Vietnam Journal of Mathematics **29**:1 (2001) 79-83

Vietnam Journal
of
MATHEMATICS

Short Communication

Characterization of Singular Integral Equations

Nguyen Tan Hoa

*Department of Mathematics, Gia Lai Teachers' Training College,
Gia Lai, Vietnam*

Received May 25, 1999
Revised March 6, 2000

1. Introduction

Consider singular integral equations of the form

$$K\varphi := (K_0 + T)\varphi = f, \tag{$*$}$$

where

$$(K_0\varphi)(t) := a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau, \quad (T\varphi)(t) := \int_\Gamma T(t,\tau)\varphi(\tau)d\tau.$$

It is known that the characteristic equation and its associated characteristic equation admit effective solutions. In general, equations of the form $(*)$ do not admit effective solutions. However, there are some sufficient conditions which are given by Samko and Mau (see [2]) such that the equation $(*)$ can be solved effectively. In order to get other sufficient conditions for kernel $T(t,\tau)$, we consider a problem on characterization of singular integral equations, i.e. we find the operators T such that equations $(*)$ can be reduced to either $K_0\varphi = g$ or the generalized characteristic equation $(K_0 + T_0)\varphi = g$ where T_0 is a compact operator with the kernel $T_0(t,\tau)$ satisfying sufficient conditions which are given by the authors mentioned above.

This report deals with characterization of the singular integral equations with a regular part that has degenerated kernel to the characteristic equation.

Let Γ be a simple regular closed arc and let X be the space $H^\mu(\Gamma)$ $(0 < \mu < 1)$, $L(X)$ be the space of all linear operators acting on X. Denote by D^+ the domain bounded by Γ and D^- its complement including the point at infinity.

Consider complete singular integral equations of the form

$$(K\varphi)(t) := a(t) + b(t)(S\varphi)(t) + \lambda \int_\Gamma T_n(t,\tau)\varphi(\tau)d\tau = f(t), \tag{1}$$

where

$$(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau,$$

$T_n(t,\tau) = \sum_{k=1}^n a_k(t)b_k(\tau)$; $\varphi(t), f(t), a(t), b(t), a_k(t), b_k(t) \in X$ $(k = 1,\dots,n)$, $a(t) \pm b(t) \neq 0$ for all $t \in \Gamma$, $\{a_k(t)\}_{k=1,\dots,n}$ is a linearly independent system, $b_k(t) \neq 0$ $(k = 1,\dots,n)$, $0 \neq \lambda \in \mathbb{C}$.

Denote

$$(K_0\varphi)(t) = a(t)\varphi(t) + b(t)(S\varphi)(t),$$

$$(R\varphi)(t) = \frac{1}{a^2(t) - b^2(t)}\Big[a(t)\varphi(t) - \frac{b(t)Z(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{Z(\tau)}\frac{d\tau}{\tau - t}\Big],$$

where

$$Z(t) = e^{\Gamma(t)}\sqrt{\frac{a^2(t) - b^2(t)}{t^\kappa}}, \quad \Gamma(t) = \frac{1}{2\pi i}\int_\Gamma \frac{\ln\Big(\tau^{-\kappa}\frac{a(\tau)-b(\tau)}{a(\tau)+b(\tau)}\Big)}{\tau - t}d\tau, \quad \kappa = \mathrm{Ind}K_0.$$

Denote

$$\kappa_0 = \begin{cases} \kappa & \text{if } \kappa > 0, \\ 0 & \text{if } \kappa \le 0, \end{cases} \qquad F = I - RK_0.$$

Lemma 1. *The following equality holds*

$$(F\varphi)(t) = -\sum_{k=0}^{\kappa_0} u_k(\varphi)\varphi_k(t) \qquad \text{on } X,$$

where $\varphi_0(t) = 0$, $\varphi_j(t) = \big[a^2(t) - b^2(t)\big]^{-1} b(t)Z(t)t^{j-1}$ $(j = 1,\dots,\kappa_0)$ *and* $u_k(\varphi)$ $(k = 0,\dots,\kappa_0)$ *are linear functionals which are defined by*

$$u_k(\varphi) = \begin{cases} 0 & \text{if } k = 0, \\ \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{\tau^{\kappa_0 - k}}{e^{\Gamma^-(\tau)}}\Big[-\varphi(\tau) + \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau_1)}{\tau_1 - \tau}d\tau_1\Big]d\tau & \text{if } k = 1,\dots,\kappa_0, \end{cases}$$

where $\Gamma^-(t)$ *is a boundary value of the function* $\Gamma(z)$ *in* D^-.

Let $\mathcal{A} = \big[K_{jk}\big]_{j,k=1}^{n+\kappa_0}$ be an $(n+\kappa_0) \times (n+\kappa_0)$ matrix that is defined by complex numbers K_{jk}, where

$$K_{jk} = \begin{cases} 1 + K'_{jk} & \text{if } j = k, \\ K'_{jk} & \text{if } j \neq k, \end{cases} \qquad (j,k = 1,\dots,n+\kappa_0)$$

and

$$K'_{jk} = \begin{cases} \lambda \int\limits_{\Gamma} b_j(t)(Ra_k)(t)dt & \text{if } j,k = 1,\dots,n, \\ \int\limits_{\Gamma} b_j(t)\varphi_{k-n}(t)dt & \text{if } j = 1,\dots,n,\ k = n+1,\dots,n+\kappa_0, \\ \lambda u_{j-n}(Ra_k) & \text{if } j = n+1,\dots,n+\kappa_0,\ k = 1,\dots,n, \\ u_{j-n}(\varphi_{k-n}) & \text{if } j,k = n+1,\dots,n+\kappa_0. \end{cases} \tag{2}$$

Let $\mathcal{A}^k(\varphi)$ be an $(n+\kappa_0)\times(n+\kappa_0)$ matrix, obtained from $\mathcal{A}$ replacing the k^{th}-column by the $\gamma(\varphi)$ column, where

$$\gamma(\varphi) = \left[(\gamma_1(\varphi), \gamma_2(\varphi), \dots, \gamma_{n+\kappa_0}(\varphi)\right]^T,$$

$$\gamma_j(\varphi) = \begin{cases} \int\limits_{\Gamma} b_j(t)(R\varphi)(t)dt & \text{if } j = 1,\dots,n, \\ u_{j-n}(R\varphi) & \text{if } j = n+1,\dots,n+\kappa_0. \end{cases} \tag{3}$$

Set $\Delta = \det \mathcal{A}$ and $\Delta_k(\varphi) = \det \mathcal{A}^k(\varphi)$.

Theorem 1. *If $\Delta \neq 0$, then the equation $(\widetilde{K}K\varphi)(t) = (\widetilde{K}f)(t)$ is the characteristic equation, where*

$$\widetilde{K} = I - T_1, \quad (T_1\varphi)(t) = \lambda \sum_{k=1}^{n} \frac{\Delta_k(\varphi)}{\Delta} a_k(t).$$

Proof. It is easy to check that $\widetilde{K} \in L(X)$ and $\mathrm{dom}\widetilde{K} = \mathrm{dom}R \supset \mathrm{Im}K$. We have

$$\begin{aligned} (\widetilde{K}K\varphi)(t) &= (I - T_1)(K\varphi)(t) \\ &= a(t)\varphi(t) + b(t)(S\varphi)(t) \\ &\quad + \lambda \sum_{k=1}^{n} \alpha_k a_k(t) - \lambda \sum_{k=1}^{n} \frac{\Delta_k(K\varphi)}{\Delta} a_k(t), \end{aligned}$$

where

$$\alpha_k = \int\limits_{\Gamma} b_k(t)\varphi(t)dt, \quad k = 1,\dots,n.$$

Using (2), (3) and Lemma 1, we obtain

$$\begin{aligned} \gamma_j(K\varphi) &= \begin{cases} \int\limits_{\Gamma} b_j(t)\left[\varphi(t) + \sum_{k=1}^{n+\kappa_0} \beta_k \psi(t)\right]dt & \text{if } j = 1,\dots,n, \\ u_{j-n}\left[\varphi(t) + \sum_{k=1}^{n+\kappa_0} \beta_k \psi(t)\right] & \text{if } j = n+1,\dots,n+\kappa_0 \end{cases} \\ &= \beta_j + \sum_{k=1}^{n+\kappa_0} \beta_k K'_{jk} = \sum_{k=1}^{n+\kappa_0} \beta_k K_{jk}, \quad j = 1,\dots,n+\kappa_0, \end{aligned}$$

where

$$\beta_k = \begin{cases} a_k & \text{if } k = 1, \dots, n, \\ u_{k-n} & \text{if } k = n+1, \dots, n+\kappa_0, \end{cases}$$
$$\psi_k(t) = \begin{cases} \lambda(Ra_k)(t) & \text{if } k = 1, \dots, n, \\ \varphi_{k-n}(t) & \text{if } k = n+1, \dots, n+\kappa_0. \end{cases}$$

Thus

$$\Delta_k(K\varphi) = \beta_k \Delta, \quad k = 1, \dots, n+\kappa_0$$

and

$$\sum_{k=1}^{n} \frac{\Delta_k(K\varphi)}{\Delta} a_k(t) = \sum_{k=1}^{n} \beta_k a_k(t) = \sum_{k=1}^{n} \alpha_k a_k(t).$$

This implies

$$(\widetilde{K}K\varphi)(t) = a(t)\varphi(t) + b(t)(S\varphi)(t) = (\widetilde{K}f)(t).$$

The theorem is proved. ■

Consider now the case $\Delta = 0$.

Suppose that r is the rank of matrix $\mathcal{A}$ and $\overline{\mathcal{A}} = \left[K_{\nu_j \mu_k}\right]_{j,k=1}^{r}$ is a submatrix of $\mathcal{A}$ such that

$$\Delta' = \det \overline{\mathcal{A}} \neq 0,$$

where
$$\begin{aligned} &\nu_k < \nu_j, \ \mu_k < \mu_j \ \text{if } k < j \ (j, k = 1, \dots, r), \\ &\nu_1, \nu_2, \dots, \nu_e, \ \mu_1, \mu_2, \dots, \mu_m \in \{1, 2, \dots, n\}, \\ &\nu_{e+1}, \nu_{e+2}, \dots, \nu_r, \ \mu_{m+1}, \mu_{m+2}, \dots, \mu_r \in \{n+1, n+2, \dots, n+\kappa_0\}. \end{aligned}$$

Let $\overline{\mathcal{A}}^{\mu_k}(\varphi)$ be an $r \times r$ matrix, obtained from $\overline{\mathcal{A}}$ replacing the k^{th}-column by the $[\gamma_{\nu_1}(\varphi), \gamma_{\nu_2}(\varphi), \dots, \gamma_{\nu_r}(\varphi)]^T$-column, where $\gamma_{\nu_j}(\varphi)$ $(j = 1, \dots, r)$ are defined by (3) and set $\Delta'_{\mu_k}(\varphi) = \det \overline{\mathcal{A}}^{\mu_k}(\varphi)$.

The set of all equations of the form

$$(K_0\varphi)(t) + \lambda \sum_{k=1}^{s} d_k(t) v_k(\varphi) = f(t)$$

will be denoted by $H^s_{K_0}$, where $\{d_k(t)\}_{k=1,\dots,s}$ is a linearly independent system in X, $0 \neq v_k \in X^*$ $(k = 1, \dots, s)$ are linear functionals, $f(t) \in X$ is a given function, $0 \neq \lambda \in \mathbb{C}$.

Denote

$$H^0_{K_0} = \{(K_0\varphi)(t) = f(t) \mid f(t) \in X\},$$
$$\widetilde{H}^s_{K_0} \doteq \bigcup_{l=0}^{s} H^l_{K_0}.$$

Evidently, every equation of the form (1) belongs to $H^n_{K_0}$.

By similar arguments as in the proof of Theorem 1, we obtain

Theorem 2. *If $\Delta' \neq 0$, then the equation $(\widetilde{\widetilde{K}}K\varphi)(t) = (\widetilde{\widetilde{K}}f)(t)$ belongs to $\widetilde{H}_{K_0}^{n+\kappa_0-r}$, where*

$$\widetilde{\widetilde{K}} = I - T_2, \quad (T_2\varphi)(t) = \lambda \sum_{k=1}^{m} \frac{\Delta'_{\mu_k}(\varphi)}{\Delta'} a_{\mu_k}(t).$$

Corollary 1. *Suppose that $u_k(\varphi) = e_k$ $(k = 0, \dots, \kappa_0)$, $e_k \in \mathbb{C}$ are the given complex numbers. If $\Delta' \neq 0$ then the equation $(\widetilde{\widetilde{\boldsymbol{K}}}K\varphi) = (\widetilde{\widetilde{\boldsymbol{K}}}f)(t)$ belongs to $\widetilde{H}_{K_0}^{n-m}$, where*

$$\widetilde{\widetilde{\boldsymbol{K}}} = I - T'_2,$$

$$(T'_2\varphi)(t) = \lambda \sum_{k=1}^{m} \frac{\Delta'_{\mu_k}(\varphi)}{\Delta'} a_{\mu_k}(t) + \sum_{j=m+1}^{r} \frac{\Delta'_{\mu_j}(\varphi)}{\Delta'} \varphi_{\mu_j - n}(t).$$

Acknowledgement. The author would like to express his deep gratitude to professor Nguyen Van Mau for helpful suggestion in the preparation of this paper.

References

1. F. D. Gakhov, *Boundary Value Problems*, 3rd ed., Nauka, Moscow, 1977 (Russian).
2. Ng. V. Mau, *Boundary Value Problems and Controllability of Linear Systems with Right Invertible Operators*, Dissertationes Math., CCCXVI, PANIM, Warszawa, 1992.
3. Ng. V. Mau, *Generalized Algebraic Elements and Linear Singular Integral Equations with Transformed Arguments*, WPW, Warszawa, 1989.
4. D. Prezeworska-Rolewicz, *Algebraic Analysis*, PWN and Reidel, Warszawa - Dordrecht, 1988.
5. D. Prezeworska-Rolewicz, *Equations with Transformed Argument, An Algebraic Approach*, Elsevier Scientific Publishing, Amsterdam and PWN, Warszawa, 1973.