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Survey

Detection of Cavities in Solids and Solutions of Elliptic Equations and Systems*

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Abstract. The author makes a survey of some recent developments on the problem of detecting cavities inside a solid body from measurements performed on its surface. Three groups of problems are considered. The first group takes its origin from the problem of detecting cracks inside a solid by the electric method and in its simplest form, it is equivalent to the problem of identifying the domain of the Laplace equation. The second group is related to the problem of detecting cracks in the interior of an elastic body. The third group is related to the problem of detecting mass inhomogeneities inside the Earth from gravity anomalies or gravity gradients measured on the surface. All of these problems are nonlinear. They are approximated by various methods: the method of finite dimensional approximations and the method of linearization.

We consider the problem of identifying the location and shape of cavities in a solid body from measurements performed on part of its surface. The problem is equivalent to that of identifying the domain of an elliptic equation or an elliptic system. The measured surface data are Cauchy data for elliptic equations and Cauchy like data for elliptic systems. Specifically, we shall consider the following three problems.

Problem 1. *Domain identification and construction of solutions for elliptic equations.*

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Let Ω be a domain (i.e., an open connected set) in $\mathbb{R}^n$, $n = 2$ or 3, with finitely many interior holes or cavities. The outer boundary Γ of Ω is known. The cavities are simply connected domains ω_i, $i = 1, \dots, n$. Suppose the following elliptic equation holds in Ω:

$$Au = 0 \text{ in } \Omega \tag{$*$}$$

subject to Cauchy conditions on an open portion Γ_0 of Γ and to the condition that either u or the conormal derivative of u vanishes on the interior boundary $\gamma = \cup_{i=1}^n \partial\omega_i$.

The problem is to identify γ and to construct the solution of $(*)$.

Problem 2. *Cavity identification in an elastic body and construction of solutions.*

The problem is to identify cavities in a solid body from displacements and surface stresses given on part of the body' s surface assuming the surfaces of the cavities to be stress free or clamped, and to construct solutions.

Problem 3. *Cavity identification by gravimetric methods.*

Let the Earth be represented by a half-space or half-plane. The problem is to identify interior holes, assumed to be finite in number, from gravity anomalies or gravity gradients measured on the surface.

Before considering each one of the foregoing problems individually, we note that the problems were described in our talk at the Mathematical Conference in Hanoi in 1997 [6]. Many significant new results have been proved since then, and it is the purpose of this paper to present a survey of the subject with emphasis on developments occuring after 1997. The survey is essentially restricted to the reseach activities of our group. We should mention that cavities that degenerate into lines or surfaces are called mathematical cracks or simply cracks. We note that in our terminology, a crack may have a nonempty interior. For the detection of cracks by the electric method, the pioneering paper of Friedman and Vogelius [28] should be mentioned and furthermore for the detection of cracks in elastic bodies by the mechanical method, Andrieux, Ben Abda and Bui [4] have introduced the method of jump functionals. Since we are directly concerned with regular cavities, we can only refer the reader to these important papers.

The remainder of the paper consists of three sections, devoted respectively to the three problems under consideration.

1. Identification of Domain and Regularization of Solution of Elliptic Equations

We first consider the question of uniqueness. In the case of a plane domain Ω with one hole ω, $\overline{\omega} \subset \Omega$ such that the outer boundary Γ of Ω is known and represented by a Jordan curve and that $\gamma = \partial\omega$ is a smooth Jordan curve, it has been shown in [8] that under Cauchy conditions on an open smooth portion Γ_0

of Γ and the condition $u = 0$ on γ, there exist at most one plane domain Ω and one harmonic function u on Ω satisfying these boundary conditions.

The foregoing uniqueness results was recently extended to the case of semi-linear elliptic equations in plane domains with finitely many holes (see [32]).

Now we consider a variant of the problem. As above, let Ω be a plane domain with outer boundary Γ. We assume that Ω contains interior holes. Let the holes be represented by simply connected domains $\omega_i, \ldots, \omega_n$ with pairwise disjoint closures.

$$\overline{\omega_i} \cap \overline{\omega_j} = \emptyset, \ i \neq j. \tag{1.1}$$

Let

$$\gamma = \bigcup_{i=1}^{n} \partial\omega_i. \tag{1.2}$$

We assume that each $\partial\omega_i$ is piecewise C^1. Consider a differential operator of the form

$$A(\mathbf{x}, u)u = \sum_{i,j=1}^{2} D_j(a_{ij}(\mathbf{x}, u)D_i u) - c(\mathbf{x})F(u)$$

with $F \in C^1(\mathbb{R})$, $a_{ij} \in C^1(\mathbb{R}^2 \times \mathbb{R})$ and (a_{ij}) satisfying the ellipticity condition

$$\sum a_{ij}(\mathbf{x}, \sigma)\xi_i\xi_j \geq C_0(\xi_i^2 + \xi_j^2)$$

for a $C_0 > 0$, for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and for all $(\mathbf{x}, \sigma) \in \mathbb{R}^2 \times \mathbb{R}$.

Consider the equation

$$A(\mathbf{x}, u)u = 0 \tag{1.3}$$

subject to the boundary conditions

$$u|_{\Gamma_0} = f, \tag{1.4}$$

$$\sum_{i,j=1}^{2} a_{ij}(\mathbf{x}, u(\mathbf{x}))n_j(\mathbf{x})D_i u(\mathbf{x}) = g(\mathbf{x}), \ \ \mathbf{x} \in \Gamma_0, \tag{1.5}$$

where Γ_0 is a smooth open subset of Γ, $\mathbf{n}(\mathbf{x}) = (n_1(\mathbf{x}), n_2(\mathbf{x})$ being the unit outer normal to $\Gamma \cup \gamma$ at $\mathbf{x}$ and

$$\sum_{i,j=1}^{2} a_{ij}(\mathbf{x}, u(\mathbf{x}))n_j(\mathbf{x})D_i u(\mathbf{x}) = 0, \ \ \mathbf{x} \in \gamma_* = \gamma \backslash \{y_1, \ldots, y_k\}, \tag{1.6}$$

where $\{y_1, \ldots, y_k\}$ is a finite subset of γ such that γ_* is of C^1 type. The functions f, g, c and F are to satisfy the conditions

$$\begin{aligned} &c \in \mathrm{C}(\mathbb{R}^2), \ c(\mathbf{x}) > 0 \text{ a.e., or } c \equiv 0 \text{ in } \Omega, \\ &f, g \in \mathrm{C}(\overline{\Gamma_0}), \ F(0) = 0, \ F'(v) > 0, \ \forall v \neq 0. \end{aligned} \tag{1.7}$$

It will be assumed further that

$$\Gamma, \ \partial\omega_i \text{ are piecewise } C^1 \text{ Jordan curves} \tag{1.8}$$

$$\Gamma \cap \partial\omega_i = \emptyset,\ i = 1, \dots, n. \tag{1.9}$$

Then, we have

Theorem 1 [33]. *Let* (1.3) - (1.9) *hold. If we have either* $f \neq$ const *or* $g \neq 0$, *then there exists at most one pair* (Ω, u), $u \in C(\overline{\Omega}) \cap C^2(\Omega \cup \Gamma \cap \gamma_*) \cap H^1(\Omega)$ *for which* (1.3) - (1.6) *hold.*

It should be remarked that the foregoing analysis does not apply to the 3-dimensional case. In the latter case, we would need the surfaces to be piecewise analytic. This will be discussed in a future work. Note that if on γ, the condition $\partial u/\partial n = 0$ is replaced by the condition $u = 0$, then uniqueness holds for domains in $\mathbb{R}^n$ for all $n \geq 2$. It is of interest to note that if, in particular, (1.3) is the Laplace equation, then u denotes the electric potential in the problem of cavity detection by the electric method cf [17], and the condition $\partial u/\partial n = 0$ on γ and $u = 0$ on γ correspond to a nonconducting surface γ and an infinitely conducting surface γ respectively.

1.1. Regularization by quasi-reversibility

After the problem of detection of the cavity (or cavities), there remains the question of determining the solution.

Consider a linear form of Eq. (1.3)

$$Au \equiv \sum_{i,j=1}^{2} D_i(a_{ij} D_j u) + b_1 D_1 u + b_2 D_2 u - h = 0,\ (x,y) \in \Omega, \tag{1.10}$$

where Ω is a bounded plane domain. Let Γ be a bounded smooth subset of $\partial\Omega$. The problem now is that of finding a function $u = u(x,y)$ satisfying (1.10) subject to the conditions

$$u|_\Gamma = f,\ \sum_{i,j=1}^{2} (D_j u) n_i \Big|_\Gamma = g, \tag{1.11}$$

where $\mathbf{n} = (n_1, n_2)$ is the unit outer normal vector to $\partial\Omega$. As is well-known, this is an ill-posed problem. In Lattès–Lions' monograph [31, Chapter 4], a special form of (1.10) - (1.11) is regularized by the method of quasi-reversibility. The solutions of (1.10) - (1.11) are approximated by a family (u_ε), $\varepsilon \downarrow 0$, of solutions of well-posed problems. In [31] (loc. cit.), it is proved that if a solution u exists, then the family (u_ε) converges to u as $\varepsilon \downarrow 0$, but the case of nonexistence of a solution is not considered. It is noted, however, that solutions usually do not exist. In fact, the set of boundary data (f, g) for which the problem (1.10) - (1.11) has no solution is dense in $L^2(\Gamma) \times L^2(\Gamma)$. Indeed, if (1.10) - (1.11) has a solution u in $H^2(\Omega)$, say, then f is in $H^{3/2}(\Gamma)$. Thus if f, g are step functions, then (1.10) - (1.11) has no solution in $H^2(\Omega)$ (which is a natural solution space). Now, in practice f, g are results of experimental measurements and thus are given as finite sets of points. It is realistic, therefore, not to assume existence of a solution, and this is the approach of Klibanov and Santosa [30] (1991). However, in the latter work, the given Cauchy data are assumed to be highly smooth (in fact, as traces of Sobolev functions of higher order) whereas as pointed out above, in practice, they are just finite sets of points that are patched up into L^2-functions. In the more recent work [9], the authors take the given Cauchy data as L^2-functions, and for approximate solutions, they used the quasireversibility method (QR method, for short), and regularized the given data into smooth functions. Thus, their QR approach departs from the usual ones in that not only the equation is perturbed but the given data are also regularized. The problem is regularized as it is given, without any existence assumption. Estimates of error between the regularized solution and an exact solution corresponding to an exact right hand side are derived. The analysis is based on an estimate of Carleman type derived in the course of the proof.

1.2. Formulation as a moment problem: Tikhonov regularization

We shall, instead of a general elliptic equation, consider the Laplace equation

$$\Delta u = 0 \tag{1.12}$$

and a special plane domain Ω bounded by two C^1-Jordan curves L and ℓ, such that ℓ is interior to L. We assume that L contains a segment L_0 parallel to the x-axis (this assumption does not affect the generality since by an appropriate conformal mapping, we can have a domain with this property). Let u be a harmonic function on Ω such that $u \in C^1(\Omega \cup L_0)$. We propose to determine $\partial u/\partial n$ on the interior curve ℓ from the values of $\partial u/\partial n$ on L and values of u on an appropriate discrete subset of L. This problem will be formulated as a moment problem that will be regularized by the Tikhonov method. We have

Theorem 2 [7]. *Let $N(x, y; \xi, \eta)$ be the Neumann function for the Laplace equation in the domain Ω described above. Let u be a harmonic function on Ω such that $u \in C(\overline{\Omega})$ and is piecewise C^1 on $\partial\Omega$. Let $v = \partial u/\partial n$. Then u admits the representation*

$$\begin{aligned} u(x,y) = a &+ \int_{L_0} N(x,y;\xi,\eta)v(\xi,\eta)ds(\xi,\eta) \\ &+ \int_{L\setminus L_0} N(x,y;\xi,\eta)v(\xi,\eta)ds(\xi,\eta) \\ &+ \int_{\ell} N(x,y;\xi,\eta)v(\xi,\eta)ds(\xi,\eta), \end{aligned} \tag{1.13}$$

where a is a constant to be determined and ds is the arc length differential along $\partial\Omega$.

Let v be given on L and u given on an infinite bounded sequence of points $(x_n, y_n) \in L_0$ such that $y_n = k\ \forall n$, and $x_i \neq x_j$ for $i \neq j$. Then the moment problem for a and v

$$\int_{\ell} N(x_n,k;\xi,\eta)v(\xi,\eta)ds(\xi,\eta) = u(x_n,k) - a - \int_{L_0} N(x_n,k;\xi,\eta)v(\xi,\eta)ds(\xi,\eta) - \int_{L\setminus L_0} N(x_n,k;\xi,\eta)v(\xi,\eta)ds(\xi,\eta),\ n=1,2,... \quad (1.14)$$

admits at most one solution.

The theorem is proved in [7] (loc. cit.) using the reflection principle and the identity theorem for analytic functions.

The moment problem (1.14) for a and v is regularized by the Tikhonov method with error estimates given, a result in [7]. It is noted that J. Cheng et al. [25] recently formulated Cauchy' s problem for Laplace' s equation as a moment problem and developed a numerical algorithm for its solution.

1.3. Cavity identification: finite dimensional approximation

We consider a star-shaped cavity in a plane domain Ω. The surface of the cavity is assumed to be insulating (corresponding, e.g., to the case of a cavity filled with air) and to be made up of two C^1-arcs meeting each other at two points a_1, a_2, the edges of the cavity. The quantities to be determined are therefore two C^1 functions, the graphs of which intersect each other at a_1, a_2 and two vectors emanating from the origin of $\mathbb{R}^2$ with a_1, a_2 as the respective end points. A regularized construction of the cavity surface satisfying the foreging conditions is given in [17] using finite dimensional approximations and minimization methods.

2. Elastic Continuation and Crack Identification in an Elastic Body

We are concerned in this section with the problem of elastic continuation and the problem of identification of cracks in a plane elastic body. Elastic continuation is an important ingredient in the identification of cracks and moreover, by itself, it is of independent interest. This section is divided into three subsections: the first subsection deals with the problem of elastic continuation, the second subsection treats of the regularization of the problem of identification of the stress field in an elastic body from displacements and surface stresses given on an open portion of the boundary. We shall refer to the latter problem as a Cauchy like problem in Elasticity. The final subsection is devoted to a regularization problem for crack identification.

2.1. Problem of elastic continuation

Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, that is occupied by an isotropic elastic solid. Let $\mathbf{u} = \mathbf{u}(\mathbf{x})$ be the displacement field in Ω. Then $\mathbf{u}(\mathbf{x})$ satisfies the following Lamé system of equations

$$L\mathbf{u}(\mathbf{x}) = 0,\ \ \mathbf{x} \text{ in } \Omega, \quad (2.1)$$

where
$$Lu(\mathbf{x}) \equiv \mu(\mathbf{x}).\Delta\mathbf{u}(\mathbf{x}) + [\lambda(\mathbf{x}) + \mu(\mathbf{x})]\nabla(\nabla \cdot \mathbf{u}(\mathbf{x})) + (\nabla \cdot \mathbf{u}(\mathbf{x}))\nabla\lambda(\mathbf{x}) + [\nabla\mathbf{u}(\mathbf{x}) + \mathbf{u}(\mathbf{x})]^T\nabla\mu(\mathbf{x}).$$

The following theorem is proved in [16] (see also [11])

Theorem 3. *Let Γ_0 be a C^2-portion of $\partial\Omega$. Let $\mu \in C^3(\overline{\Omega})$, $\lambda \in C^2(\overline{\Omega})$ satisfying*

$$\mu(\mathbf{x}) > 0,\ \lambda(\mathbf{x}) + 2\mu(\mathbf{x}) > 0,\ \forall\mathbf{x} \text{ in } \overline{\Omega}.$$

We set
$$\sigma(\mathbf{u})_{i,j} = \lambda\delta_{ij}\nabla \cdot \mathbf{u} + \mu(\nabla\mathbf{u})^T_{ij}, \tag{2.2}$$

where $\delta_{ij} = 1$ for $i \neq j$ and 0 otherwise, and M_{ij} denotes the (i,j)-component of an $n \times n$ matrix M. Let $\mathbf{n}(\mathbf{x}) = (n_1(\mathbf{x}), \dots, n_n(\mathbf{x}))^T$ be the outer unit normal vector to $\partial\Omega$ at $\mathbf{x}$. If $(H^2(\Omega))^n$ satisfies

$$\begin{aligned}(Lu)(\mathbf{x}) &= 0\ \forall\mathbf{x} \in \Omega,\\ \mathbf{u}(\mathbf{x}) = 0,\ \sigma(\mathbf{u}(\mathbf{x}))\mathbf{n}(\mathbf{x}) &= 0,\ \mathbf{x} \in \Gamma_0,\end{aligned} \tag{2.3}$$

then $\mathbf{u}(\mathbf{x}) = 0\ \forall x \in \Omega$.

The uniqueness of elastic continuation was proved by Dehman and Robbiano [26] under the condition of C^∞-smoothness on the coefficients and by lengthy pseudo-differental caculus. The method used in [16] (loc. cit.) is more compact, thanks to a physically natural transformation and under rather mild regularity conditions on the coefficients. It should be noted that a theorem on uniqueness of elastic continuation had been proved by E. Almansi [3] in 1907 [1] . Theorem 3 was recently extended to semilinear elasticity [34]. In the latter paper, the authors consider an elastic body represented by a bounded domain Ω in $\mathbb{R}^3$. It is assumed that the elastic moduli depend also on the displacement

$$\lambda = \lambda(\mathbf{x}, \mathbf{u}). \tag{2.4}$$

Let $\mathbf{x} = (x_1, x_2, x_3)$ be in Ω. For $(i,j,k) = (1,2,3)$, $(2,3,1)$, $(3,1,2)$, let σ_i, τ_{jk} denote the components of the normal stress and of the shear stress corresponding to the x_i-direction. Consider the following system

$$\frac{\partial\sigma_i}{\partial x_i} + \frac{\partial\tau_{ij}}{\partial x_j} + \frac{\partial\tau_{ik}}{\partial x_k} = -X_i\,, \tag{2.5}$$

X_i being the body force in the i-th direction,

$$\begin{aligned}\mathbf{u}|_\Gamma &= \mathbf{u}^0,\\ n_i\sigma_i + n_j\tau_{ij} + n_k\tau_{ik} &= \overline{X}_i \text{ on } \Gamma,\end{aligned} \tag{2.6}$$

where Γ is an open portion of the boundary $\partial\Omega$, $\overline{X}_i$ is the surface stress in the i-th direction. It is assumed that

[1]The author would like to thank Prof. E. Obolashvili for calling his attention to Almansi's paper, and Prof. H. Begher for his interest in the problem.

$$\lambda \in C^2(\mathbb{R}^3 \times \mathbb{R}^3),\ G \in C^2(\mathbb{R}^3). \tag{2.7}$$

Then the following has been proved

Theorem 4 [34]. *Let* Γ *be* C^1*-smooth, let* λ, G *satisfy* (2.7). *Then the system* (2.5) - (2.7) *has at most one solution* $\mathbf{u}$ *in* $(C^3(\Omega \cup \Gamma))^3$.

2.2. Identification of cracks in elastic bodies

We consider the problem of determining the location and shape of an interior cavity in a plane nonhomogeneous elastic body. The problem admits of an electrical method as shown in Sec. 1. In this section, the method used is purely mechanical. It is shown that displacements and surface stresses given on an open portion of the boundary uniquely determine the location and shape of a crack. Only the case of one crack is considered here, the case of finitely many cracks being considered in a forthcoming paper [13]. This uniqueness result is the point of departure of computational attempts.

Let Ω be a plane elastic body assumed to be nonhomogeneous and limited by a known outer boundary Γ and an unknown inner boundary γ represented by a Jordan curve. Let ω be the interior of γ

$$\gamma = \partial\omega.$$

Plane stress [36] is assumed. Let the displacements and the surface stresses be specified on an open portion Γ_0 of the outer boundary. Before specifying conditions on the surface stresses on γ, we note that it is in general not smooth. In fact, it can have corners or sharp edges at which points the stresses can become infinite. In the case of rectilinear cracks, Williams found that for an isotropic medium, the stresses go to infinity as $1/\sqrt{r}$ for $r \to 0$, where r is the distance from the crack edge. The same sort of singularity prevails is the orthotropic case cf Ang & Williams [5]. Accordingly we shall assume the surface of the crack to be smooth except at a finite number points $\{z_1, \dots, z_k\}$ corresponding to the edges. Thus setting

$$\gamma_* = \gamma \setminus \{z_1, \dots, z_k\},$$

we have that the surface stresses vanish on γ_*. Near the edges, the following conditions are assumed to hold

$$\int_\alpha^\beta \sigma_x(r,\theta)rd\theta,\ \int_\alpha^\beta \sigma_y(r,\theta)rd\theta,\ \int_\alpha^\beta \tau_{xy}(r,\theta)rd\theta \to 0 \text{ as } r \to 0, \tag{2.8}$$

where r is the distance from the crack edge and $\beta - \alpha > 0$ is the notch angle. Note that if $\beta - \alpha = 2\pi$, then we have a crack degenerating into a line. Physically, the condition (2.8) means that there is no net force exerted on the edge. It is satisfied, in particular, if there exists an $\alpha \in (0,1)$ such that

$$\sigma_x,\ \sigma_y,\ \tau_{xy} \sim r^{-\alpha} \text{ for } r \to 0.$$

Before stating our uniqueness theorem, we list below our standing assumptions. For each z in $\gamma = \partial\omega$, there exists a $\delta_z > 0$ such that for any $\delta \in (0, \delta_z)$, the following holds

$$B(z,\delta) \setminus \overline{\omega},\ B(z,\delta) \cap \omega,\ B(z,\delta) \cap \partial\omega \text{ are connected} \qquad (2.9)_a$$

and

$$B(z,\delta) \cap \partial\omega \subset \partial[B(z,\delta) \setminus \overline{\omega}] \cap \gamma = \partial[B(z,\delta) \cap \omega] \cap \gamma. \qquad (2.9)_b$$

We can now state our uniqueness theorem.

Theorem 5 [12]. *Let the assumption* (2.8) - (2.9) *hold, let* Γ_0 *be a smooth open subset of the outer boundary* Γ *and let the elastic moduli be in* $C^3(\mathbb{R}^2)$. *If either*

there is no body force and the surface stresses on Γ_0 *do not identically vanish*

or

the body forces do not identically vanish,

then there exists at most one pair $(\Omega, (u, v))$ *with* u, v *in* $C^3(\Omega \cup \Gamma_0 \cup \gamma_*) \cap C(\overline{\Omega}) \cap H^1(\Omega)$, u, v *being the dispacements in the* x *and* y *directions, that satisfies the equation of elastic equilibrium* (2.5), *the Cauchy conditions on* Γ_0 *and the stress free conditions on* γ_*.

• *Remark.* Let the conditions be the same as those of preceeding theorem except that the surface of the cavity, instead of being stress free, is clamped. Then uniqueness still holds (in fact, we can have a stronger statement, namely, that there may be finitely many cavities, see [14] for details).

2.3. Regularization by quasi-reversibility

A Cauchy like problem for Lamé system consists in determining the stress field in an elastic body Ω from displacements and surface stresses given on an open portion of its boundary is an ill-posed problem. Uniqueness of the problem is proved in [16] (see theorem 3 above). The problem has been regularized by various methods: the quasi-reversibility method [10] in the n-D, $n = 2, 3$, and in the 2-D case the method of moments [15].

3. Cavity Detection by Gravimetric Methods

We are concerned in this section with the problem of detecting cavities in the interior of the Earth using gravimetric methods. Inhomogeneities of mass density in the Earth generate gravity anomalies that can be measured on its surface. Under certain conditions on the geometry of the inhomogeneity, measured gravity anomalies uniquely determine its location and shape. On the other hand, the method of gravity gradient presents certain advantages (see [37]).

A cavity inside the Earth can be seen as a hole filled with air or water, in any case, with a material so light that its density is practically zero. Accordingly, the problem is reduced to that of finding the geometry of inhomogeneity and a cavity is thus a set of zeros of the density.

We shall adopt a flat Earth model. Thus we deal with a half-space or a half-plane. We shall consider successively the 3-D case and the 2-D case.

3.1. Uniqueness and finite dimensional approximation: the 3-D case

Let B be a hole inside the half-space $\mathbb{R}^3_- = \{(x,y,z) \in \mathbb{R}^3 : z \leq 0\}$. Let Ω be a bounded domain in $\mathbb{R}^2$ and $H > 0$ and let $\sigma(\xi,\eta)$ be an unknown continuous function on Ω satisfying the conditions

$$\begin{aligned} 0 < \sigma(\xi,\eta) < H \ \text{ for } \ (\xi,\eta) \in \Omega, \\ \sigma(\xi,\eta) = 0 \ \text{ for } \ (\xi,\eta) \in \partial\Omega. \end{aligned} \tag{3.1}$$

Let B be a body immersed in the half-space $\mathbb{R}^3_-$. Suppose B has a known density $\rho_1 \geq 0$ and is described by the formula

$$B = \{(x,y,z) \in \mathbb{R}^3_- : -H \leq z \leq \sigma(\xi,\eta) - H, \forall (x,y) \in \Omega \text{ and } \sigma \text{ satisfies } (3.1)\}.$$

Let the (known) density of $\mathbb{R}^3_- \setminus B$ be $\rho_2 > \rho_1$. Denote by V the potential of the masses of density $\rho = \rho_2 - \rho_1$ filling the domain B. Such a system generates a gravity anomaly on the surface $z = 0$ of $\mathbb{R}^3_-$ given by the formula

$$\delta g = -\left.\frac{\partial V}{\partial z}\right|_{z=0} \tag{3.2}$$

We thus have

$$\begin{aligned} &\int_\Omega \{(x-\xi)^2 + (y-\eta)^2 + (\sigma(\xi,\eta) - H)^2\}^{-1/2} d\xi d\eta \\ &\qquad - \int_\Omega \{(x-\xi)^2 + (y-\eta)^2 + H^2\}^{-1/2} d\xi d\eta = c\delta g, \end{aligned} \tag{3.3}$$

where c is an appropriate constant. After some rearrangements we have

$$\int_\Omega \{(x-\xi)^2 + (y-\eta)^2 + (\sigma(\xi,\eta) - H)^2\}^{-1/2} d\xi d\eta = f(x,y), \ (x,y) \in \mathbb{R}^2 \tag{3.4}$$

for some (known) continuous function f, and $\sigma(x,y)$ is the unknown function. It is shown in [18] that Eq. (3.4) admits at most one solution $\sigma(\xi,\eta)$ satisfying (3.1).

Theorem 6 [18]. *There exists at most one solution* $\sigma(\xi,\eta)$ *of Eq.* (3.4) *in* $C(\overline{\Omega})$ *satisfying* (3.1).

It is noted that in the foregoing theorem, the boundary of the hole is only required to be continuous. For more complex cavity geometries, we refer to the monograph of Isakov [29] which contains a wealth of information and results on the subject.

Eq. (3.4) is an ill-posed nonlinear integral equation. The equation is approximated by stable finite dimensional equations. Convergence of the approximate solutions when the data f is exact is proved. In the case of noise, it is shown that an approximate solution can be chosen arbitrarily close to the solution of the original equation if the right hand side f is sufficiently close to the exact data. Details are given in [19].

3.2. The 2-D case: uniqueness and linearization

In subsection 3.1 above, we have considered the problem of detecting mass inhomogeneities as a 3-D body. We have used the method of gravity anomalies, and furthermore the problem is regularized by finite dimensional approximation. In this subsection, we use, instead, the method of gravity gradient.

For our problem, we consider the Earth represented by the half-plane (x, z), $-\infty < z \leq H$ where $H > 0$ and let the body be presented by $0 \leq z \leq \sigma(x)$.

Let ρ be relative density of B, i.e., the difference between the density of B and that of the surrounding medium. We take ρ to be equal to 1. Denote by $U = U(x, z)$ the gravity potential created by ρ,

$$U(x, z) = 2\pi \int_B \ln\left[(x-\xi)^2 + (z-\zeta)^2\right] dv. \tag{3.5}$$

Then the gravity anomaly created by ρ is

$$-\frac{\partial U}{\partial z} = 2\pi \int_B \frac{x-\xi}{(x-\xi)^2 + (z-\zeta)^2} dv$$

and the gravity gradient on the surface $z = H$ is

$$\begin{aligned} -\left.\frac{\partial^2 U}{\partial z^2}\right|_{z=H} &= -2\pi \int_B \frac{\partial}{\partial \zeta} \frac{H-\zeta}{(x-\xi)^2 + (H-\zeta)^2} dv \\ &= -2\pi \left\{ \int_0^1 \frac{H-\sigma(\xi)}{(x-\xi)^2 + (H-\sigma(\xi))^2} d\xi - H \int_0^1 \frac{d\xi}{(x-\xi)^2 + H^2} \right\} \end{aligned} \tag{3.6}$$

which we rewrite, after some rearrangements, as

$$\int_0^1 \frac{H-\sigma(\xi)}{(x-\xi)^2 + (H-\sigma(\xi))^2} = f(x). \tag{3.7}$$

Uniqueness for this equation in $\sigma(\xi)$ is proved in [20] (see [21] for a more general uniqueness theorem). We shall assume

$$0 < \sigma(\xi) < \alpha < H, \quad 0 < x < 1. \tag{3.8}$$

Setting

$$\phi(x) = H - \sigma(x), \quad x \in (0, 1), \tag{3.9}$$

we see that the function

$$h(x) = \int_0^1 \frac{\phi(\xi) d\xi}{(x-\xi)^2 + \phi^2(\xi)} \tag{3.10}$$

can, in view of (3.8), be extended to complex analytic function on a strip of width $< H-\alpha$ around the real axis of the complex plane. Hence h is completely defined by its values on an interval $(-\infty, -M)$ for any $M > 0$, i.e., Eq. (3.7) is equivalent to the following equation

$$\int_0^1 \frac{\phi(\xi)d\xi}{(x-\xi)^2+\phi^2(\xi)} = f(x),\ x \le -M, \tag{3.11}$$

where ϕ is a continuous function on $[0,1]$, $\phi(0) = \phi(1) = H$ and $H - \alpha \le \phi(x) < H$ for $0 < x < 1$.

Now for large M and $x \ge 0$, we have the expansion

$$\frac{\phi(\xi)}{(M+x-\xi)^2+\phi^2(\xi)} = \phi(\xi)(M+x+\xi)^{-2}\left[1+\left(\frac{\phi(\xi)}{M+x+\xi}\right)^2\right]^{-1} \approx \frac{\phi(\xi)}{(M+x+\xi)^2} \tag{3.12}$$

And we consider the following linear integral equation in ϕ:

$$\int_0^1 \frac{\phi(\xi)}{(M+\xi+x)^2}\,d\xi = f(-M-x),\ x > 0. \tag{3.13}$$

By taking $x = 1, 2, \ldots$, we arrive at the following moment problem

$$\int_0^1 \frac{\phi(\xi)}{(M+\xi+x)^2}\,d\xi = f(-M-n) \equiv \mu_n,\ \ n = 1, 2, \ldots \tag{3.14}$$

It is shown that Eq. (3.14) admits at most one continuous solution $\phi(x)$, $0 \le x \le 1$ (Theorem 2 of [20]). The moment problem (3.14) is known as an ill-posed problem. It is regularized by the Tikhonov method (see [21]).

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