Vietnam Journal of Mathematics **28**:4 (2000) 295-310

Vietnam Journal of MATHEMATICS

Some Sufficient Conditions for the Existence of Equilibrium Points Concerning Multivalued Mappings

Nguyen Ba Minh and Nguyen Xuan Tan
Institute of Mathematics, P.O. Box 631, Bo Ho 10000, Hanoi, Vietnam

Received April 9, 1998
Revised April 9, 2000

Abstract. Some results on the continuity with respect to a cone of multivalued mappings are shown and applied to consider sufficient conditions for the existence of weak equilibrium points concerning multivalued mappings. Some applications of these results to the existence of efficient points of subsets in reflexive Banach space and of solutions of vector optimization problems are also discussed.

1. Introduction

Let X, Y be topological locally convex Hausdorff spaces, $D \subset X$ a closed convex subset, $C \subset Y$ a closed convex pointed cone. Given a multivalued mapping $F : D \times D \to 2^Y$ with $F(x,y) \neq \emptyset$ for all $x, y \in D$, we are interested in considering the problem of finding

$$\bar{x} \in D \text{ such that } F(\bar{x}, y) \not\subseteq -\text{int}C, \text{ for all } y \in D. \tag{1}$$

The point $\bar{x}$ is called a weak equilibrium point (or, a solution) of the weak equilibrium problem with respect to C, denoted by (WEP,C). The problem of finding

$$\bar{x} \in D \text{ such that } F(\bar{x}, y) \not\subseteq -(C \setminus \{0\}), \text{ for all } y \in D, \tag{2}$$

is called an equilibrium problem with respect to C and denoted by (EP,C). Such a point $\bar{x}$ is said to be an equilibrium point (or, a solution) of the problem (EP,C). It is clear that if $\text{int}\, C = \emptyset$, then every point $x \in D$ is an equilibrium point of (WEP,C). In this case it is not interesting to consider (1). Therefore, in this paper, we only study (1) with $\text{int}\, C \neq \emptyset$. Further, if $\bar{x}$ is an equilibrium point of (EP,C), then it is also a weak equilibrium point of (WEP,C). And, if

$\widetilde{C}$ is another closed convex point cone in Y with $\widetilde{C} \subset C$ and $\bar{x}$ is an (a weak equilibrium point of (EP,C) ((WEP,C)), then $\bar{x}$ is also an (a weak) equilibriu point of (EP,$\widetilde{C}$) ((WEP,$\widetilde{C}$), respectively). Further, if C satisfies condition: there is a closed convex pointed cone $\widetilde{C}$ such that $C \setminus \{0\} \subseteq \operatorname{int} \widetilde{C}$ and $\bar{x} \in D$ i a solution of (WEP,C), then $\bar{x}$ is a solution of (EP,C). Such a condition on C i satisfied, for example, if it has a convex compact base (see the proof in [12]). I this paper we consider (WEP,C) only.

Before presenting the main results we generalize the well-known Banac Steinhaus theorem [3] for a family of convex functions. We prove necessary an sufficient conditions for a multivalued mapping to become a lower C-continuou multivalued mapping (Theorem 2.3). Then, using the new Banach-Steinha theorem for convex functions we show that if C has the polar C', which is polyhedral cone and $F : D \to 2^Y$ is C-convex and lower C-continuous wit $F(x) + C$ convex for all $x \in D$, then it is weak lower C-continuous (Theore 3.4).

Further, we extend the result in [12] to the multivalued case. The problem (1), (2) concerning singlevalued mappings are studied by Ky Fan [6], Blum an Oettli [4, 5], Oettli [10] and others. Our results are extensions of their results. [10] Oettli proved some results on the existence of equilibrium points concernin vector functions by using the method of scalarization. This method cannot applied to our case with the mapping F of the form $G + H$, where G and satisfy different conditions. We also weaken the coercive conditions (Conditio A8/ in Theorem 3.1 below) by other conditions (see Theorem 3.3), using t weak topology in a reflexive Banach space X. The weak lower C-continuo property of F in Theorem 2.4 is used to study (WEP,C) in the case where has a cone C with C' a polyhedral cone.

Finally, we apply the obtained results to show the existence of efficient poin of subsets in reflexive Banach spaces and of solutions of vector optimization pro lems. The results can be also applied to the existence of solutions of variation inequalities, Nash equilibrium problem... concerning multivalued mappings.

2. Preliminaries

Let X be a topological locally convex space, $D \subset X$ a convex set. By R denote the space of real numbers with the usual topology, $\overline{R} = R \cup \{\pm\infty\}$. Fi of all we recall the following definitions.

Definition 2.1. *A function $f : D \to \overline{R}$ is called a convex (concave) functi if $f(\alpha x+(1-\alpha)y) \leq \alpha f(x)+(1-\alpha)f(y)$ ($f(\alpha x+(1-\alpha)y) \geq \alpha f(x)+(1-\alpha)f($* respectively), *holds for all $x, y \in \operatorname{dom} f = \{x \in D \mid f(x) < +\infty\}$ and $\alpha \in [0,$*

Definition 2.2. *Let $\{f_\alpha \mid \alpha \in I\}$ be a family of functions on D, where I a nonempty parameter set. We say that this family is upper (lower) equise continuous at $x_0 \in D$ if for every $\varepsilon > 0$ there is a neighborhood U of x_0 in such that $f_\alpha(x) \leq f_\alpha(x_0) + \varepsilon$ ($f_\alpha(x) \geq f_\alpha(x_0) - \varepsilon$,* respectively), *holds for $x \in U \cap D$ and $\alpha \in I$.*

Further, let Y be another topological locally convex space with a cone C and F be a multivalued mapping from D to Y (denoted by $F: D \to 2^Y$) which means that $F(x)$ is a set in Y for each $x \in D$.

We denote by $\operatorname{dom} F$ the set of all $x \in D$ such that $F(x) \neq \emptyset$. In this paper without loss of generality we can assume $\operatorname{dom} F = D$.

Definition 2.3. a) *We say that F is upper (lower) C-continuous at $x_0 \in D$ if for each neighborhood V of the origin in Y, there is a neighborhood U of x_0 in X such that $F(x) \subset F(x_0) + V + C$ ($F(x_0 \subset F(x) + V - C$,* respectively) *holds for all $x \in U \cap \operatorname{dom} F$.*

b) *We say that F is C-continuous at x_0 if it is upper and lower C-continuous at that point, and F is upper (respectively, lower,...) C-continuous on D if it is upper (respectively, lower,...) C-continuous at every point of D.*

c) *We say that F is weakly upper (lower) C-continuous at x_0 if the neighborhood U of x_0 as above is in the weak topology of X.*

Definition 2.4. *F is said to be C-convex if $F(\alpha x + (1-\alpha)y) \subset \alpha F(x) + (1-\alpha)F(y) - C$ holds for all $x, y \in D$ and $\alpha \in [0,1]$.*

Let Y' denote the topological dual space of Y and

$$C' = \{\xi \in Y' \mid \langle \xi, y\rangle \geq 0, \text{ for all } y \in C\}.$$

Here by $\langle \xi, y\rangle$ we mean $\xi(y)$.

This C' is called the polar cone of the cone C. For given $F: D \to 2^Y$ and $\xi \in C'$ we define the function $G_\xi : D \to \overline{R}$ by $G_\xi(x) = \sup_{y \in F(x)} \langle \xi, y\rangle$, $x \in D$.

Proposition 2.1. a) *If F is C-convex, then G_ξ is a convex function.*

b) *If F is lower C-continuous at $x_0 \in \operatorname{dom} F$, then G_ξ is a lower semicontinuous at x_0.*

Proof. a) Let $x_i \in D$, $i = 1, 2$, and $\alpha \in [0,1]$. Since

$$F(\alpha x_1 + (1-\alpha)x_2) \subset \alpha F(x_1) + (1-\alpha)F(x_2) - C,$$

it follows that

$$\begin{aligned} G_\xi(\alpha x_1 + (1-\alpha)x_2) &= \sup_{y \in F(\alpha x_1 + (1-\alpha)x_2)} \langle \xi, y\rangle \\ &\leq \sup_{y \in \alpha F(\alpha x_1) + (1-\alpha)F(x_2) - C} \langle \xi, y\rangle \\ &\leq \sup_{y \in \alpha F(\alpha x_1) + (1-\alpha)F(x_2)} \langle \xi, y\rangle \\ &\leq \alpha \sup_{y \in F(x_1)} \langle \xi, y\rangle + (1-\alpha) \sup_{y \in F(x_2)} \langle \xi, y\rangle \\ &= \alpha G_\xi(x_1) + (1-\alpha) G_\xi(x_2). \end{aligned}$$

This shows that G_ξ is a convex function.

b) Let $\varepsilon > 0$ be given. Since $\xi \in C'$, there is a neighborhood V of the origin in Y such that $\xi(V) \subset (-\varepsilon, \varepsilon)$. As F is lower C-continuous at $x_0 \in \operatorname{dom} F$, it follows that there exists a neighborhood U of x_0 in X such that

$$F(x_0) \subset F(x) + V - C, \text{ for all } x \in U \cap D.$$

This implies

$$\begin{aligned} G_\xi(x_0) &= \sup_{y \in F(x_0)} \langle \xi, y \rangle \\ &\leq \sup_{y \in F(x)} \langle \xi, y \rangle + \sup_{v \in V} \langle \xi, v \rangle + \sup_{y \in -C} \langle \xi, y \rangle \\ &\leq G_\xi(x) + \varepsilon. \end{aligned}$$

Hence

$$G_\xi(x) \geq G_\xi(x_0) - \varepsilon, \text{ for all } x \in U \cap D.$$

Therefore, G_ξ is lower semicontinuous at x_0.
This completes the proof of the proposition. ■

Further, we recall that a barrel space is a topological locally convex space, in which any nonempty closed, symmetric, convex and absorbing set is a neighborhood of the origin (see, for example [11]). The following theorem is an extension of the well-known Banach–Steinhaus theorem for a family of convex functions.

Theorem 2.2. *Assume that X is a barrel space, I is an index set and $f_\alpha : X \to \overline{R}$, $\alpha \in I$, is convex and lower semicontinuous on some neighborhood U_0 of $x_0 \in \operatorname{dom} f_\alpha$ for all $\alpha \in I$. In addition, suppose that for any $x \in X$ there is a constant $\gamma > 0$ such that $f_\alpha(x) \leq \gamma$ for all $\alpha \in I$. Then the family $\{f_\alpha \mid \alpha \in I\}$ is upper equisemicontinuous at x_0.*

Proof. By setting $\bar{f}_\alpha(x) = f_\alpha(x + x_0) - f_\alpha(x_0)$, if necessary, we may assume that $x_0 = 0$ and $f_\alpha(0) = 0$ for all $\alpha \in I$. For given $\varepsilon > 0$ we put

$$A_\alpha = \{x \in X \mid f_\alpha(x) \leq \varepsilon\}.$$

Since $0 \in A_\alpha$, we conclude that $A_\alpha \neq \emptyset$.

Without loss of generality, we may assume that U_0 is a closed convex neighborhood of the origin. From the convexity and the lower semicontinuity of f_α on U_0, it follows that $U_0 \cap A_\alpha$ is a nonempty closed convex set for all $\alpha \in I$. Further, putting $U = \bigcap_{\alpha \in I}(U_0 \cap A_\alpha \cap (-A_\alpha))$, we conclude that U is nonempty closed, symmetric and convex. Now, we claim that U is absorbing. Indeed, let $x \in X$. By the hypotheses of the theorem there is a constant $\gamma > 0$ such that $f_\alpha(x) \leq \gamma$ and $f_\alpha(-x) \leq \gamma$, for all $\alpha \in I$. Without loss of generality, we can assume $\gamma > \varepsilon$. Hence

$$f_\alpha(\frac{\varepsilon}{\gamma}x) = f_\alpha(\frac{\varepsilon}{\gamma}x + (1 - \frac{\varepsilon}{\gamma})0) \leq \frac{\varepsilon}{\gamma}f_\alpha(x) + (1 - \frac{\varepsilon}{\gamma})f_\alpha(0) = \frac{\varepsilon}{\gamma}f_\alpha(x) \leq \varepsilon.$$

Since U_0 is absorbing, there is a constant $\rho > 0$ such that $\varepsilon x/\rho$, $-\varepsilon x/\rho \in U_0$. For $\gamma_0 = \max\{\gamma, \rho\}$, we conclude that $(\varepsilon/\gamma_0)x \in A_\alpha \cap U_0$. By a similar argument, we obtain $(-\varepsilon/\gamma_0)x \in A_\alpha \cap U_0$, for all $\alpha \in I$, and then $(\varepsilon/\gamma_0)x \in U$. It means that U is absorbing. Since X is a barrel space, U is a neighborhood of the origin in X. For $x \in U$ we have

$$f_\alpha(x) \leq \varepsilon = f_\alpha(0) + \varepsilon, \text{ for all } \alpha \in I.$$

This means that the family $\{f_\alpha \mid \alpha \in I\}$ is upper equisemicontinuous at the origin in X.

This completes the proof of the theorem. ■

The following theorems extend Theorem 5.5 in [8].

Theorem 2.3. *Suppose that Y is a Banach space and $F : D \to 2^Y$ is a multivalued mapping with $F(x) - C$ convex for all $x \in D$. Then F is lower C-continuous at x_0 if and only if the family $\{G_\xi \mid \xi \in C', \|\xi\| = 1\}$ is lower equisemicontinuous at x_0.*

Proof. We first assume that F is lower C-continuous at x_0. Let $\varepsilon > 0$ be given. By Banach–Steinhaus theorem, the family $\{\xi \in C' \mid \|\xi\| = 1\}$ is equicontinuous. Therefore there is a neighborhood V of the origin in Y such that $\xi(y) \in (-\varepsilon, \varepsilon)$ holds for all $y \in V$ and $\xi \in C'$, $\|\xi\| = 1$. From the lower C-continuity of F at x_0 there exists a neighborhood U of x_0 in X such that

$$F(x_0) \subset F(x) + V - C, \text{ for all } x \in U \cap D.$$

It follows that

$$\begin{aligned} G_\xi(x_0) &= \sup_{y \in F(x_0)} \langle \xi, y \rangle \\ &\leq \sup_{y \in F(x)} \langle \xi, y \rangle + \sup_{y \in V} \langle \xi, y \rangle + \sup_{y \in -C} \langle \xi, y \rangle \\ &\leq \sup_{y \in F(x)} \langle \xi, y \rangle + \varepsilon = G_\xi(x) + \varepsilon \end{aligned}$$

holds for all $x \in U \cap D$ and $\xi \in C'$, $\|\xi\| = 1$. This means that the family $\{G_\xi \mid \xi \in C', \|\xi\| = 1\}$ is lower equisemicontinuous at x_0.

Now, assume that this family is lower equisemicontinuous at x_0, but F is not lower C-continuous at x_0. This implies that there exists a convex neighborhood V of the origin in Y such that one can find a net $\{x_\alpha\}$ from X with $\lim x_\alpha = x_0$ and

$$F(x_0) \not\subseteq F(x_\alpha) + V - C, \text{ for all } \alpha.$$

Then, we can take $y_\alpha \in F(x_0)$ with

$$y_\alpha \notin F(x_\alpha) + V - C.$$

Since the set $\operatorname{cl}(F(x_\alpha) + V/2 - C)$ is closed and convex, applying a separation theorem one can find some ξ_α from the topological dual space of Y with unit norm such that

$$\xi_\alpha(y_\alpha) \geq \xi_\alpha(y)$$

for all $y \in F(x_\alpha) + V/2 - C$. Assume that there is α_0 such that $\xi_{\alpha_0} \notin C'$. Thus, we can find $y_0 \in C$ with $\xi_{\alpha_0}(y_0) < 0$. Since $\gamma y_0 \in C$ for all $\gamma > 0$, we have

$$\xi_{\alpha_0}(y_0) \geq \xi_{\alpha_0}(z_{\alpha_0}) + \xi_{\alpha_0}(v_{\alpha_0}) - \gamma \xi_{\alpha_0}(y_0)$$

for some $z_{\alpha_0} \in F(x_{\alpha_0})$ and $v_0 \in V/2$. Letting $\gamma \to +\infty$, the right-hand side tends to $+\infty$ and we have a contradiction. Therefore, we deduce that $\xi_\alpha \in C'$ for all α.

For an arbitrary $\delta > 0$ there exist $\bar{y}_\alpha \in F(x_\alpha)$, $\bar{v}_\alpha \in V/2$ and $\bar{c}_\alpha \in C$ such that

$$\langle \xi_\alpha, \bar{y}_\alpha \rangle > \sup_{y \in F(x_\alpha)} \langle \xi_\alpha, y \rangle - \frac{\delta}{3}$$

$$\langle \xi_\alpha, \bar{v}_\alpha \rangle > \sup_{v \in \frac{V}{2}} \langle \xi_\alpha, v \rangle - \frac{\delta}{3}$$

$$\langle \xi_\alpha, \bar{c}_\alpha \rangle > \sup_{c \in -C} \langle \xi_\alpha, c \rangle - \frac{\delta}{3}.$$

Hence, for $z_\alpha = \bar{y}_\alpha + \bar{v}_\alpha + \bar{c}_\alpha \in F(x_\alpha) + V/2 - C$, we have

$$\begin{aligned}\xi_\alpha(y_\alpha) &\geq \xi_\alpha(z_\alpha)\\ &\geq \sup_{y \in F(x_\alpha)} \langle \xi_\alpha, y \rangle + \sup_{v \in \frac{V}{2}} \langle \xi_\alpha, v \rangle + \sup_{c \in -C} \langle \xi_\alpha, c \rangle \ - \delta\\ &= G_{\xi_\alpha}(x_\alpha) + \sup_{v \in V/2} \langle \xi_\alpha, v \rangle - \delta.\end{aligned}$$

Since the family $\{\xi_\alpha \mid \xi_\alpha \in C', \|\xi\| = 1\}$ is equicontinuous at the origin, we can assume

$$\sup_\alpha \sup_{v \in V/2} \langle \xi_\alpha, v \rangle = \delta_0 > 0.$$

Consequently,

$$G_{\xi_\alpha}(x_0) \geq G_\xi(x_\alpha) + \delta_0 - \delta, \text{ for all } \xi_\alpha \in C', \ \|\xi_\alpha\| = 1.$$

Taking $\varepsilon = (\delta_0 - \delta)/2$ for $\delta < \delta_0$, we obtain

$$G_{\xi_\alpha}(x_0) > G_{\xi_\alpha}(x_\alpha) + \varepsilon,$$

or

$$G_{\xi_\alpha}(x_\alpha) < G_{\xi_\alpha}(x_0) - \varepsilon, \text{ for all } \xi_\alpha \in C', \ \|\xi_\alpha\| = 1.$$

It contradicts the lower equisemicontinuity of the family $\{G_\xi \mid \xi \in C', \|\xi\| = 1\}$. This completes the proof of the theorem. ∎

Next, we recall that a cone $C \subset Y$ is said to be a polyhedral cone if $C = \text{cone}\,(\text{conv}\,\{y_1, \dots, y_n\})$, i.e. a cone spanned by the convex hull of n independent vectors $\{y_1, \dots, y_n\}$.

Theorem 2.4. *Let D, X, Y be as above and let $C \subset Y$ be a convex cone with C' a polyhedral cone. Assume that $F : D \to 2^Y$ is C-convex and lower C-continuous with $F(x) + C$ convex for all $x \in D$. Then F is weakly lower C-continuous.*

Proof. Assume that $C' = \text{cone}\,(\text{conv}\,\{\xi_1, \dots, \xi_n\})$. It is clear that, for $i = 1, 2, \dots, n$, G_{ξ_i} is a convex and lower semicontinuous function from D to $\overline{R}$. Therefore, it is weakly lower semicontinuous from D to $\overline{R}$. Suppose that $x_0 \in \text{dom}\,F$. We show that F is weakly lower C-continuous at x_0. Indeed, for any given $\varepsilon > 0$ and $i = 1, 2, \dots, n$, we can find a neighborhood U_i of x_0 in the weak topology of X such that

$$G_{\xi_i}(x) \geq G_{\xi_i}(x_0) - \beta_0\varepsilon, \text{ for all } x \in U_i \cap D,$$

where $\beta_0 = \min \| \sum_{i=1}^n \lambda_i \beta_i \|$, $\sum_{i=1}^n \lambda_i = 1$. Since $\{\xi_i, i = 1, \dots, n\}$ is an independent system and $\sum_{i=1}^n \lambda_i = 1$, we have $\beta_0 > 0$. Putting $U = \bigcap_{i=1}^n U_i$, we obtain $G_{\xi_i}(x) \geq G_{\xi_i}(x_0) - \beta_0\varepsilon$, for all $x \in U \cap D$ and $i = 1, \dots, n$.
This shows that the family $\{G_{\xi_i} \mid i = 1, \dots, n\}$ is weakly lower equisemicontinuous at x_0. Now, we claim that

$$G_\xi(x) \geq G_\xi(x_0) - \varepsilon, \text{ for all } x \in U \text{ and } \xi \in C', \ \|\xi\| = 1.$$

Indeed, for $\xi \in C'$ $\|\xi\| = 1$, we can write $\xi = \beta \sum_{i=1}^n \lambda_i \xi_i$ for some $\beta > 0$. We have $1 = \|\xi\| = \beta \| \sum_{i=1}^n \lambda_i \xi_i \|$. Therefore,

$$\beta = \frac{1}{\| \sum_{i=1}^n \lambda_i \xi_i \|} \leq \frac{1}{\beta_0}, \text{ or } \beta_0 \beta \leq 1.$$

Since

$$G_{\xi_i}(x) = \sup_{y \in F(x)} \langle \xi_i, y \rangle \geq \sup_{y \in F(x_0)} \langle \xi_i, y \rangle - \beta_0 \varepsilon, \ i = 1, \dots, n, \ x \in U \cap D,$$

multiplying both sides of these inequalities by $\beta\lambda_i$ and taking the sum of them we get

$$\begin{aligned} G_\xi(x) &= \sup_{y \in F(x)} \langle \sum_{i=1}^n \beta \lambda_i \xi_i, y \rangle \\ &\geq \sup_{y \in F(x_0)} \langle \sum_{i=1}^n \beta \lambda_i \xi_i, y \rangle - \beta\beta_0\varepsilon \\ &\geq G_\xi(x_0) - \varepsilon, \text{ for all } x \in U \cap D. \end{aligned}$$

This shows that the family $\{G_\xi \mid \xi \in C', \|\xi\| = 1\}$ is weakly lower equisemicontinuous at x_0. Applying Theorem 2.3, we conclude that F is weakly lower C-continuous at x_0.
This completes the proof of the theorem. ■

Proposition 2.5. *If $F : D \to 2^Y$ is lower C-continuous on D, then the set $A = \{x \in D \mid F(x) \cap \text{int} C = \emptyset\}$ is closed.*

Proof. Without loss of generality, we assume that $\text{int}\, C \neq \emptyset$. Let $x_0 \in \overline{A}$, the closure of A. We claim that $x_0 \in A$. Indeed, let $x_n \in A$ and $x_n \to x_0$. We assume on the contrary $x_0 \notin A$. This means that $F(x_0) \cap \text{int}\, C \neq \emptyset$ and therefore we can find a point $y_0 \in F(x_0)$ and a neighborhood V of the origin in Y such that $y_0 + V \subset \text{int}\, C$. Since F is lower C-continuous at x_0, one can find a neighborhood U of x_0 in X such that

$$F(x_0) \subset F(x) + V - C, \text{ for all } x \in U \cap D.$$

Therefore,

$$y_0 \in F(x) + V - C, \ \text{ for all } x \in U \cap D,$$

or,

$$\begin{aligned} 0 \in F(x) - y_0 + V - C \subset F(x) - & \operatorname{int} C - C \\ \subset F(x) - & \operatorname{int} C, \text{ for all } x \in U \cap D. \end{aligned}$$

This shows that $F(x) \cap \operatorname{int} C \neq \emptyset$ for all $x \in U \cap D$. It contradicts the fact that $x_n \in A$ and $x_n \to x_0$. So $x_0 \in A$ and A is closed.
This completes the proof of the proposition. ∎

Next, we introduce the following definitions:

Definition 2.5. *We say that $G : D \times D \to 2^Y$ is a monotone mapping if $G(x,y) + G(y,x) \subseteq -C$ holds for all $x, y \in D$.*

Definition 2.6. *Let K and D be nomempty convex subsets in X with $K \subset D$. Then $\operatorname{core}_D K$, the core of K relative to D, is defined through $a \in \operatorname{core}_D K$ if and only if $a \in K$ and $K \cap (a, y] \neq \emptyset$ for all $y \in D \setminus K$, where $(a, y] = \{x \in X \mid x = \alpha a + (1-\alpha)y$ for all $\alpha \in [0,1)\}$.*

Finally, we recall the following definition:

Definition 2.7. *Assume that B is a nonempty set in Y.*

1. *A point $x \in B$ is called an efficient or, Pareto-minimal point of B with respect to C if $x - y \in C$, for some $y \in B$, then $y - x \in C$.*
   The set of all the efficient points of B is denoted by $\operatorname{Min}(B \mid C)$.
2. *In the case where* $\operatorname{int} C$ *is nonempty, $x \in B$ is called a weakly efficient point of B with respect to C if $x \in$* $\operatorname{Min}(B \mid \{0\} \cup \operatorname{int} C)$.

The set of all efficient points of B is denoted by $\operatorname{WMin}(B|C)$.

3. The Main Results

Let X, Y be topological locally convex Hausdorff spaces, $D \subset X$ a nonempty closed convex set, $C \subset Y$ a closed convex pointed cone in Y. We generalize Theorem 3.1 in [12] to the case concerning multivalued mappings by the following theorem:

Theorem 3.1. *Let X, Y, D and C be as above. Let $G : D \times D \to 2^Y$ and $H : D \times D \to Y$ be mappings satisfying the following conditions:*

A1. *$0 \in G(x,x)$ for all $x \in D$,*

A2. *G is a monotone mapping with $G(x,y)$ compact for all $x, y \in D$,*

A3. *For any fixed $x, y \in D$ the mapping $g : [0,1] \to 2^Y$ defined by*

$$g(t) = G(ty + (1-t)x, y)$$

is upper $(-C)$-continuous at $t = 0$,

A4. *For any fixed $x \in D$, the mapping $G(x, \cdot) : D \to 2^Y$ is C-convex and lower C-continuous on D,* A5. *$H(x,x) = 0$ for all $x \in D$,*

A6. *For any fixed $y \in D$, the mapping $H(\cdot,y) : D \to Y$ is $(-C)$-continuous on D,*

A7. *For any fixed $x \in D$, the mapping $H(x,\cdot) : D \to Y$ is C-convex,*

A8. *There exists a convex compact subset $K \subset D$ such that for any $x \in D \setminus \mathrm{core}_D K$ one can find a point $a \in \mathrm{core}_D K$ such that*

$$G(x,a) + H(x,a) \subset -C.$$

Then there exists a point $\bar{x} \in D$ such that

$$G(\bar{x},y) + H(\bar{x},y) \not\subseteq -\mathrm{int}\, C$$

for all $y \in D$, i.e. $\bar{x}$ is a solution of the (WEP,C).

Remark. If the set D is compact, then Assumption A8 is satisfied vacuously with $K = D$, since then $D \setminus \mathrm{core}_D K = \emptyset$. As in the proof of Theorem 1 in [4], we prove this theorem over three lemmata.

Lemma 3.2. *Let D, K, G and H satisfy the assumptions of Theorem* 3.1. *Then there exists a point $\bar{x} \in K$ such that $\big(G(y,\bar{x}) - H(\bar{x},y)\big) \cap \mathrm{int}\, C = \emptyset$ for all $y \in K$.*

Proof. Indeed, for any $y \in K$ we set

$$S(y) = \{x \in D \,|\, \big(G(y,x) - H(x,y)\big) \cap \mathrm{int}\, C = \emptyset\}.$$

Since $y \in S(y)$, it follows that $S(y) \neq \emptyset$ for all $y \in D$. By Proposition 2.5, $S(y)$ is a closed set in D.

Now, let $\{y_i \,|\, i \in N\}$ be a finite subset of K and $I \subset N$ be an arbitrary nonempty finite subset $I = \{1, 2, \ldots, n\}$, where N denotes the set of natural numbers. Take $z \in \mathrm{conv}\{y_i \,|\, i \in I\}$, we have

$$z = \sum_{i=1}^{n} \lambda_i y_i \ \text{ with } \ \lambda_i \geq 0,\ i \in I,\ \sum_{i=1}^{n} \lambda_i = 1.$$

We show that $z \in \bigcup_{i=1}^{n} S(y_i)$. We assume on the contrary that $z \notin \bigcup_{i=1}^{n} S(y_i)$. This implies

$$(G(y_i,z) - H(z,y_i)) \cap \mathrm{int}\, C \neq \emptyset \text{ for all } i = 1, 2, \ldots, n.$$

Hence

$$\sum_{i=1}^{n} \lambda_i (G(y_i,z) - H(z,y_i)) \cap \mathrm{int}\, C \neq \emptyset. \tag{3}$$

By Assumptions A2 and A4, we have

$$\sum_{i=1}^{n} \lambda_i G(y_i,z) \subseteq \sum_{i,j=1}^{n} \lambda_i \lambda_j G(y_i,y_j) - C = \sum_{\substack{i,j=1 \\ i<j}}^{n} \lambda_i \lambda_j (G(y_i,y_j) + G(y_j,y_i)) - C.$$

Hence

$$\sum_{i=1}^{n} \lambda_i G(y_i, z) \subset \sum_{i,j=1}^{n} \lambda_i \lambda_j (G(y_i, y_j) + G(y_j, y_i)) \subset -C. \tag{4}$$

Further, it follows from Assumptions A5 and A7 that

$$0 = H(z, z) \in \sum_{i=1}^{n} \lambda_i H(z, y_i) - C$$

and then

$$-\sum_{i=1}^{n} \lambda_i H(z, y_i) \in -C. \tag{5}$$

A combination of (4) and (5) yields

$$\sum_{i=1}^{n} \lambda_i (G(y_i, z) - H(z, y_i)) \subset -C.$$

Together with (3), this implies

$$\sum_{i=1}^{n} \lambda_i (G(y_i, z) - H(z, y_i)) \cap \operatorname{int} C \subset -C \cap \operatorname{int} C = \emptyset,$$

and we have a contradiction. So $z \in \bigcup_{i=1}^{n} S(y_i)$ and then

$$\operatorname{conv}\{y_i \mid i = 1, \dots, n\} \subseteq \bigcup_{i=1}^{n} S(y_i).$$

Applying the standard version of KKM's lemma (see, for example [1, Theorem 24, Ch. 6]), we deduce $\bigcap_{i=1}^{n} S(y_i) \neq \emptyset$. In other words, any finite subset $S(y_i)$, $y_i \in K$, has a nonempty intersection. These subsets are nonempty closed subsets of the compact set K, so the entire family has nonempty intersection. This means that

$$\bigcap_{y \in K} S(y) \neq \emptyset.$$

Taking $\bar{x} \in \bigcap_{y \in K} S(y)$, we can easily verify that

$$\big(G(y, \bar{x}) - H(\bar{x}, y)\big) \cap \operatorname{int} C = \emptyset$$

holds for all $y \in K$.
This completes the proof of the lemma. ∎

Lemma 3.3. *Let D, K, G and H satisfy the assumptions of Theorem* 3.1. *Then* (i) *implies* (ii), *where*

(i) $\bar{x} \in K$, $(G(y, \bar{x}) - H(\bar{x}, y)) \cap \operatorname{int} C = \emptyset$ *holds for all* $y \in K$.
(ii) $\bar{x} \in K$, $G(\bar{x}, y) + H(\bar{x}, y) \not\subseteq -\operatorname{int} C$ *holds for all* $y \in K$.

Proof. Let $\bar{x} \in K$ be such that $(G(y, \bar{x}) - H(\bar{x}, y)) \cap \operatorname{int} C = \emptyset$ for all $y \in K$. For fixed $y \in K$, we set $x_t = ty + (1 - t)\bar{x}$, $t \in [0, 1]$. It is clear that $x_t \in K$ for all $t \in [0, 1]$ and therefore $(G(x_t, \bar{x}) - H(\bar{x}, x_t)) \cap \operatorname{int} C = \emptyset$. Since

$$0 \in G(x_t, x_t) \subset (1 - t)G(x_t, \bar{x}) + tG(x_t, y) - C,$$

it follows that

$$\begin{aligned}0 &\in (1-t)(G(x_t,\bar{x}) - H(\bar{x},x_t)) + (1-t)H(\bar{x},x_t) + tG(x_t,y) - C\\ &\subset (1-t)(G(x_t,\bar{x}) - H(\bar{x},x_t)) + (1-t)tH(\bar{x},y) + tG(x_t,y) - C.\end{aligned}$$

If

$$(1-t)tH(\bar{x},y) + tG(x_t,y) \subseteq -\operatorname{int} C, \text{ for some } t \in [0,1]$$

then

$$(G(x_t,\bar{x}) - H(\bar{x},x_t)) \cap \operatorname{int} C \neq \emptyset, \text{ for some } t \in [0,1],$$

and we have a contradiction. It implies

$$(1-t)tH(\bar{x},y) + tG(x_t,y) \not\subseteq -\operatorname{int} C.$$

Hence

$$(1-t)H(\bar{x},y) + G(x_t,y) \not\subseteq -\operatorname{int} C, \text{ for } t \neq 0. \tag{6}$$

We claim that (6) is also true for $t = 0$. Indeed, we define the mapping $F : [0,1] \to 2^Y$ by setting

$$F(t) = (1-t)H(\bar{x},y) + G(x_t,y), \ t \in [0,1].$$

By Assumption A3, F is upper $(-C)$-continuous at $t = 0$. Suppose that $F(0) \subset -\operatorname{int} C$. Since $F(0) = H(\bar{x},y) + G(\bar{x},y)$ is a compact set, there is a neighborhood V of the origin in Y such that $F(0) + V \subset -\operatorname{int} C$. The upper $(-C)$-continuity of F at $t = 0$ implies that there exists $\delta > 0$ such that

$$F(t) \subset F(0) + V - C, \text{ for all } t \in (-\delta,\delta) \cap [0,1].$$

It implies

$$F(t) \subset -\operatorname{int} C - C = -\operatorname{int} C, \text{ for all } t \in [0,\delta),$$

then

$$(1-t)H(\bar{x},y) + G(x_t,y) \subset -\operatorname{int} C, \text{ for all } t \in [0,\delta),$$

and we have a contradiction. Thus we obtain $F(0) \not\subseteq -\operatorname{int} C$. Therefore

$$H(\bar{x},y) + G(\bar{x},y) \not\subseteq -\operatorname{int} C.$$

Since $y \in K$ is arbitrary, then we conclude that (ii) holds.
This completes the proof of the lemma. ■

Lemma 3.4. *Let D and K satisfy the assumptions of Theorem* 3.1. *Let $\Phi : D \to 2^Y$ be a multivalued C-convex mapping. Let $x_0 \in \operatorname{core}_D K$ be such that $\Phi(x_0) \subseteq -C$, $\Phi(y) \not\subseteq -\operatorname{int} C$ for all $y \in K$. Then $\Phi(y) \not\subseteq -\operatorname{int} C$ for all $y \in D$.*

Proof. Assume on contrary that there exists $y \in D \backslash K$ such that $\Phi(y) \subseteq -\operatorname{int} C$. Let $z \in (x_0, y)$, $z = \alpha x_0 + (1-\alpha)y$ for some $\alpha \in [0,1)$. We have

$$\begin{aligned}\Phi(z) &= \Phi(\alpha x_0 + (1-\alpha)y) \subseteq \alpha\Phi(x_0) + (1-\alpha)\Phi(y) - C\\ &\subseteq -C + (-\operatorname{int} C) - C = -\operatorname{int} C.\end{aligned}$$

Therefore

$$\Phi(z) \subseteq -\operatorname{int} C, \text{ for all } z \in (x_0, y].$$

Since $x_0 \in \mathrm{core}_D K$ there exists $z_0 \in (x_0, y] \cap K$. Hence $\Phi(z_0) \subseteq -\mathrm{int}\, C$. This contradicts the fact that $\Phi(y) \not\subseteq -\mathrm{int}\, C$ for all $y \in K$, and the lemma is proved. ∎

Proof of Theorem 3.1. By Lemma 3.2 there exists $\bar{x} \in K$ with

$$(G(\bar{x}, y) - H(\bar{x}, y)) \cap \mathrm{int}\, C = \emptyset, \text{ for all } y \in K.$$

Applying Lemma 3.3, we get

$$G(\bar{x}, y) + H(\bar{x}, y) \not\subseteq -\mathrm{int}\, C, \text{ for all } y \in K. \tag{7}$$

Further, we define the multivalued mapping $\Phi : D \to 2^Y$ by

$$\Phi(y) = G(\bar{x}, y) + H(\bar{x}, y),\ y \in D.$$

Assumptions A4 and A7 show that Φ is C-convex and it follows from (7) that

$$\Phi(y) \not\subseteq -\mathrm{int}\, C, \text{ for all } y \in K.$$

If $\bar{x} \in \mathrm{core}_D K$, we set $x_0 = \bar{x}$, otherwise, we set $x_0 = a$, where a is from Assumption A8. Then, we always have $\Phi(x_0) \subseteq -C$. Using Lemma 3.4, we conclude that

$$\Phi(y) \not\subseteq -\mathrm{int}\, C, \text{ for all } y \in D.$$

It follows that

$$G(\bar{x}, y) + H(\bar{x}, y) \not\subseteq -\mathrm{int}\, C, \text{ for all } y \in D.$$

This proves the theorem. ∎

In the sequel, we weaken the coercivity requirement A8 for the case of reflexive Banach space. We assume that X is a reflexive Banach space with the norm denoted by $\|\cdot\|$.

Theorem 3.5. *Let X be a reflexive Banach space, D, C, Y be as above. Let G and H satisfy Assumptions* A1 - A7 *in Theorem* 3.1 *with the lower C-continuity and $(-C)$-continuity of $G(x,\cdot)$ and $H(\cdot, y)$ in Assumptions* A4 *and* A6 *replaced by the weak lower C-continuity and the weak $(-C)$-continuity, respectively.*

In addition, assume that there is a point $a \in D$ such that, for every sequence $\{x_n\}$ with $\lim_{n\to+\infty} \|x_n\| = +\infty$, one of the following conditions holds:

(H1) *There is $n_0 > 0$ such that $G(x_{n_0}, a) + H(x_{n_0}, a) \subseteq -C$.*

(H2) *There are $n_0 > 0$ and $y \in D$ with $\|y - a\| < \|x_{n_0} - a\|$ such that $G(x_{n_0}, y) + H(x_{n_0}, y) \subseteq -C$.*

(H3) *There are n_0 and $y \in D$ such that $G(y, x_n) - H(x_n, y) \subseteq C$, for all $n \geq n_0$.*

Then there is $\bar{x} \in D$ such that $G(\bar{x}, y) + H(\bar{x}, y) \not\subseteq -\mathrm{int}\, C$.

Proof. Let $D_n = \{x \in D \mid \|x - a\| \leq n\}$, for $n = 1, 2, \ldots$ Since X is a reflexive Banach space, D_n, $n = 1, 2, \ldots$, are convex weakly compact subsets in X. Applying Theorem 3.1 with $D = D_n$ and using the weak topology in X, we conclude that there exist $x_n \in D_n$, $n = 1, 2, \ldots$, such that

$$G(x_n, y) + H(x_n, y) \nsubseteq -\operatorname{int} C \text{ for all } y \in D_n.$$

If $\|x_n - a\| < n$ for some n, then $x_n \in \operatorname{core} D_n$. We define the mapping $F : D \to 2^Y$ by setting

$$F(x) = G(x_n, x) + H(x_n, x), \; x \in D. \tag{8}$$

It is clear that F is a C-convex multivalued mapping and

$$F(x_n) = G(x_n, x_n) + H(x_n, x_n) \subset -C$$
$$F(x) \nsubseteq -\operatorname{int} C, \text{ for all } x \in D_n.$$

Applying Lemma 3.4, we deduce

$$F(x) \nsubseteq -\operatorname{int} C, \text{ for all } x \in D.$$

This means that x_n is a solution of the problem (WEP,C). Therefore, it remains to investigate the case $\|x_n - a\| = n$ for all $n \geq 1$.

First, we assume that Condition (H1) holds. We show that x_{n_0} is a solution to the problem (WEP,C). Indeed, for any $x \in D \backslash D_n$ there is a positive number $t \in [0, 1]$ such that $ta + (1-t)x \in D$. Hence

$$G(x_{n_0}, ta + (1-t)x) + H(x_{n_0}, ta + (1-t)x)$$
$$\subseteq t(G(x_{n_0}, a) + H(x_{n_0}, a)) + (1-t)(G(x_{n_0}, x) + H(x_{n_0}, x)) - C.$$

Since $G(x_{n_0}, a) + H(x_{n_0}, a) \subset -C$, it follows that $G(x_{n_0}, ta + (1-t)x) + H(x_{n_0}, ta + (1-t)x) \subset (1-t)(G(x_{n_0}, x) + H(x_{n_0}, x)) - C$ if $G(x_{n_0}, x) + H(x_{n_0}, x) \subseteq -\operatorname{int} C$. Then

$$G(x_{n_0}, ta + (1-t)x) + H(x_{n_0}, ta + (1-t)x) \subseteq -\operatorname{int} C,$$

a contradiction with the fact that x_{n_0} is a solution of the problem (WEP,C) on D_n. Consequently, $G(x_{n_0}, x) + H(x_{n_0}, x) \nsubseteq -\operatorname{int} C$ for all $x \in D$, hence x_{n_0} is a solution to the problem (WEP,C) on D.

Under Condition (H2), we observe that

$$G(x_{n_0}, y) + H(x_{n_0}, y) \subset -C$$

and $\|y - a\| < \|x_{n_0} - a\| = n_0$. We define the function F as in (8). Then $y \in \operatorname{core}_D D_{n_0}$ and $F(y) \subset -C$, $F(x) \nsubseteq -\operatorname{int} C$ for all $x \in D_{n_0}$. Applying Lemma 3.4 we have

$$F(x) \nsubseteq -\operatorname{int} C, \text{ for all } x \in D.$$

It follows that

$$G(x_{n_0}, x) + H(x_{n_0}, x) \nsubseteq -\operatorname{int} C, \text{ for all } x \in D$$

or, x_{n_0} is a solution to the problem (WEP,C).

Finally, if Condition (H3) is satisfied, we show that Condition (H2) also holds. Indeed, since

$$G(x_n, y) + G(y, x_n) \subset -C,$$

we conclude that

$$G(x_n, y) + H(x_n, y) \subset H(x_n, y) - G(y, x_n) - C \subset -C.$$

For n sufficiently large, we have $\|y-a\| < \|x_n - a\|$. Thus Condition (H2) is also satisfied. This completes the proof of the theorem. ■

The above conditions (H1) - (H3) are similar to Conditions (H1) - (H3) in [9, Corollary 4.5].

Corollary 3.6. *Let X be a reflexive Banach space, $D \subset X$ be a closed convex set. Let Y be a Banach space and $C \subset X$ be a closed convex pointed cone with C' a polyhedral cone. Let G and H satisfy Assumptions* A1 $\rightarrow$ A7 *in Theorem* 3.1 *with the $(-C)$-continuity of $H(\cdot, y)$ in Assumption A6 replaced by the weak $(-C)$-continuity of $H(\cdot, y)$. In addition, assume that $G(x,y)+C$ is convex for all $(x,y) \in D\times D$ and there is a point $a \in D$ such that for every sequence $\{x_n\}$ with $\lim_{n\to\infty}\|x_n\| = +\infty$, one of Conditions* (H1) - (H3) *holds. Then there is $\bar{x} \in D$ such that $G(\bar{x},y)+H(\bar{x},y) \not\subseteq -\operatorname{int} C$.*

Proof. Theorem 2.4 ensures that, for any fixed $x \in D$, $G(x,\cdot)$ is C-convex and weakly lower C-continuous. Therefore, the proof of this corollary follows immediately from Theorem 3.5.

Next, as in [12], the above results can be applied to different problems. Here we only show some applications to vector optimization problems.

Corollary 3.7. *Let D be a convex closed subset of a reflexive Banach space X with a closed convex pointed cone C. Each of the following conditions is sufficient for* $\mathrm{WMin}(D\,|\,C) \neq \emptyset$*: There is a point $a \in D$ such that for every sequence $\{x_n\} \subset D$ with $\lim_{n\to+\infty}\|x_n\| = +\infty$,*

(G1) *there is $n_0 > 0$ such that $x_{n_0} \in a + C$,*

(G2) *there are $n_0 > 0$ and $y \in D$ such that $\|y-a\| < \|x_{n_0} - a\|$ and $x_{n_0} \in y + C$,*

(G3) *there are $n_0 > 0$ and $y \in D$ with $x_n \in y + C$, for all $n \geq n_0$.*

Proof. The corollary follows immediately from Corollary 3.6 with $X = Y$, $G(x,y) = y - x$, $H \equiv 0$.

Corollary 3.8. *Let X be a topological locally convex Hausdorff space, $D \subset X$ a nonempty closed convex set. Let Y be a Banach space and $C \subset Y$ be a closed convex pointed cone with C' polyhedral cone. Let $f : D \longrightarrow Y$ be a singlevalued C-convex, C-continuous mapping. In addition, assume that there exists a nonempty convex weakly compact subset $K \subset D$ such that for any $x \in D \setminus \operatorname{core}_D K$ one can find a point $a \in \operatorname{core}_D K$ with $f(x) \in f(a) + C$. Then there exists a point $\bar{x} \in D$ such that $f(\bar{x}) \in \mathrm{WMin}(f(D)|C)$.*

Proof. We define the mapping $G : D\times D \rightarrow Y$ by $G(x,y) = f(y) - f(x)$, $x, y \in D$. Using Theorem 2.4, we conclude that for any fixed $x \in D$, $G(x,\cdot) : D \rightarrow Y$ is C-convex and weakly C-continuous. Therefore, considering the weak topology in X, we apply Theorem 3.1 with G and $H \equiv 0$.
This completes the proof of the corollary. ■

Corollary 3.9. *Let X, D, Y, C be as in Corollary* 3.6. *Let $f : D \rightarrow Y$ be a* *singlevalued C-convex and C-continuous mapping. Then each of the following conditions is sufficient for* $\text{WMin}\,(f(D)\,|\,C) \neq \emptyset$*: There is a point* $a \in D$ *such that for every sequence* $\{x_n\} \subset D$ *with* $\lim_{n>\infty} \|x_n\| = +\infty$,

(I1) *there is* $n_0 > 0$ *such that* $f(x_{n_0}) \in f(a) + C$,

(I2) *there are* $n_0 > 0$ *and* $y \in D$ *such that* $\|y - a\| < \|x_{n_0} - a\|$ *and* $f(x_{n_0}) \in f(y) + C$,

(I3) *there are* $n_0 > 0$ *and* $y \in D$ *such that* $f(x_n) \in f(y) + C$, *for all* $n \geq n_0$.

Proof. We define the mapping $G:\ D \times D \to Y$ by $G(x, y) = f(y) - f(x),\ \ x, y \in D$, and apply Corollary 3.6 with G as above and $H = 0$ to conclude that there exists a point $\bar{x} \in D$ such that $G(\bar{x}, y) \notin -\text{int}\, C$ for all $y \in D$. This implies

$$f(y) - f(\bar{x}) \notin -\text{int}\, C, \text{ for all } y \in D,$$

or

$$f(\bar{x}) \in \text{WMin}\,(f(D)\,|\,C).$$

This completes the proof of the corollary. ∎

To conclude the paper we make the following remark.

Remark. Instead of the lower C-continuity of $G(x, \cdot)$ for any fixed $x \in G$ and the $(-C)$-continuity of $H(\cdot, y)$ for any fixed $y \in D$ in A4 and A6 we assume that for any given $y \in D$, the set

$$S(y) = \{x \in D\,|\,(G(y, x) - H(x, y)) \cap (C \setminus \{0\}) = \emptyset\}$$

is closed. Then, the conclusions of Theorems 3.1, 3.5, Corollary 3.6 remain true for (EP,C). The proofs are exactly as the above one with $\text{int}\, C$ replaced by $C \setminus \{0\}$ everywhere. Therefore the conclusions of Corollaries 3.7 - 3.9 also remain true for $\text{Min}\,(D\,|\,C)$ and $\text{Min}\,(f(D)\,|\,C)$.

References

1. J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, 1990.
2. J. P. Aubin and I. Ekland, *Applied Nonlinear Analysis*, Wiley Interscience, 1984.
3. S. Banach, *Théorie des Opérations Linéaire*, Monographie Matematyczne, PWN, Warszawa, 1932.
4. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *The Math. Student* **64** (1993) 1-23.
5. E. Blum and W. Oettli, Variational principles for equilibrium problems, *Parametric Optimization and Related Topics III* (to appear).
6. K. Fan, A Minimax inequality and application, in: *Inequalities III*, O. Shisha (Ed.), Academic Press, New York, 1972, pp. 103.
7. V. Jeyaluma, W. Oettli, and M. Natividad, A solvability theorem for a class of quasiconvex mappings with applications to optimization, *J. Math. Anal. Appl.* **179** (1993) 537-546.
8. D. T. Luc, Theory of vector optimization, *Lecture Notes in Economics and Mathematical Systems*, 319, Springer Verlag, Berlin-Heidelberg, 1989.

9. D. T. Luc, Stampacchia's theorem for densely pseudomonotone variational inequality, *Preprint 96/24, Inst. Math.*, Hanoi, Vietnam.
10. W. Oettli, A remark on vector-valued equilibria and generalized monotonicity, *Acta Math. Vietnam.* **22** (1997) 213-222.
11. H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, (2nd edition), 1971.
12. N. X. Tan and P. N. Tinh, On the existence of equilibrium points of vector functions, *Numer. Funct. Anal. Optimization* **19** (1998) 141-156.