Vietnam Journal of Mathematics **28**:4 (2000) 341–380

Vietnam Journal of MATHEMATICS

Bounds for Cohomological Hilbert-Functions of Projective Schemes Over Artinian Rings*

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Received April 14, 2000

Abstract. Let X be a projective scheme over an artinian commutative ring R_0. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$- modules. We present a sample of bounding results for the so called cohomological Hilbert functions

$$h^i_{X,\mathcal{F}}:\mathbb{Z}\to\mathbb{N}_0,\ n\mapsto h^i_{X,\mathcal{F}}(n)=\text{length}_{R_0}H^i(X,\mathcal{F}(n))$$

of $\mathcal{F}$. Our main interest is to bound these functions in terms of the so called cohomology diagonal $\left(h^j_{X,\mathcal{F}}(-j)\right)_{j=0}^{\dim(\mathcal{F})}$ of $\mathcal{F}$. Our results present themselves as quantitative versions of the vanishing theorems of Castelnuovo-Serre and of Severi-Enriques-Zariski-Serre. In particular we get polynomial bounds for the (Castelnuovo) regularity at arbitrary levels and for the (Severi) coregularity at any level below the global subdepth $\delta(\mathcal{F}):=\min\{\text{depth}(\mathcal{F}_x)\mid x\in X,\ x\text{ closed}\}$ of $\mathcal{F}$. We also show that the cohomology diagonal of $\mathcal{F}$ provides minimal bounding systems for the mentioned regularities and coregularities.

As a fundamental tool we use an extended version of the method of linear systems of general hyperplane sections.

1. Introduction

Let C be the class of all pairs $(X,\mathcal{F})$ in which

*Research supported by Swiss National Science Foundation, Project No. 20-52762.97.

$X = \mathrm{Proj}(R)$ is the projective scheme induced by a positively graded homogeneous noetherian ring $R = \oplus_{n \geq 0} R_n$ with artinian base ring R_0. (1.1)

$\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules. (1.2)

For any pair $(X, \mathcal{G}) \in \mathcal{C}$ and any $i \in \mathbb{N}_0$, the i-th *cohomology module* $H^i(X, \mathcal{G})$ *of* X *with coefficients in* $\mathcal{G}$ is a finitely generated R_0-module (cf. [29, III, Theorem 5.2], [43, §66, Theorem 1]) and hence of finite length. So, for each $i \in \mathbb{N}_0$ and each pair $(X, \mathcal{F}) \in \mathcal{C}$, we may introduce the i-th *cohomological Hilbert function of (X with respect to)* $\mathcal{F}$

$$h^i_{X,\mathcal{F}} = h^i_{\mathcal{F}} : \mathbb{Z} \to \mathbb{N}_0 \quad (i \in \mathbb{N}_0) \tag{1.3}$$

defined by

$$h^i_{X,\mathcal{F}}(n) := h^i_{\mathcal{F}}(n) := l_{R_0}\big(H^i(X, \mathcal{F}(n))\big) \quad (n \in \mathbb{Z}), \tag{1.4}$$

where $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ is the n-th *twist of* $\mathcal{F}$, and $l_{R_0}(N)$ denotes the length of the R_0-module N.

There are some general constraints on the cohomological Hilbert functions $h^i_{\mathcal{F}} : \mathbb{Z} \to \mathbb{N}_0$ given by classical vanishing theorems. So, the *vanishing Theorem of Grothendieck* [29, III, Theorem 2.7] says that

$$h^i_{\mathcal{F}} \equiv 0 \quad \text{for all} \quad i > \dim(\mathcal{F}), \tag{1.5}$$

where $\dim(\mathcal{F}) := \dim(\mathrm{supp}(\mathcal{F}))$ denotes the dimension of the support of $\mathcal{F}$. Another constraint on the functions $h^i_{\mathcal{F}}$ is given by the *Vanishing Theorem of Castelnuovo-Serre* (cf. [44, §66, Theorem 2(b)]).

$$h^i_{\mathcal{F}}(n) = 0 \quad \text{for all} \quad i > 0 \quad \text{and all} \quad n \gg 0. \tag{1.6}$$

There is yet another important constraint on the cohomological Hilbert functions $h^i_{\mathcal{F}}$. To formulate it, let us introduce the *subdepth of (X with respect to)* $\mathcal{F}$:

$$\delta(\mathcal{F}) := \min\{\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \mid x \in X, \text{ closed}\}. \tag{1.7}$$

Now, the *Vanishing Theorem of Severi-Enriques-Zariski-Serre* (cf. [43, §76, Theorem 4], [24]) claims:

$$h^i_{\mathcal{F}}(n) = 0 \quad \text{for all} \quad i < \delta(\mathcal{F}) \quad \text{and all} \quad n \ll 0, \tag{1.8}$$

$$h^{\delta(\mathcal{F})}_{\mathcal{F}}(n) \neq 0 \quad \text{for all} \quad n \ll 0. \tag{1.9}$$

Moreover, there is an observation which essentially goes back to *Mumford* [39] and which gives a further constraint on our functions:

Let $k \in \mathbb{N}_0$ and let $r \in \mathbb{Z}$ be such that $h^i_{\mathcal{F}}(r-i) = 0$ for all $i > k$.
Then $h^i_{\mathcal{F}}(n-i) = 0$ for all $i > k$ and all $n \geq r$. (1.10)

This observation gives rise to the concept of *regularity of* $\mathcal{F}$ *above level* k.

$$\mathrm{reg}_k(\mathcal{F}) := \inf\{r \in \mathbb{Z} \mid h^i_{\mathcal{F}}(n-i) = 0, \quad \forall n \geq r, \quad \forall i > k\}. \tag{1.11}$$

which in the case $k = 0$ coincides with the so called *Castelnuovo-Mumford* *regularity of $\mathcal{F}$* (cf. [39]). Observe that by (1.5) and (1.6) we have $\mathrm{reg}_k(\mathcal{F}) < \infty$. Similar to (1.10) we also have:

$$\text{Let } k \in \mathbb{N}_0 \text{ and let } c \in \mathbb{Z} \text{ be such that } h^i_{\mathcal{F}}(c-i) = 0 \text{ for all } i \leq k. \text{ Then } h^i(n-i) = 0 \text{ for all } i \leq k \text{ and all } n \leq c. \tag{1.12}$$

The observation made in (1.12) gives rise to the concept of *coregularity of $\mathcal{F}$ at and below level k*:

$$\mathrm{coreg}^k\ (\mathcal{F}) := \sup\{c \in \mathbb{Z} \mid h^i_{\mathcal{F}}(n-i) = 0, \quad \forall n \leq c, \quad \forall i \leq k\}. \tag{1.13}$$

By (1.8) and (1.9) we have $\mathrm{coreg}^k(\mathcal{F}) > -\infty$ if and only if $k < \delta(\mathcal{F})$.
The constraints on the functions $h^i_{\mathcal{F}}$ which are given in (1.5), (1.6), (1.8), (1.9), (1.10) and (1.12) merely concern the vanishing or non-vanishing of these functions in certain ranges. But there are also constraints on the rate of growth of these functions. Let us only mention two such constraints.

$$\text{If } \dim(\mathcal{F}) = d, \text{ then } h^d_{\mathcal{F}}(n+1) \leq \max\{0,\ h^d_{\mathcal{F}}(n) - d\} \text{ for all } n \in \mathbb{Z}; \tag{1.14}$$

$$h^0_{\mathcal{F}}(n-1) \leq \max\{0, h^0_{\mathcal{F}}(n) - \delta(\mathcal{F})\} \text{ for all } n \in \mathbb{Z}. \tag{1.15}$$

The principal aim of this paper is to establish bounds on the cohomological Hilbert functions $h^i_{\mathcal{F}}$ for $i = 0, \cdots$ which depend only on the *"diagonal values"* $h^j_{\mathcal{F}}(-j)$ $(j = 0, \cdots)$ of these functions. Obviously these bounds should be such that they are in accordance with the above constraints. It turns out that such bounds naturally split up into two types:

I: Bounds of *Castelnuovo type* which, for each $i > 0$, bound the values $h^i_{\mathcal{F}}(n)$ in the range $n \geq -i$ in terms of the *diagonal values* $\left(h^j_{\mathcal{F}}(-j)\right)_{j \geq i}$ *at and above level i*.

II: Bounds of *Severi type* which, for each $i < \delta(\mathcal{F})$, bound the values $h^i_{\mathcal{F}}(n)$ in the range $n \leq -i$ in terms of the diagonal values $\left(h^j_{\mathcal{F}}(-j)\right)_{j \leq i}$ *at and below level i*.

As for the bounds of Castelnuovo type we shall define *bounding functions*.

$$\begin{aligned} \underline{B}^{(i)}_{(d+1)} &: \mathbb{N}_0^{d+1-i} \times \mathbb{Z}_{\geq -i} \longrightarrow \mathbb{N}_0 \\ & \qquad\qquad\qquad\qquad\qquad (0 < i \leq d) \\ \underline{C}^{(i)}_{(d+1)} &: \mathbb{N}_0^{d+1-i} \longrightarrow \mathbb{Z}_{\geq -i} \end{aligned}$$

such that (cf. Corollary 4.7):

For each pair $(X, \mathcal{F}) \in \mathcal{C}$ with $\dim(\mathcal{F}) \leq d$:

$$\begin{aligned} &\text{(a) } h^i_{\mathcal{F}}(n) \leq \underline{B}^{(i)}_{(d+1)}\left(h^i_{\mathcal{F}}(-i), \cdots, h^d_{\mathcal{F}}(-d); n\right), \quad \forall n \geq -i; \\ &\text{(b) } h^i_{\mathcal{F}}(n) = 0, \ \forall n \geq \underline{C}^{(i)}_{(d+1)}\left(h^i_{\mathcal{F}}(-i), \cdots, h^d_{\mathcal{F}}(-d)\right); \\ &\text{(c) } \mathrm{reg}_{i-1}(\mathcal{F}) \leq \underline{C}^{(i)}_{(d+1)}\left(h^i_{\mathcal{F}}(-i), \ldots, h^d_{\mathcal{F}}(-d)\right) + i. \end{aligned} \tag{1.16}$$

Clearly, in view of statement (1.16)(b) the constraint (1.6) is reflected by our bounds. Moreover, the above bounding functions are defined such that $\underline{B}^{(i)}_{(d+1)}(0,\dots,0;n) = 0$ for all $n \geq -i$ and $\underline{C}^{(i)}_{(d+1)}(0,\dots,0) = -i$, so that the constraint (1.10) is respected. Finally, according to our definitions we shall have $\underline{B}^{(d)}_{(d+1)}(h;n) := \max\{h - d(n+d), 0\}$, and this is in accordance with the constraint mentioned under (1.14).

Actually,we shall deduce the above bounds from a class of sharper bounds, which in addition depend on the global subdepth $\delta(\mathcal{F})$ of the coherent sheaf $\mathcal{F}$ (cf. Corollary 4.6).

Concerning the bounds of Severi type, we shall define bounding functions

$$B^{(i)}_{(l)} : \mathbb{N}_0^{i+1} \times \mathbb{Z}_{\leq -i} \longrightarrow \mathbb{N}_0 \qquad (0 < i \leq l)$$

$$C^{(i)}_{(l)} : \mathbb{N}_0^{i+1} \longrightarrow \mathbb{Z}_{\geq -i}$$

such that (cf. Corollary 5.3).

For each pair $(X, \mathcal{F}) \in \mathcal{C}$ with $l \leq \delta(\mathcal{F})$:

$$\begin{aligned} &\text{(a) } h^i_{\mathcal{F}}(n) \leq B^{(i)}_{(l)}(h^0_{\mathcal{F}}(0), \dots, h^i_{\mathcal{F}}(-i); n), \ \forall n \leq -i; \\ &\text{(b) } h^i_{\mathcal{F}}(n) = 0, \ \forall n \leq C^{(i)}_{(l)}(h^0_{\mathcal{F}}(0), \dots, h^i_{\mathcal{F}}(-i)); \\ &\text{(c) } \operatorname{coreg}^i(\mathcal{F}) \geq C^{(i)}_{(l)}(h^0_{\mathcal{F}}(0), \dots, h^i_{\mathcal{F}}(-i)) + i. \end{aligned} \tag{1.17}$$

Here, statement (1.17)(b) reflects the constraint given in (1.8), whereas the bounding functions are defined such that we have accordance with the constraints (1.12) and (1.15).

So, conceptually the bounds mentioned under (1.16) present a quantitative version of the vanishing theorem of Castelnuovo Serre (as formulated in (1.6)) which is in accordance with Mumford's observation (1.10) and the constraint (1.14). Smilarly, the bounds mentioned under (1.17) present a quantitative version of the vanishing theorem of Severi-Enriques-Zariski-Serre (cf. (1.8)) which is in accordance with the constraints (1.12) and (1.15).

As our bounds hold for arbitrary pairs $(X, \mathcal{F}) \in \mathcal{C}$, we call them *a priori bounds*. As the bounds depend only on the "diagonal values" $h^j_{\mathcal{F}}(-j)$, we sometimes refer to them as *diagonal bounds*.

Our bounding functions $\underline{B}^{(i)}_{(d+1)}$, $\underline{C}^{(i)}_{(d+1)}$, $B^{(i)}_{(l)}$, $C^{(i)}_{(l)}$, are defined recursively. Sometimes one might prefer to dispose on explicite bounds - even if they are weaker. As an example of this latter type of bounds we shall establish the following estimates (cf. Remark 6, Remark 10):

For each pair $(X, \mathcal{F}) \in \mathcal{C}$ with $\dim(\mathcal{F}) \leq d$

$$\operatorname{reg}_{i-1}(\mathcal{F}) \leq \Big(2 \sum_{j=i}^{d} \binom{d-i}{j-i} h^j_{\mathcal{F}}(-j)\Big)^{2^{d-i}}. \tag{1.18}$$

For each pair $(X, \mathcal{F}) \in \mathcal{C}$ with $i < \delta(\mathcal{F})$

$$\operatorname{coreg}^i(\mathcal{F}) \geq -(2\sum_{j=0}^{i}\binom{i}{j}h^j_{\mathcal{F}}(-j))^{2^i}, \tag{1.19}$$

We also shall see, that the diagonal values $h^j_{\mathcal{F}}(-j)$, which occur in our bounding results (1.16) and (1.17), form a *minimal bounding system*: If we allow that one of the diagonal values $h^j_{\mathcal{F}}(-j)$ becomes arbitrary large, whereas the other ones are bounded, $\operatorname{reg}_{i-1}(\mathcal{F})$ and $\operatorname{coreg}^i(\mathcal{F})$ need not be bounded (cf. Remark 5, Construction and Remark 5.4).

The aim of this paper is two-fold:

- To extend the existing results on a priori bounds for cohomological Hilbert functions which were established over algebraically closed ground fields (cf. [5, 7, 9]) to the case of arbitrary artinian gound rings.
- To give a self-contained introduction to the subject of a-priori bounds for cohomological Hilbert functions addressed to a reader familiar with basic knowledge of local cohomology as presented in [14].

Clearly, in view of the second aspect, our paper partly will have expository character. Moreover, we attack our task from the algebraic point of view: We namely first establish bounding results for the cohomological Hilbert functions of finitely generated graded modules over noetherian positively graded homogeneous rings $R = R_0 \oplus R_1 \oplus \cdots$ with artinian ground ring R_0 (cf. Theorem 4.2, Corollaries 4.3 - 4.5, Theorem 5.2). Then we use the *Serre-Grothendieck correspondence* to translate these results from the language of local cohomology to the language of sheaf cohomology.

In Sec. 2 of the present paper we list a few facts on cohomological Hilbert functions of finitely generated graded modules over positively graded noetherian rings with artinian base ring. Our basic reference here is [14].

The main technical tool we shall use, relies on the *method of linear systems of hyperplane sections* which was used to deduce our earlier bounding results over algebraically closed ground fields (see [5, 7, 8, 9, 13]). To make use of this idea we need some preparatory results which allow to adapt the method of linear hyperplane sections to the case of arbitrary artinian ground rings. These results shall be presented in Sec. 3. In view of the expository aspect of our paper we allowed ourselves to give a complete prove of the *lifting result* Proposition 3.2. Some of the ideas presented in this section were already used in [4].

In Sec. 4 we deduce our bounds of Castelnuovo type, as mentioned under (1.16). These results extend and complete what already was done in [14] where the bounds of (1.16) where deduced in the special case $i = 1$ by means of the *"Lemma of Mumford and Le Potier"*. If $i > 1$, this latter lemma is not available, but a substitute for it (which gives sharper bounds and applies for all $i \geq 0$) is our adapted version of the method of linear systems of hyperplane sections.

In Sec. 5 we give the bounds of Severi type as mentioned under (1.17). Here, using multiple Segre products of projective lines and the Künneth formulas, we show that the diagonal values $h^0_{\mathcal{F}}(0)$, $h^1_{\mathcal{F}}(-1), \ldots, h^i_{\mathcal{F}}(-i)$ form a minimal bounding system for the coregularity $\operatorname{coreg}^i(\mathcal{F})$, even on rather special subclasses of $\mathcal{C}$. It should be noted, that this minimality result is closely related to the problem of an "axiomatic characterization" of *cohomological patterns*

$$\mathcal{P}(\mathcal{F}) = \{(i,n) \in \mathbb{N}_0 \times \mathbb{Z} \mid h^i_{\mathcal{F}}(n) \neq 0\}$$

of pairs $(X, \mathcal{F}) \in \mathcal{C}$. Let us also mention in this context the non-rigidity theorem of Evans-Griffiths [21, (4.13)] (in which the occuring isomorphisms need not be graded).

One appearent disadvantage of our bounds of Severi type is the fact that they apply only if $i < \delta(\mathcal{F})$. On way to get satisfactory bounding results (in the range $n \leq -i$) for $i \geq \delta(\mathcal{F})$ is to estimate the so called *cohomological deficiency functions*. For a pair $(X, \mathcal{F}) \in \mathcal{C}$ and for $i \in \mathbb{N}_0$, the i-th *cohomological deficiency function of (X with respect to)* $\mathcal{F}$ is defined by

$$\Delta^i_{X,\mathcal{F}} = \Delta^i_{\mathcal{F}} \ := \ h^i_{\mathcal{F}} - p^i_{\mathcal{F}} \ , \tag{1.20}$$

where $p^i_{\mathcal{F}}$ is the i-th *cohomological Hilbert polynomial* of $\mathcal{F}$, i.e. the unique polynomial in $\mathbb{R}[t]$ such that

$$p^i_{\mathcal{F}}(n) = h^i_{\mathcal{F}}(n), \ \ \forall n \ll 0.$$

These deficiency functions where investigated by the second author in [36]. We shall present bounds on these functions and apply them in the context of the present paper in a later investigation [12].

Obviously bounding cohomology or Castelnuovo-regularity for special pairs $(X, \mathcal{F})$ has a long tradition in algebraic geometry which goes back far beyond the time in which cohomology was available. The starting point was the study of linear systems on projective varieties. Here, for the "Castelnuovo root" of the subject the interest is focussed to pairs $(X, \mathcal{F}) \in \mathcal{C}$ for which $X = \mathbb{P}^r$ is a projective space over an (algebraically closed) field K and $\mathcal{F} = \mathcal{J} \subseteq \mathcal{O}_{\mathbb{P}^r}$ is a coherent sheaf of ideals (cf. [16, 26, 34, 41, 45]). The study of the cohomological behavior of such pairs became the driving force of many investigations and is of basic importance for the theory of Hilbert schemes (cf. [22, 25, 27, 32, 37, 38]) and for computational algebraic geometry (cf. [2, 3]). The algebraic aspect of the theory (cf. [17, 42]) initiated a considerable number of investigations on graded rings and their cohomological invariants (cf. [30, 46, 47]).

Another point of interest is the study of pairs $(X, \mathcal{F}) \in \mathcal{C}$, where $X \subseteq \mathbb{P}^r$ is a smooth or normal variety and $\mathcal{F} = \mathcal{L}$ is an invertible sheaf. This "Severi root" of the subject had its pre-cohomological origin in the study of canonical linear systems and their generalizations (cf. [20, 44, 48]) and lead to fundamental vanishing results for cohomology groups (cf. [33, 40]).

Finally, let us also mention the "vector bundle root" of the whole subject, whose interest focusses on pairs $(\mathbb{P}^r, \mathcal{E}) \in \mathcal{C}$, for which $\mathcal{E}$ is locally free over $\mathbb{P}^r$ (cf. [5, 13, 19, 28]). Here, the interplay between topological and cohomological properties of bundles is one of the main points of interest.

Clearly, along each of the above three main lines one finds bounding and vanishing results for cohomology which are formulated in terms of invariants suitable to the special class of pairs $(X, \mathcal{F}) \in \mathcal{C}$ under consideration. Obviously, it is not surprising that these specific bounding and vanishing results are sharper than our a priori bounds. On the other hand, it turns out that our basic method – the use of linear systems of hyperplane sections – can be successfully applied for certain fairly specific bounding results (cf. [1, 6, 8, 10, 11]).

2. Preliminaries on Cohomological Hilbert Functions

Let $R = \bigoplus_{n\geq 0} R_n$ be a noetherian, positively graded homogeneous ring such that R_0 is an artinian ring. Set $R_+ = \bigoplus_{n>0} R_n$, the "irrelevant ideal" of R. Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated, graded R-module. Let $i \in \mathbb{N}$ and let $H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to R_+. Moreover let $D_{R_+}(\bullet) := \operatorname{inj\,lim}_n \operatorname{Hom}_R(R^n_+, \bullet)$ be the R_+-transform functor. For $i \in \mathbb{N}_0$ let $\mathcal{R}^i D_{R_+}(\bullet)$ denote the i-th right derived functor of $D_{R_+}(\bullet)$. It is well-known that the R-modules $H^i_{R_+}(M)$ and $\mathcal{R}^i D_{R_+}(M)$ carry a natural grading for all $i \in \mathbb{N}_0$ (see [14, (12.3.3), (12.2.10), (12.4.5)]). Moreover, for all $n \in \mathbb{Z}$ the n-th homogeneous parts $H^i_{R_+}(M)_n$ and $\mathcal{R}^i D_{R_+}(M)_n$ of these graded modules are finitely generated R_0-modules and thus of finite length (see [14, (15.1.5), (17.1.4)]). Therefore, it makes sense to introduce the functions (cf. [14, (16.1.1), (17.1.4)]):

$$h^i_M : \mathbb{Z} \longrightarrow \mathbb{N}_0 \text{ and } d^i_M : \mathbb{Z} \longrightarrow \mathbb{N}_0, \ (i \in \mathbb{N}_0) \tag{2.1}$$

which are defined by

$$h^i_M(n) := l_{R_0}(H^i_{R_+}(M)_n) \tag{2.2}$$

$$d^i_M(n) := l_{R_0}(\mathcal{R}^i D_{R_+}(M)_n) \tag{2.3}$$

where $l_{R_0}(N)$ is used to denote the length of the R_0-module N. In view of the natural exact graded sequence

$$0 \longrightarrow H^0_{R_+}(M) \longrightarrow M \longrightarrow D_{R_+}(M) \longrightarrow H^1_{R_+}(M) \longrightarrow 0$$

and the natural graded isomorphisms $\mathcal{R}^i D_{R_+}(M) \cong H^{i+1}_{R_+}(M)$ for all $i \in \mathbb{N}$ (see [14, (12.4.2), (12.4.5)]), we now get for all $n \in \mathbb{Z}$

$$d^0_M(n) = l_{R_0}(M_n) - h^0_M(n) + h^1_M(n), \tag{2.4}$$

$$d^i_M(n) = h^{i+1}_M(n), \quad (\forall i > 0). \tag{2.5}$$

Observe that, for each $i \in \mathbb{N}_0$ we have (cf. [14, (15.1.5)])

$$h^i_M(n) = 0 \text{ for all } n \gg 0. \tag{2.6}$$

As a consequence of this we get by (2.4) and (2.5)

$$d^0_M(n) = l_{R_0}(M_n),\ d^i_M(n) = 0 \quad (\forall n \gg 0,\ \forall i > 0). \tag{2.7}$$

It is well-known that $h^i_M(n)$ is represented by a polynomial for $n \ll 0$.

To be precise: for each $i \in \mathbb{N}_0$ there is a unique polynomial $p^i_M \in \mathbb{Q}[x]$ of degree $< i$ such that

$$h^i_M(n) = p^i_M(n) \text{ for all } n \ll 0 \tag{2.8}$$

(see [14, 17.1.9]). We call p^i_M the i-th *cohomological Hilbert-polynomial of M.*

Two concepts which are needed for our purpose are the notions of end and beginning of a graded R-module $T = \oplus_{n\in\mathbb{Z}} T_n$:

$$\begin{aligned} &\text{(a) } \operatorname{end}(T) := \sup\{t \in \mathbb{Z} \mid T_t \neq 0\};\\ &\text{(b) } \operatorname{beg}(T) := \inf\{t \in \mathbb{Z} \mid T_t \neq 0\}. \end{aligned} \tag{2.9}$$

For $k \in \mathbb{N}_0$, we define the *regularity of M at and above level k* (see [14, (15.2.9)]) respectively the *coregularity of M at and below level k* by

$$\begin{aligned} &\text{(a) } \operatorname{reg}^k(M) := \sup_{i\geq k}\{\operatorname{end}(H^i_{R_+}(M)) + i\};\\ &\text{(b) } \operatorname{coreg}^k(M) := \inf_{i\leq k}\{\operatorname{beg}(\mathcal{R}^i D_{R_+}(M)) + i - 1\}. \end{aligned} \tag{2.10}$$

In the definitions (2.9) and (2.10) the occuring suprema and infima are supposed to be formed in the ordered set $\{-\infty\}\cup\mathbb{Z}\cup\{\infty\}$ with the additional convention that $\sup\phi = -\infty$ and $\inf\phi = \infty$.

It is important for us to notice that (see [14, (15.2.9)])

$$\operatorname{reg}^k(M) < \infty \quad \text{for all } k \in \mathbb{N}_0. \tag{2.11}$$

In view of the definition of coregularity it is natural to ask whether $-\infty < \operatorname{coreg}^k(M)$. The answer to this question is more subtle than the corresponding statement for regularities (2.11). Namely, we first have to introduce another invariant – the *minimum R_+-adjusted depth* of M – which is defined by

$$\begin{aligned} \lambda(M) &= \lambda_{R_+}(M)\\ &= \inf\Big\{\operatorname{depth}(M_\mathfrak{p}) + \operatorname{height}((\mathfrak{p}+R_+)/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\setminus \operatorname{Var}(R_+)\Big\}, \end{aligned} \tag{2.12}$$

where the infimum is formed as above under the usual conventions that depth $(0) = \infty$ and height $(R/R) = \infty$. Observe that in the definition of $\lambda(M)$ we may take the infinum over graded primes $\mathfrak{p} \notin \operatorname{Var}(R_+)$ (see [14, (13.1.17)]). Now, by Grothendieck's finiteness theorem we have (see [14, (9.5.2)]):

$$\lambda(M) = \inf\{i \in \mathbb{N}_0 \mid H^i_{R_+}(M) \text{ is not finitely generated}\}, \tag{2.13}$$

where the right-hand side infimum is understood as above. In view of (2.6) we thus get

$$\lambda(M) = \inf\{i \in \mathbb{N}_0 \mid \forall n \in \mathbb{Z} : \exists m \leq n : h^i_M(m) \neq 0\}. \tag{2.14}$$

As $l_{R_0}(M_n)$ and $h^0_M(n)$ vanish for all $n \ll 0$, (2.4) and (2.5) therefore imply

$$\lambda(M) = \inf\{i \in \mathbb{N} \mid \forall n \in \mathbb{Z} : \exists m \leq n : d^{i-1}_M(m) \neq 0\}. \tag{2.15}$$

As a consequence of this we obtain

$$-\infty < \operatorname{coreg}^k(M) \Leftrightarrow k < \lambda(M) - 1. \tag{2.16}$$

Later, various types of reduction arguments shall be performed so that we need to know how the above invariants behave if R and M are subject to certain changes. The first of these changes is taking classes modulo R_+-torsion.

So, let $\Gamma_{R_+}(M) \subseteq M$ denote the R_+-torsion submodule of M.
Then $H^0_{R_+}\big(M/\Gamma_{R_+}(M)\big) = 0$ and moreover there are natural isomorphisms of graded R-modules $H^i_{R_+}(M) \cong H^i_{R_+}\big(M/\Gamma_{R_+}(M)\big)$ for all $i \in \mathbb{N}$ and $D_{R_+}(M) \cong D_{R_+}\big(M/\Gamma_{R_+}(M)\big)$ (see [14, (2.1.7), (2.2.8)]). So, we get in view of (2.5).

$$\begin{aligned}
&\text{(a) } h^0_{M/\Gamma_{R_+}(M)}(n) = 0 \quad (\forall n \in \mathbb{Z});\\
&\text{(b) } h^i_{M/\Gamma_{R_+}(M)}(n) = h^i_M(n) \quad (\forall n \in \mathbb{Z}, \forall i \in \mathbb{N});\\
&\text{(c) } d^i_{M/\Gamma_{R_+}(M)}(n) = d^i_M(n) \quad (\forall n \in \mathbb{Z},\ \forall i \in \mathbb{N}_0).
\end{aligned} \tag{2.17}$$

As a consequence of this we have

$$\begin{aligned}
&\text{(a) } \operatorname{reg}^k\big(M/\Gamma_{R_+}(M)\big) = \operatorname{reg}^k(M) \quad (\forall k \in \mathbb{N});\\
&\text{(b) } \operatorname{coreg}^k\big(M/\Gamma_{R_+}(M)\big) = \operatorname{coreg}^k(M) \quad (\forall k \in \mathbb{N}_0);\\
&\text{(c) } \lambda\big(M/\Gamma_{R_+}(M)\big) = \lambda(M).
\end{aligned} \tag{2.18}$$

It also will be important for us to know how the above invariants change if we replace M by M/hM, where $h \in R_1$ is an M-regular element. In this situation we have an exact sequence of graded R- modules

$$0 \to M(-1) \xrightarrow{h} M \to M/hM \to 0.$$

Thus, in cohomology we obtain, for each $n \in \mathbb{Z}$, the following exact sequences of R_0-modules

$$\begin{array}{llllll}
0 & \longrightarrow & H^0_{R_+}(M)_{n-1} & \xrightarrow{h} & H^0_{R_+}(M)_n & \longrightarrow \; H^0_{R_+}(M/hM)_n\\
 & \longrightarrow & H^1_{R_+}(M)_{n-1} & \cdots & & \longrightarrow \; H^{i-1}_{R_+}(M/hM)_n\\
 & \longrightarrow & H^i_{R_+}(M)_{n-1} & \xrightarrow{h} & H^i_{R_+}(M)_n & \longrightarrow \; H^i_{R_+}(M/hM)_n\\
 & \longrightarrow & H^{i+1}_{R_+}(M)_{n-1} & \cdots & &
\end{array}$$

and

$$\begin{array}{llllll}
0 & \longrightarrow & \mathcal{R}^0D_{R_+}(M)_{n-1} & \xrightarrow{h} & \mathcal{R}^0D_{R_+}(M)_n & \longrightarrow \; \mathcal{R}^0D_{R_+}(M/hM)_n\\
 & \longrightarrow & \mathcal{R}^1D_{R_+}(M)_{n-1} & \cdots & & \longrightarrow \; \mathcal{R}^{i-1}D_{R_+}(M/hM)_n\\
 & \longrightarrow & \mathcal{R}^iD_{R_+}(M)_{n-1} & \xrightarrow{h} & \mathcal{R}^iD_{R_+}(M)_n & \longrightarrow \; \mathcal{R}^iD_{R_+}(M/hM)_n\\
 & \longrightarrow & \mathcal{R}^{i+1}D_{R_+}(M)_{n-1} & \xrightarrow{h} & \mathcal{R}^{i+1}D_{R_+}(M)_n & \cdots
\end{array} \tag{2.20}$$

For all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ we thus get the estimates

$$\begin{aligned}
&\text{(a) } h^i_M(n) \le h^i_M(n-1) + h^i_{M/hM}(n);\\
&\text{(b) } h^i_{M/hM}(n) \le h^i_M(n) + h^{i+1}_M(n-1);
\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&\text{(a)}\ d^i_M(n) \le d^i_M(n-1) + d^i_{M/hM}(n);\\ &\text{(b)}\ d^i_{M/hM}(n) \le d^i_M(n) + d^{i+1}_M(n-1);\\ &\text{(c)}\ d^{i+1}_M(n) \le d^{i+1}_M(n+1) + d^i_{M/hM}(n+1).\end{aligned} \tag{2.22}$$

For any $k \in \mathbb{N}_0$ we may conclude from (2.21)(b), respectively from (2.22)(b), that

$$\begin{aligned}&\text{(a)}\ \operatorname{reg}^k(M/hM) \le \operatorname{reg}^k(M);\\ &\text{(b)}\ \operatorname{coreg}^k(M/hM) \ge \operatorname{coreg}^{k+1}(M).\end{aligned} \tag{2.23}$$

Observing (2.16) we now may conclude from (2.23)(b)

$$\lambda(M/hM) \ge \lambda(M) - 1. \tag{2.24}$$

Another important step in the present paper is the reduction to the case where our artinian base ring R_0 is local. To pave the way for this reduction, we assume that R_0 is not necessarily local and we denote the different maximal ideals of R_0 by $\mathfrak{m}_0^{(1)}, \dots, \mathfrak{m}_0^{(r)}$. Moreover, for each $j \in \{1, \dots, r\}$ let $R'^{(j)}$ denote the positively graded homogeneous ring $R \otimes_{R_0} (R_0)_{\mathfrak{m}_0^{(j)}} = \oplus_{n\ge 0}(R_n)_{\mathfrak{m}_0^{(j)}}$ with artinian local base ring $(R_0)_{\mathfrak{m}_0^{(j)}}$. Finally, let $M'^{(j)}$ denote the finitely generated and graded $R'^{(j)}$-module $M \otimes_{R_0} (R_0)_{\mathfrak{m}_0^{(j)}} = \oplus_{n\in\mathbb{Z}}(M_n)_{\mathfrak{m}_0^{(j)}}$. Using these notations we have (see [14, (16.2.5)]):

$$\begin{aligned}&\text{(a)}\ h^i_M(n) = \textstyle\sum_{j=1}^r h^i_{M'^{(j)}}(n);\\ &\text{(b)}\ d^i_M(n) = \textstyle\sum_{j=1}^r d^i_{M'^{(j)}}(n);\end{aligned} \qquad (\forall i \in \mathbb{N}_0,\ \forall n \in \mathbb{Z}). \tag{2.25}$$

From these equalities we obtain immediately for all $k \in \mathbb{N}_0$

$$\begin{aligned}&\text{(a)}\ \operatorname{reg}^k(M) = \max\{\operatorname{reg}^k(M'^{(j)}) \mid j = 1, \dots, r\};\\ &\text{(b)}\ \operatorname{coreg}^k(M) = \min\{\operatorname{coreg}^k(M'^{(j)}) \mid j = 1, \dots, r\}.\end{aligned} \tag{2.26}$$

In view of (2.16) the last equality furnishes

$$\lambda(M) = \min\{\lambda(M'^{(j)}) \mid j = 1, \dots, r\}. \tag{2.27}$$

There is still another important step of reduction which will be performed later. It concerns the situation in which our artinian base ring R_0 is local with maximal ideal $\mathfrak{m}_0$. We suppose given an artinian local flat extension ring R_0' of R_0 with maximal ideal $\mathfrak{m}_0' = \mathfrak{m}_0 R_0'$. In this case $R' := R_0' \otimes_{R_0} R = \oplus_{n\in\mathbb{N}_0}(R_0' \otimes_{R_0} R_n)$ is a positively graded homogeneous noetherian ring with base ring R_0' and such that $R'_+ = (R_+)R'$. Moreover $M' := R' \otimes_R M = R_0' \otimes_{R_0} M = \oplus_{n\in\mathbb{Z}}(R_0' \otimes_{R_0} M_n)$ is a finitely generated graded R'-module. Now, by [14, (15.2.2) (iv), (vi)]. We get isomorphisms of R_0'-modules

$$\begin{aligned}&\text{(a)}\ H^i_{R'_+}(M')_n \cong R_0' \otimes_{R_0} H^i_{R_+}(M)_n\ , \quad (\forall n \in \mathbb{Z},\quad \forall i \in \mathbb{N}_0);\\ &\text{(b)}\ \mathcal{R}^i D_{R'_+}(M')_n \cong R_0' \otimes_{R_0} \mathcal{R}^i D_{R_+}(M)_n\ , \quad (\forall n \in \mathbb{Z},\quad \forall i \in \mathbb{N}_0).\end{aligned} \tag{2.28}$$

As R_0' is flat over R_0 with unique maximal ideal $\mathfrak{m}_0 R_0'$ we have $l_{R_0'}(R_0' \otimes_{R_0} N) = l_{R_0}(N)$ for each R_0-module N. Therefore the isomorphisms (2.28) together with (2.5) imply the equalities

$$\begin{aligned}&\text{(a)}\ \ h^i_{M'}(n) = h^i_M(n) \quad (\forall n \in \mathbb{Z},\ \forall i \in \mathbb{N}_0);\\ &\text{(b)}\ \ d^i_{M'}(n) = d^i_M(n) \quad (\forall n \in \mathbb{Z},\ \forall i \in \mathbb{N}_0).\end{aligned} \tag{2.29}$$

As a consequence of this we get for each $k \in \mathbb{N}_0$:

$$\begin{aligned}&\text{(a)}\ \ \operatorname{reg}^k(M') = \operatorname{reg}^k(M),\\ &\text{(b)}\ \ \operatorname{coreg}^k(M') = \operatorname{coreg}^k(M),\end{aligned} \tag{2.30}$$

and hence finally by (2.16)

$$\lambda(M') = \lambda(M). \tag{2.31}$$

Concerning the Krull dimension $\dim(M)$ of M we should keep in mind the relations

$$\begin{aligned}&\text{(a)}\ \ M \neq 0 \Rightarrow \dim(M) = \max\{i \in \mathbb{N}_0 \mid \exists n \in \mathbb{Z} : h^i_M(n) \neq 0\};\\ &\text{(b)}\ \ \dim(M) \leq 0 \ \Leftrightarrow\ \forall n \in \mathbb{Z} : d^0_M(n) = 0;\end{aligned} \tag{2.32}$$

(see [14, (6.1.12), (17.1.10)] and (2.4)). We convene that $\dim(0) = -\infty$ and notice for later use

$$\begin{aligned}&\text{(a)}\ \ \dim(M/\Gamma_+(M)) = \begin{cases} \dim(M), & \text{if } \dim(M) \neq 0\\ -\infty, & \text{if } \dim(M) = 0;\end{cases}\\ &\text{(b)}\ \ h \in R_1,\ M\text{-regular} \Rightarrow \dim(M/hM) = \dim(M) - 1;\\ &\text{(c)}\ \ \dim(M) = \max\{\dim(M'^{(j)}) \mid j = 1, \ldots, r\};\\ &\text{(d)}\ \ \dim(M) = \dim(M');\end{aligned} \tag{2.33}$$

where in statements (c) and (d) we respectively have used the notation introduced in (2.25) and (2.28).

There is a last and somehow more specific step of reduction which has to be performed later. To prepare it, we need to introduce some more notation and notions. Let $j \in \mathbb{N}_0$. We consider the following finite set of graded primes

$$\operatorname{Ass}^{[j]}_R(M) := \{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \dim(R/\mathfrak{p}) \leq j\}. \tag{2.34}$$

Moreover we introduce the graded ideal

$$\mathfrak{a}^{[j]} = \mathfrak{a}^{[j]}(M) := \bigcap_{\mathfrak{p} \in \operatorname{Ass}^{[j]}_R(M)} \mathfrak{p} \tag{2.35}$$

which equals R if and only if $\operatorname{Ass}^{[j]}_R(M) = \phi$. Then, we introduce the so called *j-reduction* of M as the graded R-module

$$M^{[j]} := M/\Gamma_{\mathfrak{a}^{[j]}}(M). \tag{2.36}$$

As $\operatorname{Ass}^{[j]}_R(M) = \operatorname{Var}(\mathfrak{a}^{[j]}(M)) \cap \operatorname{Ass}_R(M) = \operatorname{Ass}_R(\Gamma_{\mathfrak{a}^{[j]}}(M))$, we get

$$\operatorname{Ass}_R(M^{[j]}) = \operatorname{Ass}_R(M) \setminus \operatorname{Ass}^{[j]}_R(M), \tag{2.37}$$

(see [14, (2.1.2)]). Moreover we see that $\dim(\Gamma_{\mathfrak{a}[j]}(M)) \leq j$. So by (2.32)(a) we see that $H^{i+1}_{R_+}(\Gamma_{\mathfrak{a}[j]}(M)) = 0$ for all $i \geq j$. If we apply cohomology to the graded short exact sequence $0 \to \Gamma_{\mathfrak{a}[j]}(M) \to M \to M^{[j]} \to 0$ and keep in mind (2.5) and (2.32)(b), we thus get

$$d^i_{M^{[j]}}(n) = d^i_M(n) \quad \text{for all } i \geq j \text{ and all } n \in \mathbb{Z}. \tag{2.38}$$

Next we give a few remarks concerning the relation between local cohomology and sheaf cohomology. We keep all the previous notations and consider the scheme

$$X := \operatorname{Proj}(R). \tag{2.39}$$

For any $i \in \mathbb{N}_0$ and any coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we consider the i-th *cohomological Hilbert function* of $\mathcal{F}$

$$h^i_{X,\mathcal{F}} = h^i_{\mathcal{F}} : \mathbb{Z} \longrightarrow \mathbb{N}_0, \tag{2.40}$$

which is defined by

$$h^i_{\mathcal{F}}(n) := l_{R_0}\Big(H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n))\Big), \tag{2.41}$$

where $H^i(X, \mathcal{G})$ is used to denote the i-th Serre cohomology module of X with coefficients in the sheaf of $\mathcal{O}_X$-modules $\mathcal{G}$. For $k \in \mathbb{N}_0$ we define the *regularity of $\mathcal{F}$ above level k* by

$$\operatorname{reg}_k(\mathcal{F}) := \inf\{r \in \mathbb{Z} \mid h^i_{\mathcal{F}}(n-i) = 0, \quad \forall n \geq r,\ \forall i > k\}. \tag{2.42}$$

Observe that $\operatorname{reg}_0(\mathcal{F})$ is the so called *Castelnuovo-Mumford* regularity introduced in [39]. The *coregularity* of $\mathcal{F}$ *at and below level* k is defined as

$$\operatorname{coreg}^k(\mathcal{F}) := \sup\{c \in \mathbb{Z} \mid h^i_{\mathcal{F}}(n-i) = 0, \quad \forall\, n \leq c, \quad \forall\, i \leq k\}. \tag{2.43}$$

If M is a finitely generated graded R-module, let $\tilde{M}$ denote the coherent sheaf of $\mathcal{O}_X$-modules induced by M. In these notations, the *Serre-Grothendieck correspondence* yields (see [14, (20.4.4)]).

$$h^i_{\tilde{M}}(n) = d^i_M(n) \quad (\forall i \in \mathbb{N}_0,\ \forall n \in \mathbb{Z}). \tag{2.44}$$

As an easy consequence we get the equalities

$$\begin{aligned} &\text{(a)}\ \operatorname{reg}_k(\tilde{M}) = \operatorname{reg}^{k+2}(M), \quad \text{for all } k \in \mathbb{N}_0;\\ &\text{(b)}\ \operatorname{coreg}^k(\tilde{M}) = \operatorname{coreg}^k(M), \quad \text{for all } k \in \mathbb{N}_0. \end{aligned} \tag{2.45}$$

For a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules we also consider the invariant

$$\delta(\mathcal{F}) := \min\{\operatorname{depth}(\mathcal{F}_x) \mid x \in X,\ x \text{ closed}\} \tag{2.46}$$

which we call the *subdepth* of $\mathcal{F}$. As R_0 is artinian, we have

$$\delta(\tilde{M}) = \lambda(M) - 1 \tag{2.47}$$

for each finitely generated graded R-module M (see [14, (20.4.18), (20.4.19)]). Finally, let us keep in mind that

$$\dim(\tilde{M}) = \begin{cases} \dim(M) - 1, & \text{if } \dim(M) > 0, \\ -\infty, & \text{if } \dim(M) \leq 0; \end{cases} \tag{2.48}$$

where the dimension $\dim(\mathcal{F})$ of a coherent sheaf of $\mathcal{O}_X$-modules is defined as the dimension of the support of $\mathcal{F}$.

3. A Lifting Result for Algebraic Field Extensions to Local Rings and Linear Systems of Homomorphisms

Lemma 3.1. *Let $(B, \pi B)$ be a discrete valuation ring* (DVR). *Let L be an algebraic extension field of $B/\pi B$. Then there is an integral flat extension ring B' of B, which is a* DVR *with maximal ideal $\pi B' = (\pi B)B'$ and such that there is an isomorphism of B-algebras $L \cong B'/\pi B'$.*

Proof. Let F be the algebraic closure of $\mathrm{Quot}(B)$. Let $\mathcal{R}$ be the set of all pairs (C, ϵ_C) for which $C \subseteq F$ is an integral extension ring of B such that $(C, \pi C)$ is a DVR, and $\epsilon_C : C \longrightarrow L$ is a homomorphism of B-algebras. Let $\epsilon_B : B \longrightarrow L$ be the natural map. Since $(B, \epsilon_B) \in \mathcal{R}$, we have $\mathcal{R} \neq \phi$. For $(C, \epsilon_C), (D, \epsilon_D) \in \mathcal{R}$ we write $(C, \epsilon_C) \leq (D, \epsilon_D)$ if $C \subseteq D$ and if $\epsilon_C = \epsilon_D \restriction_C$, where "$\restriction$" is used to denote restriction.
Clearly the relation "$\leq$" defines a partial oder on $\mathcal{R}$. Let $(C_i, \epsilon_{C_i})_{i \in \mathcal{I}}$ be a chain in $\mathcal{R}$ with respect to "$\leq$" and let $C = \bigcup_{i \in \mathcal{I}} C_i$. Then C is integral over B and πC is the unique maximal ideal of C. By [35, (21D)], C_j is faithfully flat over C_i whenever $(C_i, \epsilon_{C_i}) \leq (C_j, \epsilon_{C_j})$, $(i, j \in I)$. This shows that $\pi^n C \cap C_i = \pi^n C_i$ for all $n \in \mathbb{N}$ and for all $i \in \mathcal{I}$. We thus must have $\bigcap_{n \in \mathbb{N}} \pi^n C = 0$. Therefore $(C, \pi C)$ is a DVR.
For each $e \in C$, there is some $i \in \mathcal{I}$ such that $e \in C_i$. Moreover if $j \in \mathcal{I}$ is a second index with $e \in C_j$, we have $\epsilon_{C_i}(e) = \epsilon_{C_j}(e)$. So we can define a map $\epsilon_C : C \longrightarrow L$ by setting $\epsilon_C(e) := \epsilon_{C_i}(e)$ if $i \in \mathcal{I}$ is such that $e \in C_i$. It is easy to see that the map ϵ_C is a homomorphism of B-algebras. So we have $(C, \epsilon_C) \in \mathcal{R}$. It is clear from the definition of ϵ_C that $(C_i, \epsilon_{C_i}) \leq (C, \epsilon_C)$ for all $i \in \mathcal{I}$. This shows by Zorn's lemma that $\mathcal{R}$ has a maximal member, say $(B', \epsilon_{B'})$.
Our next step is to show that $\epsilon'_B : B' \longrightarrow L$ is surjective. Assuming the opposite, we find an element $y \in L \backslash \epsilon_{B'}(B')$. Observe that $\epsilon_{B'}(B')$ is a subfield of L. Let the polynomial $f = \mathbf{x}^n + b_{n-1}\mathbf{x}^{n-1} + \cdots + b_0 \in B'[\mathbf{x}]$ be such that $\mathbf{x}^n + \epsilon_{B'}(b_{n-1})\mathbf{x}^{n-1} + \cdots \epsilon_{B'}(b_0) \in \epsilon_{B'}(B')\,[\mathbf{x}]$ is the minimal polynomial of y over $\epsilon_{B'}(B')$. Then f is of degree $n > 1$ and irreducible in $B'[\mathbf{x}]$. Now let $u \in F$ be such that $f(u) = 0$. We have $u \notin B'$.
Let $B'' = B'[u]$ and let $\varphi : B'[\mathbf{x}] \to B''$ be the homomorphism of B'-algebras which sends $\mathbf{x}$ to u. Then B'' is a finite integral extension of B'. As B' is normal, there is a canonical isomorphism $B'[\mathbf{x}] \,/\, (f) \cong B'[u] = B''$ (see [18, (4.13)] or [14, (8.1.7)]). As $\pi B'[\mathbf{x}] + (f)$ is a prime ideal in $B''[\mathbf{x}]$ we thus obtain $\pi B'' \in \mathrm{Spec}(B'')$. As B'' is integral over B', this shows that $\pi B''$ must be the unique maximal ideal of B''. This means that $(B'', \pi B'')$ is a DVR. Let $\psi : B'[\mathbf{x}] \longrightarrow L$ be the B'-homomorphism which sends $\sum a_i\, \mathbf{x}^i$ to $\sum \epsilon_{B'}(a_i) y^i$. As $\psi(f) = 0$, there exists a homomorphism of B'-algebras $\epsilon_{B''} : B'' \longrightarrow L$ such that the following diagram, in which ι denotes the inclusion map, is commutative

$$\begin{array}{ccc} B' & \xrightarrow{\epsilon_B} & L \\ \iota\downarrow & \nearrow\psi & \uparrow\epsilon_{B''} \\ B'[\mathrm{x}] & \overrightarrow{\varphi} & B'' \end{array}.$$

It is not difficult to check that $(B', \epsilon_{B'}) \leq (B'', \epsilon_{B''})$. As $B' \underset{\neq}{\subset} B''$, this contradicts the supposed maximality of $(B', \epsilon_{B'})$. So $B' \xrightarrow{\epsilon_{B'}} L$ is surjective. As L is a field, we get an isomorphism of B-algebras $B'/\pi B' \cong L$. ■

Proposition 3.2. *Let $(A, \mathfrak{m})$ be a local, noetherian, complete ring and let L be an algebraic extension field of $A/\mathfrak{m}$. Then, there is a local, noetherian, flat extension ring $(A', \mathfrak{m}')$ of A such that $\mathfrak{m}' = \mathfrak{m}A'$ and such that there is an isomorphism of A-algebras $L \cong A'/\mathfrak{m}'$.*

Proof. We have to consider two cases.

Case 1: A contains a field. Let $a_1, \ldots, a_r$ be a system of generators for $\mathfrak{m}$. By Cohen's structure theorem, A contains a field K such that $K \cong A/\mathfrak{m}$ and there is a surjective homomorphism of K-algebras from the formal power series ring $K[[\mathrm{x}_1, \ldots, \mathrm{x}_r]]$ onto A sending x_i to a_i, for $i = 1, \ldots, r$. Hence we can write $A \cong K[[\mathrm{x}_1, \ldots, \mathrm{x}_r]]/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of $K[[\mathrm{x}_1, \ldots, \mathrm{x}_r]]$.
Let $R = K[[\mathrm{x}_1, \ldots, \mathrm{x}_r]]$ and $R' := L[[\mathrm{x}_1, \ldots, \mathrm{x}_r]]$. Then R' is a flat extension ring of R (see [35, 21.D]). Let $A' = R'/\mathfrak{a}R'$. Then obviously A' is a local, noetherian, flat extension ring of A with maximal ideal $\mathfrak{m}' = (\mathrm{x}_1, \ldots, \mathrm{x}_r)\, A' = \mathfrak{m}A'$ and $A'/\mathfrak{m}' \cong L$.

Case 2: A does not contain a field. Let $a_1, \ldots, a_r$ be a system of generators for $\mathfrak{m}$. By Cohen's structure theorem , there is a DVR, say $(B, \pi B)$, such that there exists a surjective homomorphism from the power series ring $B[[\mathrm{x}_1, \ldots, \mathrm{x}_r]]$ onto A. Thus we may write $A \cong B[[\mathrm{x}_1, \ldots, \mathrm{x}_r]]/\mathfrak{a}$, with an appropriate ideal $\mathfrak{a}$ of $B[[\mathrm{x}_1, \ldots, \mathrm{x}_r]]$. Applying Lemma 3.1 we can find an integral extension DVR $(C, \pi C)$ of B such that $C/\pi C \cong L$. Set $S = B[[\mathrm{x}_1, \ldots, \mathrm{x}_r]]$ and $S' = C[[\mathrm{x}_1, \ldots, \mathrm{x}_r]]$. Then by [35, 21.D], S' is flat over S by. Let $A' = S'/\mathfrak{a}S'$. We now may conclude as previously in the case 1. ■

Proposition 3.3. *Let $(R_0, \mathfrak{m}_0)$ be an artinian local ring and let L be an algebraic extension field of $R_0/\mathfrak{m}_0$. Then, there is a flat artinian local integral extension ring $(R_0', \mathfrak{m}_0')$ of R_0 such that $\mathfrak{m}_0' = \mathfrak{m}_0 R_0'$ and such that there is an isomorphism of R_0-algebras $R_0'/\mathfrak{m}_0' \cong L$.*

Proof. By Proposition 3.2, there is a local, noetherian flat extension ring $(R_0', \mathfrak{m}_0')$ of R_0 with $\mathfrak{m}_0' = \mathfrak{m}_0 R_0'$ and such that there is an isomorphism of R_0-algebras $R_0'/\mathfrak{m}_0' \cong L$. As R_0 is artinian, there is some $n \in \mathbb{N}$ with $\mathfrak{m}_0^n = 0$. It follows that $(\mathfrak{m}_0')^n = \mathfrak{m}_0^n R_0' = 0$, and so R_0' is artinian. It remains to show that R_0' is integral over R_0. So, let $y \in R_0'$. As $R_0'/\mathfrak{m}_0 R_0' \cong L$ is an algebraic extension field of $R_0/\mathfrak{m}_0$, we find a polynomial $\mathrm{x}^m + a_{m-1}\mathrm{x}^{m-1} + \cdots + a_0 = f \in R_0[\mathrm{x}]$ with $f(y) \in \mathfrak{m}_0 R_0'$. It follows that $f(y)^n \in \mathfrak{m}_0^n R_0' = 0$. This is an integral equation for y over R_0. ■

From Proposition 3.3 we obtain

Corollary 3.4. *Let $(R_0, \mathfrak{m}_0)$ be an artinian local ring. Then there is a flat artinian local integral extension ring $(R_0', \mathfrak{m}_0')$ of R_0 such that $\mathfrak{m}_0' = \mathfrak{m}_0 R'$ and such that $R_0'/\mathfrak{m}_0'$ is an algebraically closed field.*

Remark 1. Let $R = \oplus_{n\in\mathbb{N}_0} R_0$ be a noetherian positively graded homogeneous ring such that $(R_0, \mathfrak{m}_0)$ is local and artinian. Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R- module. Moreover let $(R_0', \mathfrak{m}_0')$ be as in Corollary 3.4 and consider the noetherian positively graded homogeneous ring $R' := R_0' \otimes_{R_0} R = \oplus_{n\in\mathbb{N}_0}(R_0' \otimes_{R_0} R_n)$ and the finitely generated and graded R'-module $M' = R' \otimes_R M = \oplus_{n\in\mathbb{Z}}(R_0' \otimes_{R_0} M_n)$. Then, by (2.29), (2.30) and (2.31) we have

$$\begin{aligned} h^i_{M'}(n) &= h^i_M(n),\ d^i_{M'}(n) = d^i_M(n); \qquad (\forall n \in \mathbb{Z}, \quad \forall i \in \mathbb{N}_0); \\ \operatorname{reg}^k(M') &= \operatorname{reg}^k(M),\ \operatorname{coreg}^k(M') = \operatorname{coreg}^k(M), \qquad (\forall k \in \mathbb{N}_0); \\ \lambda(M') &= \lambda(M)\,. \end{aligned}$$

So, if studying cohomological Hilbert functions, we may replace R and M respectively by R' and M' and hence assume that $R_0/\mathfrak{m}_0$ is algebraically closed.

Later, we shall use Corollary 3.4 to extend some results on the growth of cohomological Hilbert functions of graded modules over graded algebras over an algebraically closed field K to the case, where K is replaced by an arbitrary artinian ring R_0. The main ingredient of the corresponding arguments is the notion of linear system of homomorphisms of modules over a local ring:

Let $(R_0, \mathfrak{m}_0)$ be a local ring and let P and Q be R_0-modules. We write $R_0/\mathfrak{m}_0 =: k$. By a *linear system of homomorphisms* from P to Q we mean a set $L \subseteq \operatorname{Hom}_{R_0}(P, Q)$ such that the set

$$\overline{L} := \{l \otimes_{R_0} k \mid l \in L\} \subseteq \operatorname{Hom}_k(P \otimes_{R_0} k, Q \otimes_{R_0} k) =: \overline{H}$$

does not contain 0 and such that $\overline{L} \cup \{0\}$ is a k-vector subspace of $\overline{H}$. The *dimension* $\dim(L)$ *of the linear system* L is defined to be $\dim_k(\overline{L} \cup \{0\}) - 1$. Now, let $r \in \mathbb{N}_0$, let $l_0, \dots, l_r \in \operatorname{Hom}_{R_0}(P, Q)$ and let $\mathcal{U} \subseteq R_0^{r+1}$ be a set which is mapped onto $k^{r+1}\setminus\{\underline{0}\}$ under the natural map $R_0^{r+1} \longrightarrow k^{r+1}$ given by $(\alpha_0, \dots, \alpha_r) \mapsto (\alpha_0 + \mathfrak{m}_0, \dots, \alpha_r + \mathfrak{m}_0)$. If the set $L := \{\sum_{i=0}^{r} \alpha_i l_i \mid (\alpha_0, \dots, \alpha_r) \in \mathcal{U}\} \subseteq \operatorname{Hom}_{R_0}(P, Q)$ does not meet $\operatorname{Hom}_{R_0}(P, \mathfrak{m}_0 Q) \subseteq \operatorname{Hom}_{R_0}(P, Q)$, it obviously is a linear system of dimension r. We then call L the *linear system induced* by $l_0, \dots, l_r$ and $\mathcal{U}$. These induced linear systems play an important role in our investigations. So we will study them in this section. As one basic ingredient we need the following Lemma which is found in [5, (3.1), (3.2)] and which treats the special case where R_0 is an algebraically closed field.

Lemma 3.5. *Let k be an algebraically closed field and V, W nonzero k-vector spaces of finite dimension. Let $l_0, \dots, l_n \in Hom_k(V, W)$.*

(a) *If the homomorphism* $\sum_{i=0}^{n} a_i l_i : V \longrightarrow W$ *is injective for all* $(a_0, ..., a_n) \in k^{n+1} \setminus \{\underline{0}\}$ *then* $\dim(W) \geq \dim(V) + n$.

(b) *If the homomorphism* $\sum_{i=0}^{n} a_i l_i : V \longrightarrow W$ *is surjective for all* $(a_0, \dots, a_n) \in k^{n+1} \setminus \{\underline{0}\}$ *then* $\dim(V) \geq \dim(W) + n$. ∎

We also will use the following lemma, whose proof is easy.

Lemma 3.6. *Let* $\mathfrak{a}$ *be an ideal of a ring* A *and* $h : P \longrightarrow Q$ *an injective homomorphism of* A*-modules. Then* $h^{-1}(0 :_Q \mathfrak{a}) = 0 :_P \mathfrak{a}$.

Proposition 3.7. *Let* $(R_0, \mathfrak{m}_0)$ *be an artinian local ring for which* $k = R_0/\mathfrak{m}_0$ *is an algebraically closed field. Let* P, Q *be nonzero finitely generated* R_0*-modules. Let* $l_0, ..., l_r$ *be* R_0*-homomorphisms from* P *to* Q*. Assume that there is a set* $\mathcal{U} \subseteq R_0^{r+1}$ *such that* $\mathcal{U}$ *is mapped onto* $k^{r+1} \setminus \{\underline{0}\}$ *under the canonical map* $R_0^{r+1} \longrightarrow k^{r+1}$ *given by* $(\alpha_0, ..., \alpha_r) \longmapsto (\alpha_0 + \mathfrak{m}_0, ..., \alpha_r + \mathfrak{m}_0)$ *and such that* $\sum_{i=0}^{r} \alpha_i l_i : P \longrightarrow Q$ *is injective for all* $(\alpha_0, ..., \alpha_r) \in \mathcal{U}$*. Then*

$$l_{R_0}(Q) \geq l_{R_0}(P) + r \cdot s(Q),$$

where $s(Q) := \min\{n \in \mathbb{N}_0 \mid \mathfrak{m}_0^n \cdot Q = 0\}$.

Proof. We make induction on $s(Q)$. If $s(Q) = 1$, then $\mathfrak{m}_0 Q = 0$. As $\sum_{i=0}^{r} \alpha_i l_i : P \to Q$ is injective, it follows that $\mathfrak{m}_0 P = 0$. Thus, P, Q are k-vector spaces in a natural way. Now by our hypothesis, $l_0, \dots, l_r$ are k-linear maps and $\sum_{i=0}^{r} \overline{\alpha}_i l_i : P \to Q$ is injective, for all $(\overline{\alpha}_0, ..., \overline{\alpha}_r) := (\alpha_0 + \mathfrak{m}_0, ..., \alpha_r + \mathfrak{m}_0) \in k^{r+1} \setminus \{\underline{0}\}$. Therefore, this case is proven by statement (a) of Lemma 3.5.
Now assume that $s := s(Q) > 1$ and that the proposition is true for all finitely generated target modules Q' with $s(Q') < s(Q) = s$. Clearly $0 :_P \mathfrak{m}_0^{s-1} \neq 0$ and $0 :_Q \mathfrak{m}_0^{s-1} \neq Q$. Note that $s(0 :_Q \mathfrak{m}_0^{s-1}) = s - 1$, as otherwise we had $s(0 :_Q \mathfrak{m}_0^{s-1}) \leq s-2$, hence $(0 :_Q \mathfrak{m}_0^{s-1}) \subseteq (0 :_Q \mathfrak{m}_0^{s-2})$ and thus the contradiction that $(0 :_Q \mathfrak{m}_0^{s-1}) = Q$. If we apply the induction hypothesis to the injective homomorphisms

$$\sum_{i=0}^{r} \alpha_i l_i \restriction_{0:_P \mathfrak{m}_0^{s-1}} : (0 :_P \mathfrak{m}_0^{s-1}) \longrightarrow (0 :_Q \mathfrak{m}_0^{s-1})$$

for all $(\alpha_0, \dots, \alpha_r) \in \mathcal{U}$, we obtain

$$l_{R_0}(0 :_Q \mathfrak{m}_0^{s-1}) \geq l_{R_0}(0 :_P \mathfrak{m}_0^{s-1}) + r(s-1).$$

On the other hand, by Lemma 3.6, any injective homomorphism $g : P \longrightarrow Q$ induces an injective homomorphism $g^* : P/(0 :_P \mathfrak{m}_0^{s-1}) \longrightarrow Q/(0 :_Q \mathfrak{m}_0^{s-1})$. As $s(Q/(0 :_Q \mathfrak{m}_0^{s-1})) = 1$, this implies by induction that

$$l_{R_0}(Q/(0 :_Q \mathfrak{m}_0^{s-1})) \geq l_{R_0}(P/(0 :_P \mathfrak{m}_0^{s-1})) + r. \qquad (**)$$

Combining $(*)$ and $(**)$, we get

$$l_{R_0}(Q) = l_{R_0}(P) + r \cdot s.$$ ∎

Corollary 3.8. *Let* $R_0, P, Q, l_0, \dots, l_r$ *be as in Proposition* 3.7 *but such that* P *and* Q *are not necessarily* $\neq 0$*. Then*

$$l_{R_0}(P) \leq \max\{l_{R_0}(Q) - r, 0\}. \qquad \blacksquare$$

Next, let us give a "dual version" of Proposition 3.7.

Proposition 3.9. *Let $R_0, P, Q, l_0, \ldots, l_r$ and $s(Q)$ be as in Proposition 3.7. Assume that there is a set $\mathcal{U} \subseteq R_0^{r+1}$ such that $\mathcal{U}$ is mapped onto $k^{r+1}\setminus\{\underline{0}\}$ under the natural map $R_0^{r+1} \longrightarrow k^{r+1}$; given by $(\alpha_0, \ldots, \alpha_r) \longmapsto (\alpha_0 + \mathfrak{m}_0, \ldots, \alpha_r + \mathfrak{m}_0)$ and such that $\sum_{i=0}^{r} \alpha_i l_i \; : \; P \longrightarrow Q$ is surjective for all $(\alpha_0, \ldots, \alpha_r) \in \mathcal{U}$. Then*

$$l_{R_0}(P) \geq l_{R_0}(Q) + r \cdot s(Q).$$

Proof. We make induction on $s := s(Q)$.

If $s = 1$, then $\mathfrak{m}_0 Q = 0$. Thus, each homomorphism $l_i : P \longrightarrow Q \; (i = 0, \ldots, r)$ induces a homomorphism $\overline{l_i} : P/\mathfrak{m}_0 P \longrightarrow Q$. For each $(\alpha_0, \ldots, \alpha_r) \in \mathcal{U}$ the surjective map, $\sum_{i=0}^{r} \alpha_i l_i \; : \; P \longrightarrow Q$ induces a surjective map $\sum_{i=0}^{r} \alpha_i \overline{l_i} : P/\mathfrak{m}_0 P \longrightarrow Q$. As $P/\mathfrak{m}_0 P$ and Q are k-vector spaces in natural way, we thus can apply statement (b) of Lemma 3.5 to complete the case $s(Q) = 1$.

Let $s > 1$ and assume that our statement is true for every target module Q' with $s(Q') < s$. Note that $s(\mathfrak{m}_0 Q) = s - 1$. From the surjective homomorphisms

$$\sum_{i=0}^{r} \alpha_i l_i \restriction_{\mathfrak{m}_0 P} \; : \; \mathfrak{m}_0 P \longrightarrow \mathfrak{m}_0 Q$$

for all $(\alpha_0, \ldots, \alpha_r) \in \mathcal{U}$ we get by induction

$$l_{R_0}(\mathfrak{m}_0 P) \geq l_{R_0}(\mathfrak{m}_0 Q) + r \cdot (s - 1). \qquad (*)$$

On the other hand, for all $(\alpha_0, \ldots, \alpha_r) \in \mathcal{U}$ the homomorphism $\sum_{i=0}^{r} \alpha_i \overline{l_i} : P/\mathfrak{m}_0 P \longrightarrow Q/\mathfrak{m}_0 Q$ which is induced by the surjective map $\sum_{i=0}^{r} \alpha_i l_i \; : \; P \longrightarrow Q$ is again surjective. If we apply statement (b) of Lemma 3.5 to the k-vector spaces $P/\mathfrak{m}_0 P$ and $Q/\mathfrak{m}_0 Q$ we obtain

$$l_{R_0}(P/\mathfrak{m}_0 P) \geq l_{R_0}(Q/\mathfrak{m}_0 Q) + r. \qquad (**)$$

The induction is now completed by combining $(*)$ and $(**)$. $\blacksquare$

Corollary 3.10. *Let P, Q, R_0, $l_0, \ldots, l_r$ be as in Proposition 3.9 but such that P and Q are not necessarily $\neq 0$. Then*

$$l_{R_0}(Q) \leq \max\{l_{R_0}(P) - r, 0\}. \qquad \blacksquare$$

To apply the previous results to local cohomology, we need the following result, which will ensure us, that there are appropriate induced linear systems of homomorphisms between consecutive graded parts of local cohomology modules.

Proposition 3.11. *Let $R = \bigoplus_{n \geq 0} R_n$ be a positively graded, homogeneous noetherian ring, such that $(R_0, \mathfrak{m}_0)$ is local artinian and with infinite residue field $k = R_0/\mathfrak{m}_0$. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated, graded R-module. Let $r \in \mathbb{N}_0$ be such that $r < \dim(R/\mathfrak{p})$ for all $\mathfrak{p} \in Ass_R(M)$.*

Then, there are linear forms $l_0, \dots, l_r \in R_1$ such that $\sum_{i=0}^{r} \alpha_i l_i \in R_1$ is an M-regular element for each $(r+1)$-tuple $(\alpha_0, \dots, \alpha_r) \in R_0^{r+1}$ with the property that $\alpha_j \notin \mathfrak{m}_0$ for some $j \in \{0, \dots, r\}$.

Proof. Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \mathrm{Ass}_R(M)$. As R_0 is artinian we have $\mathfrak{p}_i \cap R_0 = \mathfrak{m}_0$ so that $R/\mathfrak{p}_i$ is a homogeneous k-algebra of dimension $> r$ for all $i \in \{1, \dots, n\}$. Therefore, the linear forms of the graded ring $R/\mathfrak{p}_i$ constitute a k-vector space of dimension $> r$ and hence $\overline{H}_i := \mathfrak{p}_i \cap R_1/\mathfrak{m}_0 R_1 \subseteq R_1/\mathfrak{m}_0 R_1$ is a k-vector subspace of codimension $\geq r+1$, for all $i \in \{1, \dots, n\}$. As k is infinite, we thus find a k-vector space $\overline{L} \subseteq R_1/\mathfrak{m}_0 R_1$ of dimension $r+1$ such that $\overline{L} \cap \overline{H}_i = 0$ for all $i \in \{1, \dots, n\}$.

Now, let $l_0, \dots, l_r \in R_1$ such that their images $\overline{l}_0, \dots, \overline{l}_r \in R_1/\mathfrak{m}_0 R_1$ form a k-basis of $\overline{L}$ and let $(\alpha_0, \dots, \alpha_r) \in R_0^{r+1}$ be such that $\alpha_j \notin \mathfrak{m}_0$ for some $j \in \{0, \dots, r\}$. If we denote by $\overline{\alpha_i}$ the image of α_i in k $(i = 0, \dots, r)$, it follows that the element $\sum_{i=0}^{r} \overline{\alpha_i}\overline{l_i} \in \overline{L}$ does not vanish and thus avoids the subspace $\overline{H}_i = \mathfrak{p}_i \cap R_1/\mathfrak{m}_0 R_1 \subseteq R_1/\mathfrak{m}_0 R_1$ for all $i \in \{1, \dots, n\}$. But this implies $\sum_{i=0}^{r} \alpha_i l_i \notin \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n$ and hence proves our claim. ■

Corollary 3.12. *Let $R = \bigoplus_{n \geq 0} R_n$ be a positively graded, homogeneous noetherian ring, such that $(R_0, \mathfrak{m}_0)$ is local artinian and with infinite residue field $k = R_0/\mathfrak{m}_0$. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated, graded R-module and let $r \in \mathbb{N}_0$. Assume that $\Gamma_{R_+}(M) = 0$ and that $r < \lambda(M)$, where the invariant $\lambda(M)$ is defined as in* (2.12). *Then, the same conclusion as in Proposition* 3.11 *holds.*

Proof. Let $\mathfrak{p} \in \mathrm{Ass}\,(M)$. As $\Gamma_{R_+}(M) = 0$, we have $\mathfrak{p} \notin \mathrm{Var}(R_+)$, so that

$$\dim(R/\mathfrak{p}) = \mathrm{depth}(M_\mathfrak{p}) + \mathrm{ht}\big((\mathfrak{p} + R_+)/\mathfrak{p}\big) \geq \lambda(M) > r\,.$$

Now, we may conclude by Proposition 3.11. ■

4. A Priori Bounds of Castelnuovo Type

In the present section, we shall establish the bounds which were already mentioned under (1.16) in the introduction. We do this in a module theoretic way and then translate to sheaf cohomology.

We begin with constructing the occuring bounding functions

$$\underline{B}_{(l,d)}^{(i)} : \mathbb{N}_0^{d-i} \times \mathbb{Z}_{\geq -i} \longrightarrow \mathbb{N}_0 \quad \text{and}$$
$$\underline{C}_{(l,d)}^{(i)} : \mathbb{N}_0^{d-i} \longrightarrow \mathbb{Z}_{\geq -i}$$

for all $i, l, d \in \mathbb{N}_0$ with $0 < i < d$. We do this by induction on $d - i$ starting with $d - i = 1$ hence with $i = d - 1$. We set

$$\underline{B}_{(l,d)}^{(d-1)} : \mathbb{N}_0 \times \mathbb{Z}_{\geq -d+1} \to \mathbb{N}_0,$$
$$(e_{d-1}; n) \mapsto \max\{e_{d-1} - (d-1)(n+d-1), 0\} \qquad (4.1)$$

and

$$\underline{C}^{(d-1)}_{(l,d)} : \mathbb{N}_0 \to \mathbb{Z}_{\geq -d+1},\ e_{d-1} \mapsto -d+1+\left[\left[\frac{e_{d-1}}{d-1}\right]\right], \tag{4.2}$$

where $[[a]] := \min\{t \in \mathbb{Z} \mid t \geq a\}$ for all $a \in \mathbb{R}$.

Now, assume that $d-i>1$ and that $\underline{B}^{(j)}_{(h,s)}$ and $\underline{C}^{(j)}_{(h,s)}$ are already defined for all $j, h, s \in \mathbb{N}_0$ with $0<j<s$ and $s-j<d-i$. We set

$$\bar{l} = \max\{l, 1\} \quad \text{and} \quad k = \max\{i,\ \min\{l, d-1\}\}.$$

For each $(d-i)$-tuple $\underline{e} = (e_i, \dots, e_{d-1}) \in \mathbb{N}_0^{d-i}$ we set

$$\underline{e}' = (e_i + e_{i+1}, \dots, e_{d-2}+e_{d-1}) \in \mathbb{N}_0^{d-i-1},$$
$$c := \max\{-i, \underline{C}^{(i)}_{(\bar{l}-1,d-1)}(\underline{e}') - 1\}.$$

Then we put

$$\underline{B}^{(i)}_{(l,d)}(\underline{e};n) = \begin{cases} e_i + \sum_{-i+1\leq m\leq n} \underline{B}^{(i)}_{(\bar{l}-1,d-1)}(\underline{e}';m) & \text{if } -i \leq n \leq c \\ \max\{\underline{B}^{(i)}_{(l,d)}(\underline{e};c) - k(n-c), 0\} & \text{if } n > c \end{cases} \tag{4.3}$$

and

$$\underline{C}^{(i)}_{(l,d)}(\underline{e}) := c + \left[\left[\frac{\underline{B}^{(i)}_{(l,d)}(\underline{e};c)}{k}\right]\right]. \tag{4.4}$$

Remark 2. (A) It is immediate from the above definitions that $\underline{B}^{(i)}_{(l,d)}(\underline{e};n) = 0$ for all $n \geq \underline{C}^{(i)}_{(l,d)}(\underline{e})$, that $\underline{B}^{(i)}_{(l,d)}(\underline{e},n) > 0$ for all n with $-i<n<\underline{C}^{(i)}_{(l,d)}(\underline{e})$ and that $\underline{B}^{(i)}_{(l,d)}(\underline{e};-i) = e_i$. In particular we have $\underline{B}^{(i)}_{(l,d)}(\underline{0};n) = 0$ for all $n \geq -i$ and $\underline{C}^{(i)}_{(l,d)}(\underline{0}) = -i$ where $\underline{0} = (0, \dots, 0) \in \mathbb{N}_0^{d-i}$.

(B) Let $\underline{e} = (e_i, \dots e_{d-1}), \underline{a} = (a_i, \dots, a_{d-1}) \in \mathbb{N}_0^{d-i}$. We write $\underline{e} \geq \underline{a}$ if $e_j \geq a_j$ for all $j \in \{i, \dots, d-1\}$. Now, by induction on $d-i$, we easily see

If $\underline{e} \geq \underline{a}$, then $\underline{B}^{(i)}_{(l,d)}(\underline{e};n) \geq \underline{B}^{(i)}_{(l,d)}(\underline{a};n)$ $(\forall n \geq -i)$ and $\underline{C}^{(i)}_{(l,d)}(\underline{e}) \geq \underline{C}^{(i)}_{(l,d)}(\underline{a})$.

(C) Now, let $i, h, l, d \in \mathbb{N}_0$ with $0<i<d$ and $h \leq l$. Then, by induction on $d-i$ we easily can prove the inequalities

$$\underline{B}^{(i)}_{(l,d)}(\underline{e};n) \leq \underline{B}^{(i)}_{(h,d)}(\underline{e};n) \ \ (\forall n \geq -i) \quad \text{and} \quad \underline{C}^{(i)}_{(l,d)}(\underline{e}) \leq \underline{C}^{(i)}_{(h,d)}(\underline{e}).$$

It also follows easily by induction on $d-i$ that these inequalities become equalities if $h \leq l \leq i$. This means in particular that for all $l \in \mathbb{N}_0$ we have

$$\underline{B}^{(i)}_{(l,d)}(\underline{e};n) \leq \underline{B}^{(i)}_{(i,d)}(\underline{e};n) \ \ (\forall\, n \geq -i)$$

and

$$\underline{C}^{(i)}_{(l,d)}(\underline{e}) \leq \underline{C}^{(i)}_{(i,d)}(\underline{e}) \text{ with equalities if } l \leq i.$$

(D) It is important for us, to notice the following inequalities

$$\underline{B}^{(i)}_{(l,d)}(\underline{e};n) \geq \underline{B}^{(i+1)}_{(l,d+1)}(\underline{e};n-1) \quad \text{for all} \quad n \geq -i,$$
$$\underline{C}^{(i)}_{(l,d)}(\underline{e}) \geq \underline{C}^{(i+1)}_{(l,d+1)}(\underline{e}) + 1 .$$

We prove these inequalities by induction on $d-i$. If $d-i=1$ hence $i = d-1$, both inequalities follow easily from (4.1) and (4.2). So, let $d-i > 1$, thus $i < d-1$. By induction, $c = \max\{-i, \underline{C}^{(i)}_{(\bar{l}-1,\ d-1)}(\underline{e}')-1\} \geq c'+1$ where $c' := \max\{-(i+1), \underline{C}^{(i+1)}_{(\bar{l}-1,d)}(\underline{e}')-1\}$. By induction we also have $\underline{B}^{(i)}_{(\bar{l}-1,d-1)}(\underline{e}';m) \geq \underline{B}^{(i+1)}_{(\bar{l}-1,d)}(\underline{e}';m-1)$ for all $m \geq -i$. But now, it follows from (4.3) that $\underline{B}^{(i)}_{(l,d)}(\underline{e};n) \geq \underline{B}^{(i+1)}_{(l,d+1)}(\underline{e};n-1)$ for all $n \geq -i$ and that $\underline{B}^{(i)}_{(l,d)}(\underline{e};c) \geq \underline{B}^{(i+1)}_{(l,d+1)}(\underline{e};c')$. In view of (4.4) this completes our proof.

(E) Next, let $0 < i < d-1$. In this situation we have the inequalities

$$\underline{B}^{(i+1)}_{(l,d+1)}(\underline{e};n) \geq \underline{B}^{(i+1)}_{(l,d)}(\underline{f};n) \ \ (\forall\, n \geq -i-1), \ \ \underline{C}^{(i+1)}_{(l,d+1)}(\underline{e}) \geq \underline{C}^{(i+1)}_{(l,d)}(\underline{f}) ,$$
$$\text{where} \quad \underline{f} = (e_{i+1}, \ldots, e_{d-1}) \in \mathbb{N}_0^{d-i-1}.$$

We prove these inequalities again by induction on $d-i$. If $d-i=2$ hence $i=d-2$ we see by (4.3) that $\underline{B}^{(d-1)}_{(l,d+1)}(\underline{e};n) \geq e_{d-1}$ for all n with $-d+1 \leq n \leq c = \max\{-d+1, \underline{C}^{(d-1)}_{(\bar{l}-1,d)}(\underline{e}')-1\}$ and that $\underline{B}^{(d-1)}_{(l,d+1)}(\underline{e};n) \geq \max\{e_{d-1}-(d-1)(n-c), 0\}$ for all $n > c$. By (4.1) we thus see that $\underline{B}^{(d-1)}_{(l,d+1)}(\underline{e};n) \geq \underline{B}^{(d-1)}_{(l,d)}(\underline{f};n)$ for all $n \geq -(d-1)$. By part (A) of the present remark is follows immediately, that $\underline{C}^{(d-1)}_{(l,d+1)}(\underline{e}) \geq \underline{C}^{(d-1)}_{(l,d)}(\underline{f})$. If $d-i<2$, thus $i<d-2$, we get by induction that $\underline{B}^{(i+1)}_{(\bar{l}-1,d)}(\underline{e}';m) \geq \underline{B}^{(i+1)}_{(\bar{l}-1,d-1)}(\underline{f}';m)$ for all $m \geq -i-1$ and $\underline{C}^{(i+1)}_{(\bar{l}-1,d-1)}(\underline{e}') \geq \underline{C}^{(i+1)}_{(\bar{l}-1,d-1)}(\underline{f}')$ where $\underline{f}' := (e_{i+1}+e_{i+2}, \ldots, e_{d-2}+e_{d-1}) \in \mathbb{N}_0^{d-i-2}$. From this, the requested inequalities follow easily by (4.3) and (4.4).

(F) Let $0 < i < d-1$. Then if we combine the estimates given in part (D) and (E) of the present remark, we obtain the inequalities

$$\underline{B}^{(i)}_{(l,d)}(e_i, \ldots, e_{d-1};\ n) \geq \underline{B}^{(i+1)}_{(l,d)}(e_{i+1}, \ldots, e_{d-1};\ n-1) \quad (\forall n \geq -i);$$
$$\underline{C}^{(i)}_{(l,d)}(e_i, \ldots, e_{d-1}) \geq \underline{C}^{(i+1)}_{(l,d)}(e_{i+1}, \ldots, e_{d-1}) + 1.$$

(G) Finally, we also shall need the following estimate

$$\underline{B}^{(i)}_{(l,d)}(\underline{e}+\underline{a};n) \geq \underline{B}^{(i)}_{(l,d)}(\underline{e};n) + \underline{B}^{(i)}_{(l,d)}(\underline{a};n),\ \forall \underline{e},\underline{a} \in \mathbb{N}_0^{d-i}, \ \ \forall n \geq -i .$$

This statement is again shown by induction on $d-i$. If $d-i=1$ hence $i = d-1$, we may easily conclude by (4.1). So, let $d-i>1$. By induction we then have $\underline{B}^{(i)}_{(l-1,d-1)}(\underline{e}'+\underline{a}';m) \geq \underline{B}^{(i)}_{(\bar{l}-1,d-1)}(\underline{e}';m) + B^{(i)}_{(\bar{l}-1,d-1)}(\underline{a}';m)$ for all

$m \geq -i$. By part (A) of the present remark we also have $\underline{C}^{(i)}_{(\bar{l}-1,d-1)}(\underline{e}'+\underline{a}') \geq \underline{C}^{(i)}_{(\bar{l}-1,d-1)}(\underline{e}')$, $\underline{C}^{(i)}_{(\bar{l}-1,d-1)}(\underline{a}')$. As $\underline{e}'+\underline{a}' = (\underline{e}+\underline{a})'$ we may conclude by (4.3).

Now, we are ready to prove the central result of the present section.

Proposition 4.1. *Let $i, l, e \in \mathbb{N}_0$ with $0 < i < e$. Then, for each positively graded noetherian homogeneous ring $R = \oplus_{n\geq 0} R_n$ with artinian local base ring $(R_0, \mathfrak{m}_0)$ and for each finitely generated graded R-module M with $l < \lambda(M)$ and $\dim(M) \leq e$ we have the following estimates:*

(a) $d^i_M(n) \leq \underline{B}^{(i)}_{(l,e)}\Big(d^i_M(-i), \ldots, d^{e-1}_M(-(e-1)); n\Big)$, $\forall n \geq -i$;

(b) $d^i_M(n) = 0$, $\forall n \geq \underline{C}^{(i)}_{(l,e)}\Big(d^i_M(-i), \ldots, d^{e-1}_M(-(e-1))\Big)$;

(c) $\operatorname{reg}^{i+1}(M) \leq \underline{C}^{(i)}_{(l,e)}\Big(d^i_M(-i), \ldots, d^{e-1}_M(-(e-1))\Big) + i$.

Proof. By Corollary 3.4 and by Sec. 3 Remark 1 we may restrict ourselves to prove the stated estimates in the case where $R_0/\mathfrak{m}_0$ is algebraically closed. We now prove claim a) by induction on $e-i$. So, let $e-i=1$ hence $i = e-1$. By (2.37) we have $\dim(R/\mathfrak{p}) > e-1$ for all $\mathfrak{p} \in \operatorname{Ass}_R(M^{[e-1]})$. So by (3.12) there are linear forms $h_0, \ldots, h_{e-1} \in R_1$ such that $h_{\underline{\alpha}} := \sum_{t=0}^{e-1} \alpha_t h_t \in R_1$ is $M^{[e-1]}$-regular for all $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{e-1}) \in R_0^e \setminus (\mathfrak{m}_0 \times \ldots \times \mathfrak{m}_0) =: \mathcal{U}$. For any $\underline{\alpha} \in \mathcal{U}$, (2.33)(b) shows that $\dim(M^{[e-1]}/h_{\underline{\alpha}} M^{[e-1]}) \leq e-1$. So, by (2.32)(a) we have $h^e_{M^{[e-1]}/h_{\underline{\alpha}} M^{[e-1]}} = 0$ and hence $d^{e-1}_{M^{[e-1]}/h_{\underline{\alpha}} M^{[e-1]}} = 0$ for all $\underline{\alpha} \in \mathcal{U}$, (see (2.5) and keep in mind that $e > 1$). If we apply the exact sequence (2.20) with $h = h_{\underline{\alpha}}$ and $i = e-1$ we conclude from Corollary 3.10 that

$$d^{e-1}_{M^{[e-1]}}(n) \leq \max\Big\{d^{e-1}_{M^{[e-1]}}(n-1) - (e-1), 0\Big\}$$

for all $n \in \mathbb{Z}$. By (2.38) we thus get

$$d^{e-1}_M(n) \leq \max\Big\{d^{e-1}_M(n-1) - (e-1), 0\Big\}\ ,\ \forall n \in \mathbb{Z}\,.$$

In particular, if $n \geq -(e-1)$, a repeated application of this estimate gives

$$\begin{aligned} d^{(e-1)}_M(n) &\leq \max\{d^{(e-1)}_M(-(e-1)) - (e-1)(n+e-1), 0\} \\ &= \underline{B}^{(e-1)}_{(l,e)}\big(d^{(e-1)}_M(-(e-1)); n\big). \end{aligned}$$

This proves claim (a) if $i = e-1$.

So, let $e-i > 1$ hence $0 < i < e-1$. We set $\bar{l} = \max\{l, 1\}$ and $k = \max\{i, \min\{l, e-1\}\}$.

Assume first that $l \leq i$, so that $k = i$. Then, by (2.37) and by Proposition 3.11 there are linear forms $h_0, \ldots, h_k \in R_1$ such that $h_{\underline{\alpha}} := \sum_{t=0}^{k} \alpha_t h_t \in R_1$ is $M^{[k]}$-regular for all $\underline{\alpha} = (\alpha_0, \ldots, \alpha_k) \in R_0^{k+1} \setminus (\mathfrak{m}_0 \times \ldots \times \mathfrak{m}_0) =: \mathcal{V}$. By (2.37) and by (2.33)(b) we have $\dim(M^{[k]}/h_{\underline{\alpha}} M^{[k]}) \leq e-1$ for all $\underline{\alpha} \in \mathcal{V}$. We thus may apply induction to $M^{[k]}/h_{\underline{\alpha}} M^{[k]}$. If we use the estimates Remark 2(C) we therefore get

$$d^i_{M^{[k]}/h_{\underline{\alpha}}M^{[k]}}(n) \leq \underline{B}^{(i)}_{(i,e-1)}\big(d^i_{M^{[k]}/h_{\underline{\alpha}}M^{[k]}}(-i), \dots, d^{e-2}_{M^{[k]}/h_{\underline{\alpha}}M^{[k]}}(-(e-2)); n\big)$$
$$\leq \underline{B}^{(i)}_{(\bar{l}-1,\ e-1)}\big(d^i_{M^{[k]}/h_{\underline{\alpha}}M^{[k]}}(-i), \dots, d^{e-2}_{M^{[k]}/h_{\underline{\alpha}}M^{[k]}}(-(e-2)); n\big)$$

for all $n \geq -i$ and all $\underline{\alpha} \in \mathcal{V}$. Note that from (2.22)(b) and (2.38) we get

$$d^m_{M^{[k]}/h_{\underline{\alpha}}M^{[k]}}(-m) \leq d^m_{M^{[k]}}(-m) + d^{m+1}_{M^{[k]}}(-m-1)$$
$$= d^m_M(-m) + d^{m+1}_M(-(m+1)) \ \text{ for all } m \geq k = i.$$

Using the monotony property Remark 2(B) of the function $\underline{B}^{(i)}_{(\bar{l}-1,e-1)}$ we thus get

$$d^i_{M^{[k]}/h_{\underline{\alpha}}M^{[k]}}(m) \leq \underline{B}^{(i)}_{(\bar{l}-1,e-1)}\Big(d^i_M(-i) + d^{i+1}_M(-(i+1)), \dots$$
$$\dots, d^{e-2}_M(-(e-2)) + d^{e-1}_M(-(e-1)); m\Big) =: \underline{B}(m) \qquad (*)$$

for all $m \geq -i$ and all $\underline{\alpha} \in \mathcal{V}$. By Remark 2(A) we thus conclude in particular that

$$d^i_{M^{[k]}/h_{\underline{\alpha}}M^{[k]}}(n) = 0 \ , \forall n \geq \overline{C}^{(i)}_{(l-1,e-1)}\big(d^i_M(-i) + d^{i+1}_M(-(i+1)), \dots$$
$$\dots, d^{e-2}_M(-(e-2)) + d^{e-1}_M(-(e-1))\big) =: c' \ . \qquad (**)$$

for all $\underline{\alpha} \in \mathcal{V}$.

Using the estimates (2.22)(a) for $M^{[k]}$ with $h_{\underline{\alpha}}$ instead of h and keeping in mind (2.38) we see that $d^i_M(m) \leq d^i_M(m-1) + d^i_{M^{[k]}/h_\alpha M^{[k]}}(m)$ for all $m \in \mathbb{Z}$ and all $\underline{\alpha} \in \mathcal{V}$. So, the estimate $(*)$ gives us

$$d^i_M(m) \leq d^i_M(m-1) + \underline{B}(m) \ , \ \forall m \geq -i+1 \ . \qquad (***)$$

Let $c = \max\{-i, c' - 1\}$. Then the estimate $(***)$ yields

$$d^i_M(n) \leq d^i_M(-i) + \sum_{-i+1 \leq m \leq n} \underline{B}(m) \quad := \underline{D}(n), \quad \text{if} \quad -i \leq n \leq c \ . \qquad (\bullet)$$

Now, let $n > c$. By $(**)$ we then get $d^i_{M^{[k]}/h_{\underline{\alpha}}M^{[k]}}(n) = 0$ for all $\underline{\alpha} \in \mathcal{V}$. So, if we apply the sequence (2.20) to $M^{[k]}$ with $h_{\underline{\alpha}}$ instead of h, we get a surjective homomorphism

$$h_{\underline{\alpha}} = \sum_{t=0}^{k} \alpha_t h_t : \mathcal{R}^i D_{R_+}\left(M^{[k]}\right)_{n-1} \to \mathcal{R}^i D_{R_+}(M^{[k]})_n, \ \forall \ \underline{\alpha} \in \mathcal{V} \ .$$

So, by Corollary 3.10 we get $d^i_{M^{[k]}}(n) \leq \max\{d^i_{M^{[k]}}(n-1) - k, 0\}$. In view of (2.38) we therefore obtain

$$d^i_M(n) \leq \max\{d^i_M(n-1) - k, 0\}, \ \forall n > c \ . \qquad (\bullet\bullet)$$

A repeated application of this estimate gives

$$d^i_M(n) \leq \max\{\underline{D}(c) - k(n-c), 0\}, \ \forall n > c \ . \qquad (\bullet\bullet\bullet)$$

In view of (4.3) the estimates ($\bullet$) and ($\bullet\bullet\bullet$) prove statement (a) in the case $l \leq i$.

So, let $l > i$. In this case we have $k = \min\{d-1, l\}$ and $l = \bar{l}$. By (2.17)(c), (2.18)(c) and (2.33)(a) we may replace M by $M/\Gamma_{R_+}(M)$ and hence assume that $\Gamma_{R_+}(M) = 0$. As $k < \lambda(M)$ we conclude by Corollary 3.12 that there are elements $h_0, \dots, h_k \in R_1$ such that $h_{\underline{\alpha}} := \sum_{t=0}^{k} \alpha_t h_t \in R_1$ is M-regular for all $\underline{\alpha} = (\alpha_0, \dots, \alpha_k) \in R_0^{k+1} \setminus (\mathfrak{m}_0 \times \cdots \times \mathfrak{m}_0) = \mathcal{V}$. By (2.24) and by (2.33)(b) we now get

$$\lambda(M/l_{\underline{\alpha}}M) > l - 1 = \bar{l} - 1 \quad \text{and} \quad \dim(M/h_{\underline{\alpha}}M) \leq e - 1$$

for all $\underline{\alpha} \in \mathcal{V}$. As $i < e - 1$ we thus may apply the hypothesis of induction to $M/h_{\underline{\alpha}}M$ and obtain

$$d^{i-1}_{M/h_{\underline{\alpha}}M}(n) \leq \underline{B}^{(i)}_{(\bar{l}-1,e-1)}(d^{i}_{M/h_{\underline{\alpha}}M}(-i), \dots, d^{e-2}_{M/h_{\underline{\alpha}}M}(-(e-2)); n)$$

for all $n \geq -i$ and all $\underline{\alpha} \in \mathcal{V}$. Now, (2.22)(b) gives the inequalities

$$d^{m}_{M/h_{\underline{\alpha}}M}(-m) \leq d^{m}_{M}(-m) + d^{m+1}_{M}(-(m+1)), \ \forall m \geq i .$$

Using again the monotony property Remark 2(B) of the function $\underline{B}^{(i)}_{(\bar{l}-1,e-1)}$ we get in the notation introduced in ($*$) that

$$d^{i}_{M/h_{\underline{\alpha}}M}(m) \leq \underline{B}(m), \ \forall m \geq -i, \ \forall \underline{\alpha} \in \mathcal{V}. \qquad (*')$$

By Remark 2(A) and using the notation of ($**$) we thus get

$$d^{i}_{M/h_{\underline{\alpha}}M}(n) = 0, \ \forall n \geq c', \forall \underline{\alpha} \in \mathcal{V}. \qquad (**')$$

Now, (2.22)(a) and ($*'$) give again the previous estimate ($***$) and hence the estimate ($\bullet$).

Next, let $n > c = \max\{-i, c' - 1\}$. Then, ($**'$) shows that $d^{i}_{M/h_{\underline{\alpha}}M}(n) = 0$ for all $\underline{\alpha} \in \mathcal{V}$. So, by (2.20) we get a surjective homomorphism

$$h_{\underline{\alpha}} = \sum_{t=0}^{k} \alpha_t h_t : \mathcal{R}^i D_{R_+}(M)_{n-1} \to \mathcal{R}^i D_{R_+}(M)_n, \quad \forall \, \underline{\alpha} \in \mathcal{V} .$$

So, Corollary 2.10 furnishes again the estimate ($\bullet\bullet$) and hence the estimate ($\bullet\bullet\bullet$). Now, statement (a) follows by (4.3).

So, claim (a) is proved completely. Now, claim (b) is obvious by Remark 2(A).

Finally, by statement (b) and in view of (2.5), we have for all $j \in \{i, \dots, e-1\}$

$$h^{j+1}_{M}(n) = 0 \ , \ \forall n \geq \underline{C}^{(j)}_{(l,e)}(d^{j}_{M}(-j), \dots, d^{e-1}_{M}(-(e-1))).$$

A repeated application of the second estimate given in Remark 2(F) furnishes the inequalities

$$\underline{C}^{(j)}_{(l,e)}\Big(d^{j}_{M}(-j), \dots, d^{e-1}_{M}(-(e-1))\Big) \leq \underline{C}^{(i)}_{(l,e)}\Big(d^{i}_{M}(-j), \dots, d^{e-1}_{M}(-(e-1))\Big) + i - j$$

for all $j \in \{i, \dots, e-1\}$.

Altogether we thus see that $\operatorname{end}(H^{j+1}_{R_+}(M)) \leq \underline{C}^{(i)}_{(l,e)}(d^i_M(-j), \dots, d^{e-1}_M(-(e-1))) + i - j - 1$ for all $j \geq i$. In view of (2.10)(a), this gives statement (c). ∎

Now, we obtain the main theorem of this section.

Theorem 4.2. *Let $i, l, e \in \mathbb{N}_0$ with $0 < i < e$. Then, for each positively graded noetherian homogeneous ring $R = \oplus_{n \geq 0} R_n$ with artinian base ring R_0 and for each finitely generated graded R-module M with $l < \lambda(M)$ and $\dim(M) \leq e$ we have the following estimates.*

(a) $d^i_M(n) \leq \underline{B}^{(i)}_{(l,e)}\Big(d^i_M(-i), \dots, d^{e-1}_M(-(e-1)); n\Big)$, $\forall n \geq -i$;

(b) $d^i_M(n) = 0$, $\forall n \geq \underline{C}^{(i)}_{(l,e)}\Big(d^i_M(-i), \dots, d^{e-1}_M(-(e-1))\Big)$;

(c) $\operatorname{reg}^{i+1}(M) \leq \underline{C}^{(i)}_{(l,e)}\Big(d^i_M(-i), \dots, d^{e-1}_M(-(e-1))\Big) + i$.

Proof. Let $\mathfrak{m}^{(1)}_0, \dots, \mathfrak{m}^{(r)}_0$ be the different maximal ideals of R_0. For each $j \in \{1, \dots, r\}$ let $R'^{(j)}$ denote the positively graded noetherian homogeneous ring $(R_0)_{\mathfrak{m}^{(j)}_0} \otimes_{R_0} R$ with local artinian base ring $(R_0)_{\mathfrak{m}^{(j)}_0}$ and let $M'^{(j)}$ be the finitely generated $R'^{(j)}$-module $(R_0)_{\mathfrak{m}^{(j)}_0} \otimes_{R_0} M$. By (2.25) we have $d^i_M(n) = \sum_{j=1}^r d^i_{M'^{(j)}}(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. By (2.27) we also have $\lambda(M'^{(j)}) \geq \lambda(M) > l$ for all $j \in \{1, \dots, r\}$. By (2.33)(c) we obtain $\dim(M'^{(j)}) \leq \dim(M) \leq e$ for all $j \in \{1, \dots, r\}$. So, by Proposition 4.1 and by a repeated application of the estimate in Remark 2(G) we get for all $n \geq -i$

$$\begin{aligned}
d^i_M(n) &= \sum_{j=1}^r d^i_{M'^{(j)}}(n) \leq \sum_{j=1}^r \underline{B}^{(i)}_{(l,e)}\Big(d^i_{M'^{(j)}}(-i), \dots, d^{e-1}_{M'^{(j)}}(-(e-1)); n\Big) \\
&\leq \underline{B}^{(i)}_{(l,e)}\Big(\sum_{j=1}^r d^i_{M'^{(j)}}(-i), \dots, \sum_{j=1}^r d^{e-1}_{M'^{(j)}}(-(e-1)); n\Big) \\
&= \underline{B}^{(i)}_{(l,e)}\Big(d^i_M(-i), \dots, d^{e-1}_M(-(e-1)); n\Big).
\end{aligned}$$

This proves statement (a). Statement (b) now follows immediately by Remark 2(A). Finally, by (2.26), Proposition 4.1(c) and Remark 2(B) we get

$$\begin{aligned}
\operatorname{reg}^{i+1}(M) &= \max_{j=1}^{r}\Big\{\operatorname{reg}^{i+1}(M'^{(j)})\Big\} \\
&\leq \max_{j=1}^{r}\Big\{\underline{C}^{(i)}_{(l,e)}(d^i_{M'^{(j)}}(-i), \dots, d^{e-1}_{M'^{(j)}}(-(e-1))) + i\Big\} \\
&\leq \underline{C}^{(i)}_{(l,e)}\Big(d^i_M(-i), \dots, d^{e-1}_M(-(e-1))\Big) + i\,.
\end{aligned}$$

This proves statement (c). ∎

The statement of Theorem 4.2 will also furnish bounds on the cohomological Hilbert functions if nothing is known about the value of $\lambda(M)$. To formulate the corresponding result, we introduce a few notations.

So, let $i, d \in \mathbb{N}$ with $i < d$. For $\underline{e} := (e_i, \dots, e_{d-1}) \in \mathbb{N}_0^{d-i}$, we set

$$\underline{B}_{(d)}^{(i)}(\underline{e}; n) := \underline{B}_{(i,d)}^{(i)}(\underline{e}; n), \ \forall n \geq -i; \tag{4.5}$$

$$\underline{C}_{(d)}^{(i)}(\underline{e}) := \underline{C}_{(i,d)}^{(i)}(\underline{e}). \tag{4.6}$$

So, we get the new bounding functions

$$\underline{B}_{(d)}^{(i)} : \mathbb{N}_0^{d-i} \times \mathbb{Z}_{\geq -i} \to \mathbb{N}_0 \quad \text{and} \quad \underline{C}_{(d)}^{(i)} : \mathbb{N}_0^{d-i} \to \mathbb{Z}_{\geq -i}.$$

By Remark 2(C) we get, for each $l \in \mathbb{N}_0$ for each $n \geq -i$ and each $\underline{e} \in \mathbb{N}_0^{d-i}$

$$\begin{aligned} &\text{(a)} \ \underline{B}_{(d)}^{(i)}(\underline{e}; n) \geq \underline{B}_{(l,d)}^{(i)}(\underline{e}; n), \\ &\text{(b)} \ \underline{C}_{(d)}^{(i)}(\underline{e}) \geq \underline{C}_{(l,d)}^{(i)}(\underline{e}), \end{aligned} \tag{4.7}$$

with equality if $l \leq i$.

Using the above notation, we now have the following result.

Corollary 4.3. *Let $i, e \in \mathbb{N}$ with $i < e$. Then, for each positively graded, noetherian, homogeneous ring $R = \oplus_{n \geq 0} R_n$ with artinian base ring R_0 and for each finitely generated graded R-module M with $\dim(M) \leq e$ we have the following estimates*

(a) $d_M^i(n) \leq \underline{B}_{(e)}^{(i)}\Big(d_M^i(-i), \dots, d_M^{e-1}(-(e-1)); n\Big) \ \forall n \geq -i;$

(b) $d_M^i(n) = 0, \ \forall n \geq \underline{C}_{(e)}^{(i)}\Big(d_M^i(-i), \dots, d_M^{e-1}(-(e-1))\Big);$

(c) $\operatorname{reg}^{i+1}(M) \leq \underline{C}_{(e)}^{(i)}\Big(d_M^i(-i), \dots, d_M^{e-1}(-(e-1))\Big) + i.$

Proof. Clear from Theorem 4.2 and the inqualities (4.7). ∎

In view of (2.5) it seems natural to seek for formulations of Theorem 4.2 and Corollary 4.3 which depend entirely on the functions $n \mapsto h_M^j(n)$ as defined in (2.2). To do so, we introduce some notations. So, let $j, l, d \in \mathbb{N}_0$ with $1 < j \leq d$. For each $(d-j+1)$- tuple $\underline{e} = (e_j, \dots, e_d) \in \mathbb{N}_0^{d-j+1}$ we set

$$\overline{B}_{(l,d)}^{(j)}(\underline{e}; n) := \underline{B}_{(l,d)}^{(j-1)}(\underline{e}; n+1), \ \forall n \geq -j; \tag{4.8}$$

$$\overline{C}_{(l,d)}^{(j)}(\underline{e}) := \underline{C}_{(l,d)}^{(j-1)}(\underline{e}) - 1. \tag{4.9}$$

This defines new bounding functions

$$\overline{B}_{(l,d)}^{(j)} : \mathbb{N}_0^{d-j+1} \times \mathbb{Z}_{\geq -j} \to \mathbb{N}_0 \quad \text{and} \quad \overline{C}_{(l,d)}^{(j)} : \mathbb{N}_0^{d-j+1} \to \mathbb{Z}_{\geq -j}.$$

Using this notation, we get the following equivalent formulation of Theorem 4.2.

Corollary 4.4. *Let $j, l, d \in \mathbb{N}_0$ with $1 < j \leq d$. Then for each positively graded noetherian homogeneous ring $R = \oplus_{n \geq 0} R_n$ with artinian base ring R_0 and for each finitely generated graded R-module M with $l < \lambda(M)$ and $\dim(M) \leq d$*

(a) $h^j_M(n) \leq \overline{\underline{B}}^{(j)}_{(l,d)}(h^j_M(-j),\ldots,h^d_M(-d);n)$ for all $n \geq -j$;

(b) $h^j_M(n) = 0,\ \forall n \geq \overline{\underline{C}}^{(j)}_{(l,d)}(h^j_M(-j),\ldots,h^d_M(-d))$;

(c) $\operatorname{reg}^j(M) \leq \overline{\underline{C}}^{(j)}_{(l,d)}(h^j_M(-j),\ldots,h^d_M(-d)) + j$.

Proof. As $\lambda(M(-1)) = \lambda(M) > l$ and $\dim(M(-1)) = \dim(M) \leq d$ we may apply Theorem 4.2 to $M(-1)$ with $i = j-1$. As $h^k_M(n) = d^{k-1}_M(n) = d^{k-1}_{M(-1)}(n+1)$ for all $k \geq j$ and all $n \in \mathbb{Z}$, our claims follow easily from the corresponding estimates of Theorem 4.2. ∎

In order to give a corresponding formulation of Corollary 4.3 we suppose given $j,d \in \mathbb{N}$ with $1 < j \leq d$ and introduce the bounding functions $\overline{\underline{B}}^{(j)}_{(d)} : \mathbb{N}_0^{d-j+1} \times \mathbb{Z}_{\geq -j} \to \mathbb{N}_0$ and $\overline{\underline{C}}^{(j)}_{(d)} : \mathbb{N}_0^{d-j+1} \to \mathbb{Z}_{\geq j}$ by setting for each $\underline{e} \in \mathbb{N}_0^{d-j+1}$

$$\overline{\underline{B}}^{(j)}_{(d)}(\underline{e};n) := \underline{B}^{(j-1)}_{(d)}(\underline{e};n+1) = \overline{\underline{B}}^{(j)}_{(j-1,d)}(\underline{e};n), \quad \forall n \geq -j, \tag{4.10}$$

$$\overline{\underline{C}}^{(j)}_{(d)}(\underline{e}) := \underline{C}^{(j-1)}_{(d)}(\underline{e}) - 1 = \overline{\underline{C}}^{(j)}_{(j-1,d)}(\underline{e}), \tag{4.11}$$

where $\underline{B}^{(j-1)}_{(d)}$ and $\underline{C}^{(j-1)}_{(d)}$ are defined according to (4.5) and (4.6). Then an equivalent formulation of Corollary 4.3 is:

Corollary 4.5. *Let $j,d \in \mathbb{N}$ with $1 < j \leq d$. Then, for each positively graded noetherian homogeneous ring $R = \oplus_{n\geq 0} R_0$ with artinian base ring R_0 and for each finitely generated graded R-module M with* $\dim(M) \leq d$

(a) $h^j_M(n) \leq \overline{\underline{B}}^{(j)}_{(d)}(h^j_M(-j),\ldots,h^d_M(-d);n),\ \forall n \geq -j$;

(b) $h^j_M(n) = 0,\ \forall n \geq \overline{\underline{C}}^{(j)}_{(d)}(h^j_M(-j),\ldots,h^d_M(-d))$;

(c) $\operatorname{reg}^j(M) \leq \overline{\underline{C}}^{(j)}_{(d)}(h^j_M(-j),\ldots,h^d_M(-d)) + j$.

Proof. We conclude in the same way as Corollary 4.4 was deduced from Theorem 4.2. ∎

Remark 3. (A) The bounding functions $\underline{B}^{(i)}_{(l,d)}, \underline{C}^{(i)}_{(l,d)}$ (cf. (4.1) - (4.4)), $\underline{B}^{(i)}_{(d)}, \underline{C}^{(i)}_{(d)}$ (cf. (4.5), (4.6)), $\overline{\underline{B}}^{(j)}_{(l,d)}, \overline{\underline{C}}^{(j)}_{(l,d)}$ (cf. (4.8), (4.9)) and $\overline{\underline{B}}^{(j)}_{(d)}, \overline{\underline{C}}^{(j)}_{(d)}$ (cf. (4.10), (4.11)) are independent of the choice of the graded ring R and the graded R-module M. Therefore, we call the bounds given in Theorem 4.2, Corollaries 4.3, 4.4 and 4.5 *a priori bounds* on the cohomological Hilbert functions d^i_M, respectively h^j_M, and on the regularity $\operatorname{reg}^j(M)$. As all these estimates are given in terms of the "*values* $d^i_M(-i),\ldots,d^{\dim(M)-1}_M(-(\dim(M)-1))$ *in the diagonal at and above level* i", respectively in terms of the "*cohomological lengths* $h^j_M(-j),\ldots,h^{\dim(M)}_M(-\dim(M))$ *in the diagonal at and above level* j", we say that our bounds are *of diagonal type*.

In 1893 G. Castelnuovo proved a geometric result which can be expressed in terms of local cohomology as follows (see [16]). Let $\mathfrak{a} \subseteq \mathbb{C}\,[\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]$ be the (graded) vanishing ideal of a smooth curve $X \subseteq \mathbb{P}^3_{\mathbb{C}}$ of degree d. Then $\mathrm{reg}^2(\mathfrak{a}) \leq d-1$, with equality if X is rational. This is appearently the first regularity bound and hence the first vanishing result for cohomological Hilbert functions in the range $n \geq 0$. As the estimates given in Theorem 4.2, Corollaries 4.3, 4.4 and 4.5 bound the cohomological Hilbert functions $n \mapsto d^i_M(n)$ respectively $n \mapsto h^j_M(n)$ in the range $n > 0$, we call them *"Castelnuovo bounds"*. So, in this respect, we follow the same "policy of giving names" as found in [39, 42].

(B) Let $j, d \in \mathbb{N}$ with $1 < j \leq d$. Let $R = \oplus_{n\geq 0} R_n$ be a positively graded homogeneous noetherian ring with artinian base ring R_0. Then Corollary 4.4(a), the definition (4.8) and the last statement of Remark 2(A) yield that $h^k_M(n) = 0$ for all $k \geq j$ and all $n \geq -k$ whenever $h^j_M(-j) = \cdots = h^{\dim(M)}(-\dim(M)) = 0$ for a finitely generated graded R-module M. This is an algebraic version - which is also shown in [14, (15.2.5)] - of a vanishing result on sheaf cohomology found in [39], namely the vanishing constraint (1.10). So Corollaries 4.4 and 4.5 may be viewed as extensions on the mentioned vanishing result.

It is easy to verify that the bounding functions $B^{(d)} : \mathbb{N}_0^{d-1} \times \mathbb{Z}_{\geq -2} \to \mathbb{N}_0$ and $C^{(d)} : \mathbb{N}_0^{d-1} \to \mathbb{N}_0$ of [14, (16.2.1)] coincide respectively with the functions $\overline{\underline{B}}^{(2)}_{(d)}$ and $\overline{\underline{C}}^{(2)}_{(d)} + 2$, where $\overline{\underline{B}}^{(2)}_{(d)}$ and $\overline{\underline{C}}^{(2)}_{(d)}$ are defined according to (4.10) and (4.11). So, if we apply Corollary 4.5 with $j = 2$ (and assume moreover that R_0 is in addition local), we get back the bounding result [14, (16.2.4)]. So, Corollary 4.5 extends this latter bounding result to arbitrary values $j \in \{2, \ldots, d\}$ and Corollary 4.4 gives a refinement of this extension which becomes interesting for $j \in \{2, \ldots, \lambda(M) - 1\}$.

In geometric terms and in the notation introduced in (2.41), (2.42) and (2.46), the main results of the present section may be formulated as follows:

Corollary 4.6. *Let $i, l, d \in \mathbb{N}_0$ with $0 < i \leq d$. Then, for each artinian ring R_0, for each projective R_0- scheme X and for each coherent sheaf of $\mathcal{O}_X$ -modules $\mathcal{F}$ with $l \leq \delta(\mathcal{F})$ and $\dim(\mathcal{F}) \leq d$ we have the estimates*

(a) $h^i_{\mathcal{F}}(n) \leq \underline{B}^{(i)}_{(l,d+1)}\,(h^i_{\mathcal{F}}(-i), \ldots, h^d_{\mathcal{F}}(-d); n), \ \forall\, n \geq -i;$

(b) $h^i_{\mathcal{F}}(n) = 0, \ \forall\, n \geq \underline{C}^{(i)}_{(l,d+1)}\,(h^i_{\mathcal{F}}(-i), \ldots, h^d_{\mathcal{F}}(-d));$

(c) $\mathrm{reg}_{i-1}(\mathcal{F}) \leq \underline{C}^{(i)}_{(l,d+1)}\,(h^i_{\mathcal{F}}(-i), \ldots, h^d_{\mathcal{F}}(-d)) + i.$

Proof. Write $X = \mathrm{Proj}\,(R)$, where $R = \oplus_{n\geq 0} R_n$ is a positively graded, noetherian, homogeneous ring and write $\mathcal{F} = \tilde{M}$, where M is a finitely generated graded R-module. Then, the given estimates follow immediately from Theorem 4.2 by the equalities (2.44), (2.47), (2.48) and (2.45)(a). ∎

Corollary 4.7 *Let $i, d \in \mathbb{N}$ with $i \leq d$. Then, for each artinian ring R_0, for each projective R_0 -scheme X and for each coherent sheaf of $\mathcal{O}_X$ -modules $\mathcal{F}$ with $\dim(\mathcal{F}) \leq d$, we have the estimates*

(a) $h^i_{\mathcal{F}}(n) \leq \underline{B}^{(i)}_{(d+1)}(h^i_{\mathcal{F}}(-i), \ldots, h^d_{\mathcal{F}}(-d); n), \forall n \geq -i$;

(b) $h^i_{\mathcal{F}}(n) = 0, \forall n \geq \underline{C}^{(i)}_{(d+1)}(h^i_{\mathcal{F}}(-i), \ldots, h^d_{\mathcal{F}}(-d))$;

(c) $\mathrm{reg}_{i-1}(\mathcal{F}) \leq \underline{C}^{(i)}_{(d+1)}(h^i_{\mathcal{F}}(-i), \ldots, h^d_{\mathcal{F}}(-d)) + i$.

Proof. Follows from Corollary 4.3 in the same way as Corollary 4.6 follows from Theorem 4.2. ∎

Remark 4. (A) For projective schemes over an algebraically closed field K, the estimates given in Corollaries 4.6 and 4.7 correspond to the bounds found in [7, (6.11), (7.9)] if in these latter the "reduced linear subdimension" $\mathrm{lsdim}^{(0)}(\mathcal{F})$ and the "reduced subdepth" $\delta^{(0)}(\mathcal{F})$ are both replaced by $\delta(\mathcal{F})$. In the present paper, we did not use linear subdimensions, as these are not invariant under the changes of base rings we had to perform, and thus only make sense if $R_0 = K$ is an algebraically closed field.

(B) In view of the last statement of Remark 2(A), the bounds of Corollary 4.7 satisfy the requirement (1.10) mentioned in the introduction.

Definition 4.8 [14, (16.4.1)]. (A) Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a class of objects of $\mathcal{C}$. By a *numerical invariant for objects in* $\mathcal{D}$ or a *numerical invariant on* $\mathcal{D}$ we mean an assignement $\mu : \mathcal{D} \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ such that $\mu(U) = \mu(V)$ whenever $U, V \in \mathcal{D}$ are isomorphic in $\mathcal{C}$. We say that numerical invariant μ is *finite*, if $\mu(U) \in \mathbb{Z}$ for all $U \in \mathcal{D}$.

(B) Let $\mu_1, \ldots, \mu_s, \rho$ be numerical invariants on a class $\mathcal{D}$ of objects in the category $\mathcal{C}$. We say, that $\mu_1, \ldots, \mu_s$ form an *upper* (resp. *lower*) *bounding system for* ρ *on* $\mathcal{D}$ if the invariants $\mu_1, \ldots, \mu_s$ are finite and if there is a function $B : \mathbb{Z}^s \longrightarrow \mathbb{Z}$ such that

$$\rho(U) \leq B(\mu_1(U), \ldots, \mu_s(U)) \quad \left(\text{resp. } \rho(U) \geq B(\mu_1(U), \ldots, \mu_s(U))\right)$$

for all $U \in \mathcal{D}$.

(C) We say that the invariants $\mu_1, \ldots, \mu_s$ form a *minimal upper* (resp. *lower*) *bounding system for the invariant* ρ *on* $\mathcal{D}$ if they form an upper (resp. lower) bounding system for ρ on $\mathcal{D}$ and if no $s-1$ of the numerical invariants $\mu_i, \ldots, \mu_s$ form an upper (resp. lower) bounding system for ρ on $\mathcal{D}$.

Remark 5. (A) Corollary 4.5 tells us in particular that – in the sense of Definition 4.8(B) – the numerical invariants $h^j_{(\bullet)}(-j), \ldots, h^d_{(\bullet)}(-d)$ form an *upper bounding system* for the numerical invariant reg^j on the class of all finitely generated graded R-modules of dimension not exceeding d, $\left(j \in \{2, \ldots, d\}\right)$. It is important to notice that this is not true for $j \in \{0, 1\}$ (see [14, (16.4.4)]). By [14, (16.4.3)] the invariants $h^j_{(\bullet)}(-j), \ldots, h^d_{(\bullet)}(-d)$ form an *minimal upper bounding system* for the invariant reg^j on the class of all finitely generated graded R-modules if $j = 2$. A straightforward modification of [14 (16.4.3)] shows that the same statement holds for any $j \in \{2, \ldots, d\}$. Finally note that the bounding function

$C^{(d)}: \mathbb{N}_0^{d-1} \to \mathbb{N}_0$ is given by a polynomial of degree 2^{d-2} (see [14 (16.3.3), (16.3.4)]), so that the same holds for the bounding function $\overline{\underline{C}}^{(2)}_{(d)} = C^{(d)} - 2$.

(B) Let $i, d \in \mathbb{N}$ with $i \leq d$. Let X be a projective scheme over an artinian ring R_0. By Corollary 4.7, the invariants $h_0^2(-i), \dots, h_0^d(-d)$ form an *upper bounding system* for the invariant reg_{i-1} on the class $\mathcal{D}$ of all coherent $\mathcal{O}_X$-modules of dimention $\leq d$. By what is said in part (A) it follows by (2.45)(a), (2.44) and (2.5) that this upper bounding system in minimal.

Remark 6. (A) In many cases one might wish to replace the bounds given in Theorem 4.2, Corollaries 4.3 - 4.7 by possibly weaker but simpler bounds. This may be achieved if one replaces the bounding functions $\underline{B}^{(i)}_{(l,d)}$ and $\underline{C}^{(i)}_{(l,d)}$ by "bigger" functions which are easier to describe. In order to propose a way of doing this, we want to establish the inequalities

$$\max\{\underline{B}^{(i)}_{(l,d)}\ (\underline{e};n) \mid n \geq -i\} \leq \frac{1}{2}\left(2\sum_{j=i}^{d-1}\binom{d-i-1}{j-i}e_j\right)^{2^{d-i-1}};$$

$$\underline{C}^{(i)}_{(l,d)}(\underline{e}) \leq \left(2\sum_{j=i}^{d-1}\binom{d-i-1}{j-i}e_j\right)^{2^{d-i-1}} - i\,.$$

We do this by induction on $d-i$. The case $d-i=1$ is obvious by (4.1) and (4.2). So, let $d-i>1$ and let $P^{(i)}_{(d)}(\underline{e}) := \sum_{j=i}^{d-1}\binom{d-i-1}{j-1}e_j$. Then, using the Pascal formulas for binomial coefficients and in the notation introduced at the beginning of this section we have $P^{(i)}_{(d-1)}(\underline{e}') = P^{(i)}_{(d)}(\underline{e})$. So, by induction we have

$$\underline{B}^{(i)}_{(l-1,d-1)}(\underline{e}';m) \leq \frac{1}{2}\left(2P^{(i)}_{(d)}(\underline{e})\right)^{2^{d-i-2}} \quad \text{for all} \quad m \geq -i+1\,;$$
$$\underline{C}^{(i)}_{(l-1,d-1)}(\underline{e}') \leq \left(2P^{(i)}_{(d)}(\underline{e})\right)^{2^{d-i-2}} - i\,.$$

First, using (4.3) and observing that

$$e_i \leq P^{(i)}_{(d)}(\underline{e})$$

and

$$c \leq \max\left\{-i, \left(2P^{(i)}_{(d)}(\underline{e})\right)^{2^{d-i-2}} - i - 1\right\}$$

we get, for each $n \geq -i$,

$$\underline{B}^{(i)}_{(l,d)}(\underline{e};n) \leq e_i + \left(\left(2P^{(i)}_{(d)}(\underline{e})\right)^{2^{d-i-2}} - 1\right)\frac{1}{2}\left(2P^{(i)}_{(d)}(\underline{e})\right)^{2^{d-i-2}}$$
$$\leq \frac{1}{2}\left(2P^{(i)}_{(d)}(\underline{e})\right)^{2^{d-i-1}}.$$

But now, using (4.4) and observing that $c \leq \left(2P^{(i)}_{(d)}(\underline{e})\right)^{2^{d-i-2}} - i$ we get

$$\underline{C}^{(i)}_{(l,d)}(\underline{e}) \leq \left(2P^{(i)}_{(d)}(e)\right)^{2^{d-i-2}} - i + \frac{1}{2}\left(2P^{(i)}_{(d)}(\underline{e})\right)^{2^{d-i-1}} \leq \left(2P^{(i)}_{(d)}(\underline{e})\right)^{2^{d-i-1}} - i\,.$$

Altogether, this proves our claim.

(B) As a consequence of the above inequalities, we get particularly simple estimates, namely:
For each positively graded noetherian ring $R = \oplus_{n\geq 0} R_n$ with artinian base ring R_0 and for each finitely generated graded R-module M of dimension $d > 1$, we conclude from Corollary 4.5(c) and (4.11) that

$$\operatorname{reg}^i(M) \leq \Big(2\sum_{j=i}^{d}\binom{d-i}{j-i}h^j_M(-j)\Big)^{2^{d-i}} \quad (1 < i \leq d).$$

Similarly, in the notations and under the hypotheses of Corollary 4.7 we get

$$\operatorname{reg}_{i-1}(\mathcal{F}) \leq \Big(2\sum_{j=i}^{d}\binom{d-i}{j-i}h^j_{\mathcal{F}}(-j)\Big)^{2^{d-i}} \quad (1 \leq i \leq d).$$

5. A Priori Bounds of Severi Type

In this section, we shall establish the bounds which were already mentioned under (1.17) in the introduction. Again , we first establish the corresponding results for local cohomology modules and later translate these into sheaf theoretic terms.

We first construct the occuring bounding functions

$$B^{(i)}_{(l)} : \mathbb{N}_0^{i+1} \times \mathbb{Z}_{\leq -i} \to \mathbb{N}_0 \quad \text{and} \quad C^{(i)}_{(l)} : \mathbb{N}_0^{i+1} \to \mathbb{Z}_{\leq -i}$$

for all $i, l \in \mathbb{N}_0$ with $i < l$ by induction on i as follows. First of all, set

$$B^{(0)}_{(l)} : \mathbb{N}_0 \times \mathbb{Z}_{\leq 0} \to \mathbb{N}_0, \quad (e_0, n) \longmapsto \max\{e_0 + ln, 0\}, \tag{5.1}$$

$$C^{(0)}_{(l)} : \mathbb{N}_0 \longrightarrow \mathbb{Z}_{\leq 0}, \; e_0 \longmapsto -\Big[\Big[\frac{e_0}{l}\Big]\Big], \tag{5.2}$$

where $[[a]] := \min\{t \in \mathbb{Z} \mid t \geq a\}$ for all $a \in \mathbb{R}$.

Assume that $i \in \mathbb{N}$ and that $B^{(j)}_{(k)}$ and $C^{(j)}_{(k)}$ are already defined for all $j, k \in \mathbb{N}_0$ with $j < k$ and $j < i$. Let $l \in \mathbb{N}$ with $i < l$.

For each $(i+1)$-tuple $\underline{e} = (e_0, \dots, e_i) \in \mathbb{N}_0^{i+1}$ we set

$$\underline{e}' := (e_0 + e_1, \dots, e_{i-1} + e_i) \in \mathbb{N}_0^i \quad \text{and} \quad c := \min\{-i, C^{(i-1)}_{(l-1)}(\underline{e}')\}.$$

Then we set

$$B^{(i)}_{(l)}(\underline{e}; n) := \begin{cases} e_i + \sum_{n+1\leq m\leq -i} B^{(i-1)}_{(l-1)}(\underline{e}'; m) & \text{if } c \leq n \leq -i \\ \max\{B^{(i)}_{(l)}(\underline{e}; c) + l(n-c), 0\} & \text{if } n < c \end{cases} \tag{5.3}$$

and

$$C^{(i)}_{(l)}(\underline{e}) := c - \Bigg[\Bigg[\frac{B^{(i)}_{(l)}(\underline{e}; c)}{l}\Bigg]\Bigg]. \tag{5.4}$$

Remark 7. (A) It is immediate from the above definitions that $B^{(i)}_{(l)}(\underline{e};n) = 0$ for all $n \leq C^{(i)}_{(l)}(\underline{e})$, that $B^{(i)}_{(l)}(\underline{e};n) > 0$ for all n with $C^{(i)}_{(l)}(\underline{e}) < n < -i$ and that $B^{(i)}_{(l)}(\underline{e};-i) = e_i$. Let $\underline{0} = (0,\ldots,0) \in \mathbb{N}_0^{i+1}$. It then is obvious from the above definitions that $B^{(i)}_{(l)}(\underline{0};n) = 0 \quad \text{for all} \quad n \leq -i \quad \text{and} \quad C^{(i)}_{(l)}(\underline{0}) = -i$.

(B) Let $\underline{e} = (e_0,\ldots,e_i), \underline{a} = (a_0,\ldots,a_i) \in \mathbb{N}^{i+1}$. We write again $\underline{e} \geq \underline{a}$ if $e_j \geq a_j$ for all $j \in \{0,\ldots,i\}$. By induction on i, we easily see that

$$\underline{e} \geq \underline{a} \Rightarrow \left(B^{(i)}_{(l)}(\underline{e};n) \geq B^{(i)}_{(l)}(\underline{a};n),\ \forall n \leq -i\right) \wedge \left(C^{(i)}_{(l)}(\underline{e}) \leq C^{(i)}_{(l)}(\underline{a})\right) .$$

(C) Let $i,h,l \in \mathbb{N}_0$ with $i < h \leq l$. Then, our previous definitions give

$$B^{(i)}_{(l)}(\underline{e};n) \leq B^{(i)}_{(h)}(\underline{e};n) \quad (\forall\ n \leq -i) \quad \text{and} \quad C^{(i)}_{(l)}(\underline{e}) \geq C^{(i)}_{(h)}(\underline{e}) .$$

(D) Now, assume that $0 < i < l$. In this situation we always have

$$C^{(i-1)}_{(l)}\left(e_0,\ldots,e_{i-1}\right) \geq C^{(i)}_{(l)}\left(e_0,\ldots,e_i\right) + 1.$$

This may be seen from the above definitions as follows: If $C^{(i-1)}_{(l)}\left(e_0,\ldots,e_{i-1}\right) = -i+1$, we conclude that $C^{(i)}_{(l)}(e_0,\ldots,e_i) \leq -i$. If $C^{(i-1)}_{(l)}\left(e_0,\ldots,e_{i-1}\right) \leq -i$, then part (B) and (C) of the present remark give $-i \geq C^{(i-1)}_{(l)}\left(e_0,\ldots,e_{i-1}\right) \geq C^{(i-1)}_{(l-1)}\left(e_0,\ldots,e_{i-1}\right) \geq C^{(i-1)}_{(l-1)}(\underline{e}') = c$. By part (A) we thus have $B^{(i-1)}_{(l-1)}(\underline{e}';c+1) \neq 0$, so by (5.3) we see that $B^{(i)}_{(l)}(\underline{e};c) \neq 0$ and (5.4) gives $C^{(i)}_{(l)}(e_0,\ldots,e_{i-1}) \leq c-1 \leq C^{(i-1)}_{(l)}\left(e_0,\ldots,e_{i-1}\right) - 1$.

(E) Finally using induction on i and observing that the assignement $\underline{e} \mapsto \underline{e}'$ is linear, we can show similarly as in Sec. 4 Remark 2(G) that

$$B^{(i)}_{(l)}\left(\underline{e}+\underline{a};n\right) \geq B^{(i)}_{(l)}(\underline{e};n) + B^{(i)}_{(l)}(\underline{a};n), \quad \forall \underline{e},\underline{a} \in \mathbb{N}_0^{i+1}, \quad \forall n \leq -i.$$

Now, we are ready to prove the main result of this section.

Proposition 5.1. *Let $i,l \in \mathbb{N}_0$ with $i < l$. Then for each positively graded noetherian homogeneous ring $R = \bigoplus_{n \geq 0} R_n$ with artinian local base ring $(R_0, \mathfrak{m}_0)$ and for each finitely generated graded R-module M with $l < \lambda(M)$ we have the following estimates*

(a) $d^i_M(n) \leq B^{(i)}_{(l)}(d^0_M(0),\ldots,d^i_M(-i);\ n), \forall n \leq -i;$

(b) $d^i_M(n) = 0,\ \forall n \leq C^{(i)}_{(l)}(d^0_M(0),\ldots,d^i_M(-i));$

(c) $\operatorname{coreg}^i(M) \geq C^{(i)}_{(l)}(d^0_M(0),\ldots,d^i_M(-i)) + i.$

Proof. We proceed by induction on i. By Corollary 3.4 and by Sec. 3 Remark 1 we may assume that $k := R_0/\mathfrak{m}_0$ is an algebraically closed field. Then, by (2.17) and (2.18), we may assume that $\Gamma_{R_+}(M) = 0$. By Corollary 3.12, we can find $h_0, \ldots, h_l \in R_1$ such that $h_{\underline{\alpha}} := \sum_{t=0}^{l} \alpha_t h_t$ is M-regular for each $\underline{\alpha} = (\alpha_0, \ldots \alpha_l) \in R_0^{l+1} \setminus (\mathfrak{m}_0 \times \cdots \times \mathfrak{m}_0) := \mathcal{U}$. If we apply the exact sequence (2.20) with $h = h_{\underline{\alpha}}$ and by Corollary 3.8 we get that

$$d_M^0(n) \leq \max\{d_M^0(n+1) - l, 0\}, \quad \forall n \in \mathbb{Z}.$$

In particular, if $n \leq 0$, a repeated application of this estimate gives

$$d_M^0(n) \leq \max\{d_M^0(0) - l(-n), 0\} = B_{(l)}^{(0)}(d_M^0(0); n).$$

Moreover, by Remark 7(A), we have $d_M^0(n) = 0$ for all $n \leq C_{(l)}^{(0)}(d_M^0(0))$. Thus, the case $i = 0$ is proven.

So, let $i > 0$. Since, by (2.24),

$$\lambda(M/h_{\underline{\alpha}}M) \geq \lambda(M) - 1 > l - 1 > i - 1,$$

we can apply the hypothesis of induction to $M/h_{\underline{\alpha}}M$ and get

$$d_{M/h_{\underline{\alpha}}M}^{i-1}(n) \leq B_{(l-1)}^{(i-1)}\left(d_{M/h_{\underline{\alpha}}M}^0(0), \cdots, d_{M/h_{\underline{\alpha}}M}^{i-1}(-i+1); n\right), \ \forall n \leq -i+1$$

and

$$d_{M/h_{\underline{\alpha}}M}^{i-1}(n) = 0, \forall n \leq \ C_{(l-1)}^{(i-1)}\left(d_{M/h_{\underline{\alpha}}M}^0(0), \cdots, d_{M/h_{\underline{\alpha}}M}^{i-1}(-i+1)\right).$$

Note that from (2.22)(b) we get

$$d_{M/h_{\underline{\alpha}}M}^m(-m) \leq d_M^m(-m) + d_M^{m+1}(-m-1), \quad \forall m \in \mathbb{Z}.$$

Using the monotony property Remark 7(B) of the functions $B_{(l-1)}^{(i-1)}$ and $C_{(l-1)}^{(i-1)}$, we see that

$$\begin{aligned} d_{M/h_{\underline{\alpha}}M}^{i-1}(m) &\leq B_{(l-1)}^{(i-1)}\left(d_M^0(0) + d_M^1(-1), \ldots, d_M^{i-1}(-i+1) + d_M^i(-i); m\right) \\ &=: \overline{B}(m), \quad \forall m \leq -i+1 \end{aligned} \tag{$*$}$$

and

$$d_{M/h_{\underline{\alpha}}M}^{i-1}(n) = 0, \ \forall n \leq C_{(l-1)}^{(i-1)}(d_M^0(0) + d_M^1(-1), \ldots, d_M^{i-1}(-i+1) + d_M^i(-i)) =: c'. \tag{$**$}$$

Using the estimate (2.22)(c) we also see that

$$d_M^i(m) \leq d_{M/h_{\underline{\alpha}}M}^{i-1}(m+1) + d_M^i(m+1), \ \forall m \in \mathbb{Z}. \tag{$***$}$$

Set $c = \min\{-i, c'\}$. Then, for all n with $c \leq n \leq -i$, the estimates $(*)$ and $(***)$ yield

$$\begin{aligned} d^i_M(n) &\leq \sum_{n+1\leq m\leq -i} d^{i-1}_{M/h_{\underline{\alpha}}M}(m) + d^i_M(-i) \\ &\leq \sum_{n+1\leq m\leq -i} \overline{B}(m) + d^i_M(-i) \quad =: \overline{D}(n). \end{aligned}$$

Let $n \leq c-1$. By (**) we get $d^{i-1}_{M/h_{\underline{\alpha}}M}(n) = 0$. We thus obtain from the sequence (2.20) an injective map

$$h_{\underline{\alpha}} = \sum_{t=0}^{l} \alpha_t h_t : \mathcal{R}^i D_{R_+}(M)_n \xrightarrow{h_{\underline{\alpha}}} \mathcal{R}^i D_{R_+}(M)_{n+1}, \ \forall \underline{\alpha} \in \mathcal{U}.$$

So, Corollary 3.8 gives us $d^i_M(n) \leq \max\{d^i_M(n+1) - l, 0\}$ for all $n \leq c-1$. A repeated use of this finally shows that

$$d^i_M(n) \leq \max\{\overline{D}(c) - l(c-n); 0\}.$$

Hence,

$$d^i_M(n) \leq \begin{cases} d^i_M(-i) + \sum_{n+1\leq m\leq -i} \overline{B}(m) := \overline{D}(n) & \text{for } c \leq n \leq -i \\ \max\{\overline{D}(c) - l(c-n), 0\} & \text{for } n < c. \end{cases}$$

Therefore $d^i_M(n) \leq B^{(i)}_{(l)}\left(d^0_M(0), \ldots, d^i_M(-i); n\right)$ for all $n \leq -i$. This proves statement (a). Now it is clear from Remark 7(A) that

$$d^i_M(n) = 0, \forall n \leq C^{(i)}_{(l)}\left(d^0_M(0), \ldots, d^i_M(-i)\right).$$

This proves statement (b).
Now, statement (b) furnishes

$$d^j_M(n) = 0, \ \forall n \leq C^{(j)}_{(l)}(d^0_M(0), \ldots, d^j_M(-j)) \quad \text{for } j = 0, 1, \ldots, i.$$

A repeated use of Remark 7(D) thus gives

$$d^j_M(n) = 0 \quad \text{for all} \quad n \leq C^{(i)}_{(l)}(d^0_M(0), \ldots, d^i_M(-i)) + i - j \quad \text{for} \quad j = 0, 1, \ldots, i,$$

so that

$$\operatorname{beg}\left(\mathcal{R}^j D_{R_+}(M)\right) + j - 1 \geq C^{(i)}_{(l)}(d^0_M(0), \ldots, d^i_M(-i)) + i$$

for $j = 0, \ldots, i$. This proves statement (c). ∎

Now we prove the announced main theorem.

Theorem 5.2. *Let $i, l \in \mathbb{N}_0$ with $i < l$, let $R = \oplus_{n\geq 0} R_n$ be a positively graded noetherian homogeneous ring with artinian base ring R_0. Let M be a finitely generated graded R-module with $l < \lambda(M)$. Then:*

(a) $d^i_M(n) \leq B^{(i)}_{(l)}(d^0_M(0), \ldots, d^i_M(-i); n), \quad \forall n \leq -i;$

(b) $d_M^i(n) = 0, \quad \forall n \leq C_{(l)}^{(i)}(d_M^0(0), \dots, d_M^i(-i))$;

(c) $\operatorname{coreg}^i(M) \geq C_{(l)}^{(i)}(d_M^0(0), \dots, d_M^i(-i)) + i$.

Proof. Let $\mathfrak{m}_0^{(1)}, \dots, \mathfrak{m}_0^{(r)}$ be the different maximal ideals of R_0. For each $j \in \{1, \dots, r\}$ let $R'^{(j)}$ denote the positively graded homogeneous ring $(R_0)_{\mathfrak{m}_0^{(j)}} \otimes_{R_0} R$ with artinian local base ring $(R_0)_{\mathfrak{m}_0^{(j)}}$ and let $M'^{(j)}$ be the finitely generated graded $R'^{(j)}$-module $(R_0)_{\mathfrak{m}_0^{(j)}} \otimes_{R_0} M$. Then, by (2.25) we have $d_M^i(n) = \sum_{j=1}^r d_{M'^{(j)}}^i(n)$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$. Moreover, by (2.27) we know that $\lambda(M'^{(j)}) \geq \lambda(M)$ for $j = 1, \dots, r$. So, using Proposition 5.1(a) and Remark 7(E) we get for all $n \leq -i$

$$\begin{aligned} d_M^i(n) &= \sum_{j=1}^r d_{M'^{(j)}}^i(n) \leq \sum_{j=1}^r B_{(l)}^{(i)}\Big(d_{M'^{(j)}}^0(0), \dots, d_{M'^{(j)}}^i(-i); n\Big) \\ &\leq B_{(l)}^{(i)}\Big(\sum_{j=1}^r d_{M'^{(j)}}^0(0), \dots, \sum_{j=1}^r d_{M'^{(j)}}^i(-i); n\Big) \\ &= B_{(l)}^{(i)}(d_M^0(0), \dots, d_M^i(-i); n) . \end{aligned}$$

This proves statement (a). Now, statement (b) is clear by Remark 7(A). Finally, by (2.26), Remark 7(B) and Proposition 5.1(c) we obtain

$$\begin{aligned} \operatorname{coreg}^i(M) &= \min_{j=1}^r \Big\{ \operatorname{coreg}^i\big(M'^{(j)}\big) \Big\} \\ &\geq \min_{j=1}^r \Big\{ C_{(l)}^{(i)}(d_{M'^{(j)}}^0(0), \dots, d_{M'^{(j)}}^i(-i)) + i \Big\} \\ &\geq C_{(l)}^{(i)}(d_M^0(0), \dots, d_M^i(-i)) + i . \end{aligned}$$

This proves statement (c). ∎

Remark 8. (A) The bounding functions $B_{(l)}^{(i)}$ and $C_{(l)}^{(i)}$ are independent of the choice of R and M; so we call the bounds given in Theorem 5.2 "*a priori bounds*" for the Hilbert functions d_M^i and the invariants $\operatorname{coreg}^i(M)$. As the estimates given in Theorem 5.2 bound the functions d_M^i and the invariant $\operatorname{coreg}^i M$ in terms of the values $d_M^0(0), \dots, d_M^i(-i)$ "*in the diagonal at and below level i*", we say that our bounds are of *diagonal type*.

In 1942, Severi proved a geometric result which can be expressed in terms of local cohomology as follows (see [44]). Let R be the homogeneous coordinate ring of a smooth projective surface $X \subseteq \mathbb{P}_{\mathbb{C}}^3$ and let Ω be the $*$ canonical module of R. (cf. [15]). Then $d_\Omega^0(n) = d_\Omega^1(n) = 0$ for all $n \ll 0$. This is appearently the first vanishing result for a cohomological Hilbert functions in the range < 0. As the estimates given in Theorem 5.2 also bound the cohomological Hilbert functions $n \mapsto d_M^i(n)$ in the range $n < 0$, we call them bounds of *Severi type*. We thus follow the same "policy of giving names" as in [9].

(B) Let $i \in \mathbb{N}_0$ and let $R = \oplus_{n\geq 0} R_n$ be a positively graded, homogeneous noetherian ring with artinian base ring R_0. If M is a finitely generated graded R-module with $i < \lambda(M) - 1$, then Theorem 5.2 and the second half of Remark 7(A) show that $d_M^0(0) = \cdots = d_M^i(-i) = 0$ implies that $d_M^j(n) = 0$ for all $j \leq i$ and all $n \leq -j$. This implication may indeed easily be proved directly by induction on i, and Theorem 5 might be considered as an extension of it.

(C) Theorem 5.2 tells us in particular that the numerical invariants $d^0_{(\bullet)}(0), \dots,$ $d^i_{(\bullet)}(-i)$ form a *lower bounding system* for the numerical invariant coreg^i on the class of all finitely generated graded R-modules M with $i < \lambda(M) - 1$. In Construction and Remark 5.4 below we shall see, that there are choices of R such that $d^0_{(\bullet)}(0), \dots, d^i_{(\bullet)}(-i)$ is a minimal lower bounding system for coreg^i on the class of all finitely generated graded R-modules M with $i < \lambda(M) - 1$.

In geometric terms, and using the notation introduced in (2.41), (2.43) and (2.46) the main result of the present section may be formulated as follows.

Corollary 5.3. *Let $i, l \in \mathbb{N}_0$ with $i < l$. Then for each artinian ring R_0, for each projective R_0-scheme X and for each coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ with $l \leq \delta(\mathcal{F})$, we have the following estimates:*

(a) $h^i_{\mathcal{F}}(n) \leq B^{(i)}_{(l)}\left(h^0_{\mathcal{F}}(0), \dots, h^i_{\mathcal{F}}(-i); n\right), \forall n \leq -i;$

(b) $h^i_{\mathcal{F}}(n) = 0, \forall\, n \leq C^{(i)}_{(l)}\left(h^0_{\mathcal{F}}(0), \dots, h^i_{\mathcal{F}}(-i)\right);$

(c) $\operatorname{coreg}^i(\mathcal{F}) \geq C^{(i)}_{(l)}\left(h^0_{\mathcal{F}}(0), \dots, h^i_{\mathcal{F}}(-i)\right) + i.$

Proof. Immediate from Theorem 5.2 by the equalities (2.44), (2.45)(b), (2.47).∎

Remark 9. For projective schemes over an algebraically closed field K, the estimates given in Corollary 5.3(a,b) essentially correspond to the bounds given in [9, (4.10)], if in these latter again the "linear subdimension", is replaced by the "subdepth" $\delta(\mathcal{F})$.

We now will show that the invariants $d^0_{(\bullet)}(0), \dots, d^i_{(\bullet)}(-i)$ form a *minimal lower bounding system* for the invariant coreg^i on the class of all finitely generated R-modules M with $i < \lambda(M)$. We shall do this in the context of sheaf cohomology and perform a construction, which shall give us a more specific insight.

Construction and Remark 5.4. (A) Let K be an algebraically closed field. Let $d \in \mathbb{N}$. We write Y for the Segre-product $(\mathbb{P}^1_K)^{\times^d} = \mathbb{P}^1_K \times \cdots \times \mathbb{P}^1_K$ of d copies of the projective line $\mathbb{P}^1_K$. For all $k \in \{1, \dots, d\}$, let $p_k : Y \longrightarrow \mathbb{P}^1_K$ be the projection onto the k-th factor. Then, the tensor product

$$\mathcal{O}_Y(1) := \bigotimes_{k=1}^{d} p_k^* \, \mathcal{O}_{\mathbb{P}^1_K}(1)$$

is a very ample sheaf defined on Y by the (multiple) Segre-embedding $Y \overset{\sigma}{\hookrightarrow} \mathbb{P}^{2^d-1}_K$

given by

$$\left((c_0^{(1)}:c_1^{(1)}),\dots,(c_0^{(d)}:c_1^{(d)})\right)\mapsto$$
$$\left(c_0^{(1)}c_0^{(2)}\dots c_0^{(d)}:c_0^{(1)}c_0^{(2)}\dots c_0^{(d-1)}c_1^{(d)}:\dots:c_1^{(1)}c_1^{(2)}\dots c_1^{(d)}\right),$$

(cf. [29, II (5.11), p. 125] for $d=2$).

Now, we fix a d-tuple $\underline{r}=(r_1,\dots,r_d)\in\mathbb{Z}^d$ with $r_1\le r_2\le\cdots\le r_d$ and consider the invertible sheaf of $\mathcal{O}_Y$-modules

$$\mathcal{L}^{(\underline{r})}:=\bigotimes_{k=1}^{d}p_k^*\mathcal{O}_{\mathbf{P}^1_K}(r_k).$$

We write $\mathbb{D}:=\{1,\dots,d\}$. For all $n\in\mathbb{Z}$ we have $\mathcal{L}^{(\underline{r})}(n)\cong\bigotimes_{k=1}^{d}p_k^*\mathcal{O}_{\mathbf{P}^1_K}(r_k+n)$. So, for each $j\in\{0,\dots,d\}$ and for each $n\in\mathbb{Z}$ the Künneth formulas [23, III §6] give

$$h^j_{\mathcal{L}^{(\underline{r})}}(n)=\sum_{\mathbb{M}\subseteq\mathbb{D};\,\#\mathbb{M}=j}\Pi_{m\in\mathbb{M}}h^1_{\mathcal{O}_{\mathbf{P}^1_K}}(r_m+n)\,\Pi_{l\in\mathbb{D}\setminus\mathbb{M}}\,h^0_{\mathcal{O}_{\mathbf{P}^1_K}}(r_l+n),$$

where $\#\mathbb{M}$ denotes the cardinality of $\mathbb{M}$.

As $h^1_{\mathbf{P}^1_K}(r_m+n)=\max\{0,-r_m-n-1\}$ and $h^0_{\mathbf{P}^1_K}(r_l+n)=\max\{0,r_l+n+1\}$ we thus get

$$h^0_{\mathcal{L}^{(\underline{r})}}(n)=\begin{cases}\Pi_{l=1}^d(r_l+n+1), & \text{if } -r_1\le n\\ 0, & \text{otherwise}\end{cases}$$
$$h^d_{\mathcal{L}^{(\underline{r})}}(n)=\begin{cases}\Pi_{m=1}^d(-r_m-n-1), & \text{for } n\le -r_d-2\\ 0, & \text{otherwise}\end{cases}$$

and, for $0<j<d$:

$$h^j_{\mathcal{L}^{(\underline{r})}}(n)=\begin{cases}\Pi_{m=1}^j\;(-r_m-n-1)\,\Pi_{l=j+1}^d(r_l+n+1), & \text{if } -r_{j+1}\le n\le -r_j-2\\ 0, & \text{otherwise}.\end{cases}$$

In particular, we may conclude that

$$h^j_{\mathcal{L}^{(\underline{r})}}(n)=0\quad(\forall n\in\mathbb{Z})\Leftrightarrow r_{j+1}\ge r_j+2\ (0<j<d)$$

and that with the convention $r_0=-\infty$ we have $\operatorname{coreg}^i(\mathcal{L}^{(\underline{r})})=\min\{-r_{j+1}+j-1\mid j\le i:r_j<r_{j+1}-1\}$ for all $i\in\{0,\dots,d-1\}$.

(B) Now, fix $k\in\{0,\dots,d-1\}$. For $t\in\mathbb{N}$, we consider the invertible sheaf of $\mathcal{O}_Y$-modules

$$\mathcal{K}_{k,t}\;:=\;\mathcal{L}^{(-k,\dots,-1,\,t+k+1,\,t+k+2,\dots,\,t+d)}$$

If we apply the formulas of part (A) with $r_p=-k+p-1$ for $1\le p\le k$ and $r_p=t+m$ for $k+1\le p\le d$ we obtain

$$h^0_{\mathcal{K}_{k,t}}\ quad(n) = \begin{cases} \prod_{l=1}^{k}(-k+l+n)\prod_{l=k+1}^{d}(t+l+1+n)\,, & \text{if } k \leq n \\ 0, & \text{otherwise} \end{cases}$$

$$h^d_{\mathcal{K}_{k,t}}(n) = \begin{cases} \prod_{m=1}^{k}(k-m-n)\prod_{m=k+1}^{d}(-t-m-n-1)\,, & \text{if } n \leq -t-d-2 \\ 0, & \text{otherwise} \end{cases}$$

$$h^k_{\mathcal{K}_{k,t}}(n) = \begin{cases} \prod_{m=1}^{k}(k-m-n)\prod_{l=k+1}^{d}(t+l+1+n), & \text{if } -t-k-1 \leq n \leq -1, \\ 0, & \text{otherwise} \end{cases}$$

if $k \neq 0$ and

$$h^j_{\mathcal{K}_{k,t}}(n) = 0, \ \Big(\forall j \in \{1, \dots, d-1\}\setminus\{k\},\ \forall n \in \mathbb{Z}\Big).$$

In particular, we obtain

$$h^j_{\mathcal{K}_{k,t}}(-j) = 0 \ \text{ for all } \ j \in \{0, \dots, d\}\setminus\{k\}\,,$$
$$\operatorname{coreg}^i(\mathcal{K}_{k,t}) = \ -t-2 \ \text{ for all } \ i \in \{k, \dots, d-1\}\,.$$

(C) Now, let $k \in \{0, , \dots, d-1\}$ and $i \in \{k, \dots, d-1\}$. Then, the last two statements of part (B) show that the invariants

$$h^0_{(\bullet)}(0), \dots, h^{k-1}_{(\bullet)}(-(k-1)),\, h^{k+1}_{(\bullet)}(-(k+1)), \dots, h^d_{(\bullet)}(-d) \qquad (*)$$

do not form a lower bounding system for the invariant coreg^i, even on the class of all invertible sheaves of $\mathcal{O}_Y$-modules.

Next, let $\pi : Y \to \mathbb{P}^d_K$ a finite surjective morphism induced by $d+1$ global sections of $\mathcal{O}_Y(1)$. Then, for each $t \in \mathbb{N}$ the direct image $\pi_*\mathcal{K}_{k,t} =: \mathcal{E}_{k,t}$ is a locally free sheaf of rank $d!$ over $\mathbb{P}^d$. As $h^j_{\mathcal{E}_{k,t}}(n) = h^j_{\mathcal{K}_{k,t}}(n)$ for all $n \in \mathbb{Z}$ (see [29, III, (4.1), p.222]), we thus see that the invariants $(*)$ do not form a lower bounding system for the invariant coreg^i on the class of all vector bundles of rank $\leq d!$ over $\mathbb{P}^d_K$.

(D) Now, let R be the homogeneous coordinate ring of Y in $\mathbb{P}^{2^d-1}_K$ or the polynomial ring $K[\mathrm{x}_0, \dots, \mathrm{x}_d]$ and let $\mathcal{C}_i$ be the class of all finitely generated graded R-modules with $i \leq \lambda(M)-1$, where $i \in \{0, \dots, d-1\}$ is fixed. If M is a finitely generated graded R-module such that the induced sheaf $\tilde{M}$ is $\neq 0$ and locally free (on Y resp. on $\mathbb{P}^d$), then $\lambda(M)-1 = \delta(\tilde{M}) = d > i$ shows that M belongs to the class $\mathcal{C}_i$. Now, by Remark 6(C), the observations of part (C) and the equalities (2.44) and (2.45)(b), we see that the invariants $d^0_{(\bullet)}(0), \dots, d^i_{(\bullet)}(-i)$ form a minimal lower bounding system for the invariant coreg^i on the class $\mathcal{C}_i$.

Finally, by the base change arguments and the equalities (2.29)(b), (2.30)(b) and (2.31) it follows easily that the above statement is true whenever $R = R_0[\mathrm{x}_0, \dots, \mathrm{x}_d]$ is a polynomial ring over an arbitrary artinian ring R_0.

Remark 10. (A) As in the case of bounds of Castelnuovo type it might be useful to replace the bounds given in Theorem 5.2 and Corollary 5.3 by weaker but simpler bounds (cf. Sec. 4 Remark 6). To do so, one may use the following inequalities

$$\max\left\{B_{(l)}^{(i)}(\underline{e},n)\mid n\leq -i\right\} \;\leq\; \frac{1}{2}\Big(2\sum_{j=0}^{i}\binom{i}{j}e_j\Big)^{2^i},$$

$$C_{(l)}^{(i)}(\underline{e}) \;\geq\; -\Big(2\sum_{j=0}^{i}\binom{i}{j}e_j\Big)^{2^i}-i,$$

which can be shown by induction on i as follows: The case $i=0$ is obvious by (5.1) and (5.2). So, let $i>0$ and let $Q^{(i)}(\underline{e}) := \sum_{j=0}^{i}\binom{i}{j}e_j$. Then, in the notation introduced at the beginning of this section we obtain $Q^{(i-1)}(\underline{e}') = Q^{(i)}(\underline{e})$. So, by induction

$$B_{(l-1)}^{(i-1)}(\underline{e}';m) \;\leq\; \frac{1}{2}\big(2Q^{(i)}(\underline{e})\big)^{2^{i-1}} \quad \text{for all} \quad m\leq -i+1\,;$$

$$C_{(l-1)}^{(i-1)}(\underline{e}') \;\geq\; -\big(2Q^{(i)}(\underline{e})\big)^{2^{i-1}}-i+1\,.$$

As $e_i\leq Q^{(i)}(\underline{e})$ and $c\geq \min\{-i,-(2Q^{(i)}(\underline{e}))^{2^{i-1}}-i+1\}$, (5.3) shows that, for each $n\leq -i$,

$$\begin{aligned} B_{(l)}^{(i)}(\underline{e};n) \;&\leq\; e_i+\Big(\big(2Q^{(i)}(\underline{e})\big)^{2^{i-1}}-1\Big)\frac{1}{2}\big(2Q^{(i)}(\underline{e})\big)^{2^{i-1}} \\ &\leq \frac{1}{2}\Big(\big(2Q^{(i)}(\underline{e})\big)^{2^{i-1}}\Big)^2=\frac{1}{2}\big(2Q^{(i)}(\underline{e})\big)^{2^i}. \end{aligned}$$

But now, by (5.4)

$$C_{(l)}^{(i)}(\underline{e})\geq -\big(2Q^{(i)}(\underline{e})\big)^{2^{i-1}}-i+1-\frac{1}{2}\big(2Q^{(i)}(\underline{e})\big)^{2^i}\geq -\big(2Q^{(i)}(\underline{e})\big)^{2^i}-i.$$

Altogether, this proves the stated inequalities.

In the notations and under the hypothesis of Theorem 5.2 we conclude from Theorem 5.2(c) that

$$\operatorname{coreg}^i(M)\geq -\Big(2\sum_{j=0}^{i}\binom{i}{j}d_M^j(-j)\Big)^{2^i}\quad (0\leq i<\lambda(M)-1).$$

Using the notations and under the hypothesis of Corollary 5.3 we obtain the estimate

$$\operatorname{coreg}^i(\mathcal{F})\geq -\Big(2\sum_{j=0}^{i}\binom{i}{j}h_{\mathcal{F}}^j(-j)\Big)^{2^i}\quad (0\leq i<\delta(\mathcal{F})).$$

Acknowledgement. The first author likes to express his gratitude to the Hanoi Institute of Mathematics and to the Pedagogical University of Quy Nhon for their support and hospitality offered during the preparation of this paper.

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