a > 0,\quad \alpha_1 > \sqrt{a},\ 0 < \alpha_2 < \sqrt{a}$. Consider the following two self-functions of $\mathbb{R}^+$ which are obviously commuting: $f_i(x) :=$

$(\alpha_i x + a)/(x + \alpha_i)$, $i = 1, 2$ (In a similar way one can consider such piece-wise fractional linear type transformations). It is easy to verify (2) holds for f_1, f_2. So in view of the compactness of f_1, f_2 and 2.6 - 2.7 above one sees that f_1 and f_2 have a unique common fixed point. In general, on a suitable compact subset, say $I_a = [0, \sqrt{a}]$, the family of continuous functions $\{f(x) := (\alpha x + a)/(x + \alpha)\}$ which are commuting, obviously has a common fixed point $\sqrt{a}$.

On the other hand the uniqueness and the question on the rate of convergence of the iterated process may be well understood due to Banach's contraction principle for a suitable iterated power f^n of f. In fact the iterated approximants converge quite fast to the common fixed point $\sqrt{a}$. In particular if $a = N$, where N is not a perfect square, $x_0 = \alpha_1 = \alpha_2 = [\sqrt{N}]$, or $= [\sqrt{N}] + 1$ for the initial value of the iteration, one sees that the iterated process gives enough good rational approximations for the irrationality $\sqrt{N}$.

For instance, numerical illustration with $N = 2$ exhibits all the solutions of the well-known Pellian equation $x^2 - 2y^2 = \pm 1$. A similar assertion holds true for $N = 3, 5, 8, 10, \ldots$.

As for another numerical illutration one takes $N = 163$, $x_0 = \alpha = 12$ then the 8-th iterate gives the answer accurate to ten decimal places: $\sqrt{163} =$ 12.7671453348...

It should be noted that extending to the whole $\mathbb{R}$ one can consider also the iteration for negative powers of f. The result of the process then converges to $-\sqrt{a}$. Another interesting application is to investigate the complex behaviour of the iterated process, i.e., if one takes the initial value x_0 from the complex plane.

Acknowledgement. The author is grateful to Professor Nguyen Minh Chuong for his constant encouragement during preparation of this note. The author would like also to thank Professor Do Hong Tan for some useful remarks.

References

1. Tran Thi Lan Anh and Nguyen Minh Chuong, *Generalizations of Common Fixed Point Theorems*, Preprint 97/22, Hanoi Institute of Mathematics (1997).
2. Janos Ludvik, Ko Hwei-Mei, and Tan Kek-Keong, Edelstein's contractivity and attractors, *Proc. Amer. Math. Soc.* **76** (1979) 339-344.
3. Leader Solomon, Uniformly contractive fixed points in compact metric spaces, *Proc. Amer. Math. Soc.* **82** (1982) 153-158.
4. Liu Ze-Quing, Fixed point theorems for condensing and compact maps, *Kobe J. of Math.* **12** (1995) 139-145.
5. Shih Mau-Hsiang and Yeh Cheh-Chih, On fixed point theorems of contractive type, *Proc. Amer. Math. Soc.* **85** (1982) 465-468.