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# Invariant Property of Roughly Contractive Mappings

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*Dedicated to Professor Dr. Eberhard Zeidler on the occasion of his 60th birthday*

**Abstract.** In this paper, $r$-roughly $k$-contractive mappings $T : M \to M$ (which satisfy $d(Tx, Ty) \leq k\, d(x, y) + r$ for all $x, y \in M$ and some $k \in (0, 1)$, $r > 0$) are considered. If $M$ is not assumed to be convex, $T$ is only guaranteed to admit $\gamma$-invariant points $x^*$ (which fulfill $d(x^*, Tx^*) \leq \gamma$) with $\gamma \geq r/(1-k)$. For $M$ as a compact convex subset, $T$ possesses $\gamma$-invariant points for all $\gamma > r$. If $M$ is a closed and convex subset of some normed space $(\mathbb{R}^n, \|\cdot\|)$, then, for all $\varepsilon > 0$, there exist $\gamma$-invariant points with $\gamma = nr/(n+1) + \varepsilon$. If the normed space $(\mathbb{R}^n, \|\cdot\|)$ is strictly convex, then $T$ admits $\gamma$-invariant points with $\gamma = nr/(n+1)$. In particular, if $\|\cdot\|$ is the Euclidean norm, then there are $\gamma$-invariant points with $\gamma = (n/2(n+1))^{1/2}\, r$.

## 1. Introduction

As represented in the wonderful book [14], fixed-point theorems belong to the fundamental results of non-linear functional analysis, which have many important applications. One of them is Banach's fixed-point theorem [1] which deals with so-called contractive mappings, i.e., mappings $T : M \to M$ on some metric space $(M, d)$ satisfying the contraction condition

$$d(Tx, Ty) \leq k\, d(x, y) \text{ for all } x, y \in M, \quad \text{for a fixed } k < 1. \tag{1.1}$$

Due to the theorem mentioned, if $M$ is non-empty and complete, then such a $k$-contractive mapping admits a unique fixed point $x^* \in M$, i.e., a point satisfying

$$Tx^* = x^*,$$

and the sequence $(x_i)$ defined by the iteration

$$x_{i+1} = Tx_i, \quad i = 0, 1, 2, ...$$

(with an arbitrary start point $x_0 \in M$) converges to this fixed point.

What happens if the right-hand side of the inequality in (1.1) is modified by some positive constant $r$? That means, $T$ fulfills the condition

$$d(Tx, Ty) \leq k\, d(x, y) + r \text{ for all } x, y \in M \text{ and some fixed } k \in [0, 1),\ r > 0, \tag{1.2}$$

which is said to be an *$r$-roughly $k$-contractive mapping*. Such mappings may arise in quite natural ways. For instance, given a $k$-contractive mapping $T$ which cannot be determined exactly, one has to do with an approximation $\tilde{T}$. For

$$r := 2 \max_{x \in M} d(Tx, \tilde{T}x),$$

(1.1) implies

$$d(\tilde{T}x, \tilde{T}y) \leq d(\tilde{T}x, Tx) + d(Tx, Ty) + d(Ty, \tilde{T}y) \leq k\, d(x, y) + r$$

for all $x, y \in M$, i.e., $\tilde{T}$ is $r$-roughly $k$-contractive.

Obviously, $r$-roughly $k$-contractive mappings cannot always possess fixed points, even under the same condition for the subset $M$ as in Banach's fixed-point theorem. But they admit some so-called *$\gamma$-fixed* or *$\gamma$-invariant points*, i.e., such points $x^*$ satisfying

$$d(x^*, Tx^*) \leq \gamma \tag{1.3}$$

for some $\gamma > 0$. In this paper, we determine the best possible *roughness degree* $\gamma$ of this *rough invariant*, depending on $k$ and $r$.

Section 2 deals with roughly contractive mappings whose domains are not necessarily convex. Theorem 2.1 says that such mappings possess $\gamma$-invariant points with $\gamma \geq r/(1-k)$, and these points can be determined or approximated by the well-known iteration scheme (2.1). Example 2.1 shows that, in general, we cannot expect to obtain $\gamma$-invariant points with $\gamma < r/(1-k)$. Even if these points exist, the iteration is not suitable for seeking them, as illustrated by Example 2.2.

In Sec. 3, roughly contractive mappings on convex domains are considered. For a general normed linear space, Theorem 3.2 says that an $r$-roughly $k$-contractive mapping $T : M \to M$ on some compact convex subset $M$ admits $\gamma$-invariant points if $\gamma > r$. In finite dimensional normed spaces, we obtain a better result. If $M$ is a closed and convex subset (not necessarily bounded) of some normed space $(\mathbb{R}^n, \|\cdot\|)$ then, for all $\varepsilon > 0$, there exist $\gamma$-invariant points with $\gamma = nr/(n+1) + \varepsilon$. If the normed space $(\mathbb{R}^n, \|\cdot\|)$ is strictly convex, then $T$ admits $\gamma$-invariant points with $\gamma = nr/(n+1)$. In particular, if $\|\cdot\|$ is the Euclidean norm, then there are $\gamma$-invariant points with $\gamma = (n/2(n+1))^{1/2}\, r$ (Theorem 3.11).

## 2. Roughly Contractive Mappings on Non Convex Domains

In this section we investigate $r$-roughly $k$-contractive mappings $T : M \to M$ on some metric space $(M, d)$ where the completeness and the convexity of $M$ are not generally required. As usual, we also use the iteration

$$x_0 \in M, \quad x_{i+1} = Tx_i, \quad i = 0, 1, 2, \ldots . \tag{2.1}$$

The main result of this section is

**Theorem 2.1.** *Let $T : M \to M$ be an $r$-roughly $k$-contractive mapping on some metric space $(M, d)$ for $r > 0$ and $k \in (0, 1)$. Suppose $x_0 \in M$ and*

$$a := d(x_0, Tx_0) - \frac{r}{1-k} > 0.$$

(a) *If $\gamma > r/(1-k)$ and*

$$i \geq \log_k \left( \left( \gamma - \frac{r}{1-k} \right) a^{-1} \right), \tag{2.2}$$

*then $x_i$ provided by (2.1) is a $\gamma$-invariant point under $T$.*

(b) *If $x^* \in M$ is a cluster point of the sequence $(x_i)$, then it is a $\gamma$-invariant point under $T$ with $\gamma = r/(1-k)$.*

(c) *For every $\gamma > 0$, the set $I_\gamma$ of all $\gamma$-invariant points (of $T$) is bounded. If $\gamma \geq r/(1-k)$, then $I_\gamma$ is invariant under $T$, i.e., $TI_\gamma \subset I_\gamma$.*

*Proof.* (a) Applying (1.2) and (2.1) successively we obtain

$$\begin{aligned} d(Tx_{i-1}, Tx_i) &\leq k\, d(Tx_{i-2}, Tx_{i-1}) + r \\ &\leq k^2 d(Tx_{i-3}, Tx_{i-2}) + (1+k)r \\ &\leq \cdots \\ &\leq k^i d(x_0, Tx_0) + (1 + k + \cdots + k^{i-1})r \\ &= k^i d(x_0, Tx_0) + \frac{1-k^i}{1-k} r, \end{aligned}$$

which implies

$$d(x_i, Tx_i) \leq k^i \left( d(x_0, Tx_0) - \frac{r}{1-k} \right) + \frac{r}{1-k} = k^i a + \frac{r}{1-k}. \tag{2.3}$$

Since $a > 0$, $\gamma > r/(1-k)$ and $0 < k < 1$, (2.2)–(2.3) yield

$$d(x_i, Tx_i) \leq \left( \left( \gamma - \frac{r}{1-k} \right) a^{-1} \right) a + \frac{r}{1-k} = \gamma,$$

i.e., $x_i$ is $\gamma$-invariant under $T$.

(b) For an arbitrary $i \geq 1$, we have

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_i) + d(Tx_{i-1}, Tx^*) \\ &\leq d(x^*, x_i) + k\, d(x_{i-1}, x^*) + r \\ &\leq d(x^*, x_i) + k\,(d(x_{i-1}, Tx_{i-1}) + d(Tx_{i-1}, x^*)) + r \\ &\leq d(x^*, x_i)(1+k) + k\, d(x_{i-1}, Tx_{i-1}) + r. \end{aligned}$$

It follows from (2.3) that

$$\limsup_{i\to\infty} d(x_{i-1}, Tx_{i-1}) \le \frac{r}{1-k}.$$

Since $x^*$ is a cluster point of the sequence $(x_i)$, considering a subsequence which converges to $x^*$ leads to

$$d(x^*, Tx^*) \le k\frac{r}{1-k} + r = \frac{r}{1-k},$$

i.e., $x^*$ is $\gamma$-invariant under $T$ with $\gamma = r/(1-k)$.
(c) For every $\gamma > 0$ and for all $x, y \in I_\gamma$, it follows from

$$d(x,y) \le d(x,Tx) + d(Tx,Ty) + d(y,Ty) \le 2\gamma + k\,d(x,y) + r$$

that

$$d(x,y) \le \frac{2\gamma + r}{1-k},$$

i.e., $I_\gamma$ is bounded.
If $x \in I_\gamma$, then

$$d(Tx, T^2x) \le k\,d(x,Tx) + r \le k\gamma + r.$$

This inequality implies by $0 < k < 1$ and $\gamma \ge r/(1-k)$ that

$$d(Tx, T^2x) \le k\gamma + (1-k)\gamma = \gamma,$$

i.e., $Tx \in I_\gamma$, too. Hence, $TI_\gamma \subset I_\gamma$ for $\gamma \ge r/(1-k)$. ∎

**Corollary 2.2.** *If $M$ is a compact metric space or if it is a closed subset of some finite dimensional metric space, then each $r$-roughly $k$-contractive mapping $T : M \to M$ admits at least one $\gamma$-invariant point with $\gamma = r/(1-k)$.*

*Proof.* If $M$ is compact, then each sequence $(x_i)$ defined by (2.1) has at least one cluster point. Therefore, due to Theorem 2.1(b), this cluster point is $\gamma$-invariant under $T$ with $\gamma = r/(1-k)$.

Assume now that $M$ is a closed subset of some finite dimensional metric space. Take an arbitrary $x_0 \in M$. For $\gamma_0 := d(x_0, Tx_0)$, $x_0$ is obviously $\gamma_0$-invariant. Thus, if $\gamma_0 \le r/(1-k)$, then $x_0$ is $\gamma$-invariant under $T$ with $\gamma = r/(1-k)$. If $\gamma_0 > r/(1-k)$, then Theorem 2.1(c) ensures that all $x_i$, $i \in \mathbb{N}$, are $\gamma_0$-invariant, too, and hence, they are contained in the bounded subset $I_{\gamma_0}$ of the finite dimensional space considered. Therefore, this sequence has at least one cluster point, which belongs to $M$, because $M$ is closed. The rest follows directly from Theorem 2.1(b). ∎

Theorem 2.1 only ensures the existence of $\gamma$-invariant points of $r$-roughly $k$-contractive mappings with $\gamma \ge r/(1-k)$. In general, $\gamma = r/(1-k)$ is already the smallest possible roughness degree, as the following example illustrates.

*Example 2.1.* Let $r > 0$, $k \in (0,1)$,

$$M_1 := \left( -\infty, \frac{-r}{2(1-k)} \right), \; M_2 := \left( \frac{r}{2(1-k)}, +\infty \right), \text{ and } M := M_1 \cup M_2. \quad (2.4)$$

Define

$$Tx := \begin{cases} r/2 - kx, & \text{if } x \in M_1, \\ -r/2 - kx, & \text{if } x \in M_2. \end{cases} \quad (2.5)$$

This mapping $T$ is $r$-roughly $k$-contractive. In fact, if $\{x, y\} \subset M_1$ or $\{x, y\} \subset M_2$, then

$$|Tx - Ty| = |-kx + ky| = k\,|x - y|.$$

If $x \in M_1$ and $y \in M_2$, or if $x \in M_2$ and $y \in M_1$, we have

$$|Tx - Ty| \le \left| \frac{r}{2} - \frac{-r}{2} \right| + |kx - ky| = r + k\,|x - y|.$$

Formulae (2.4)–(2.5) imply that if $x \in M_1$, then

$$Tx > \frac{r}{2} - k\frac{-r}{2(1-k)} = \frac{r}{2(1-k)},$$

i.e., $Tx \in M_2$, and if $x \in M_2$, then

$$Tx < \frac{-r}{2} - k\frac{r}{2(1-k)} = -\frac{r}{2(1-k)},$$

i.e., $Tx \in M_1$. Hence, $T$ maps $M \subset \mathbb{R}$ into $M$. Moreover, it also follows that

$$|x - Tx| > \inf M_2 - \sup M_1 = \frac{r}{2(1-k)} - \frac{-r}{2(1-k)} = \frac{r}{1-k} \text{ for all } x \in M.$$

That means, this $r$-roughly $k$-contractive mapping $T$ cannot have any $\gamma$-invariant points with $\gamma \le r/(1-k)$. Consequently, in general, there is no better roughness degree $\gamma$ as given in Theorem 2.1(a).

As just considered, $T$ admits no $\gamma$-invariant points with $\gamma = r/(1-k)$. This means no conflict with (b) in Theorem 2.1. In fact, since

$$\limsup_{i\to\infty} |x_i - x_{i+1}| \le \frac{r}{1-k}$$

(due to (2.3)) and

$$\inf M_2 - \sup M_1 = \frac{r}{1-k},$$

the fact

$$\text{either} \quad x_i \in M_1,\ x_{i+1} \in M_2 \quad \text{or} \quad x_i \in M_2,\ x_{i+1} \in M_1$$

implies that the only cluster points of each sequence $(x_i)$ defined by (2.1) are $\pm r/(2(1-k))$, which do not belong to $M$.

Let us extend $T$ to $\overline{T}$ defined on the closure $\overline{M}$ of $M$, i.e., $Tx = \overline{T}x$ for all $x \in M$. In order for $\overline{T}$ to be $r$-roughly $k$-contractive on $\overline{M}$, it must hold true

$$\left|\overline{T}\frac{r}{2(1-k)} - \frac{r}{2(1-k)}\right| = \lim_{x \uparrow \frac{-r}{2(1-k)}} \left|\overline{T}\frac{r}{2(1-k)} - \overline{T}x\right|$$
$$\leq k\frac{r}{1-k} + r = \frac{r}{1-k}.$$

Hence, $r/(2(1-k))$ is $\gamma$-invariant under $\overline{T}$ for $\gamma = r/(1-k)$. This result corresponds with (b) in Theorem 2.1, because $r/(2(1-k))$ is a cluster point (in $\overline{M}$) of each sequence $(x_i)$. It is also an illustration for Corollary 2.2 since $\overline{M}$ is a closed subset of $\mathbb{R}$.

*Remark 2.1.* Theorem 2.1 shows that the iteration (2.1) can be used to determine or approximate $\gamma$-invariant points of $r$-roughly $k$-contractive mappings for $\gamma \geq r/(1-k)$. But, in general, this scheme is no more suitable for finding $\gamma$-invariant points with $\gamma < r/(1-k)$ even if they exist, as the following example shows.

*Example 2.2.* Let us consider the following extension of $T$ defined in (2.5)

$$\tilde{T}x := \begin{cases} r/2 - kx, & \text{if } x \in \tilde{M}_1, \\ -r/2 - kx, & \text{if } x \in \tilde{M}_2, \end{cases} \tag{2.6}$$

where

$$\tilde{M}_1 := (-\infty, 0] \quad \text{and} \quad \tilde{M}_2 := (0, +\infty).$$

Similarly as above, it can be shown that $\tilde{T}$ is an $r$-roughly $k$-contractive mapping from $\mathbb{R}$ into $\mathbb{R}$. Moreover, the difference $|x - \tilde{T}x|$ admits on the interval $[-r/2(1-k), r/2(1-k)]$ all values of $[r/2, r/(1-k)]$. That means, for all $\gamma \geq r/2$, the set $I_\gamma$ of $\gamma$-invariant points of $\tilde{T}$ is non-empty.

We now focus our consideration within the subsets

$$\tilde{M}_1' := \left(\frac{-r}{2(1-k)}, 0\right] \quad \text{and} \quad \tilde{M}_2' := \left(0, \frac{r}{2(1-k)}\right).$$

It is easy to verify that

$$x \in \tilde{M}_1' \Rightarrow \tilde{T}x \in \tilde{M}_2' \quad \text{and} \quad x \in \tilde{M}_2' \Rightarrow \tilde{T}x \in \tilde{M}_1'.$$

Therefore, $\tilde{M}_1' \subset \tilde{M}_1$, $\tilde{M}_2' \subset \tilde{M}_2$, and (2.6) yield

$$\tilde{T}x - x = \begin{cases} r/2 - (1+k)x, & \text{if } x \in \tilde{M}_1', \\ -r/2 - (1+k)x, & \text{if } x \in \tilde{M}_2', \end{cases}$$

and

$$\tilde{T}^2x - \tilde{T}x = \begin{cases} -r(1+k/2) + k(1+k)x, & \text{if } x \in \tilde{M}_1', \\ r(1+k/2) + k(1+k)x, & \text{if } x \in \tilde{M}_2'. \end{cases}$$

Thus, if $x \in \tilde{M}'_1$, then $x > -r/(2(1-k))$ and $1-k^2 > 0$ imply

$$|\tilde{T}^2x - \tilde{T}x| - |\tilde{T}x - x| = (1-k^2)x + \frac{r(1+k)}{2}$$
$$> (1-k^2)\frac{-r}{2(1-k)} + \frac{r(1+k)}{2} = 0.$$

Similarly, if $x \in \tilde{M}'_2$, then $x < r/(2(1-k))$ implies

$$|\tilde{T}^2x - \tilde{T}x| - |\tilde{T}x - x| = \frac{r(1+k)}{2} - (1-k^2)x$$
$$> \frac{r(1+k)}{2} - (1-k^2)\frac{r}{2(1-k)} = 0.$$

In any case, for each $x \in \tilde{M}' := \tilde{M}'_1 \cup \tilde{M}'_2$, we have

$$|\tilde{T}(\tilde{T}x) - \tilde{T}x| > |\tilde{T}x - x|.$$

Consequently, by starting at any point $x_0 \in \tilde{M}'$, the sequence $(x_i)$ defined by $x_{i+1} = \tilde{T}x_i$ remains in $\tilde{M}'$, but after each iteration step, the distance $|\tilde{T}x_i - x_i|$ becomes greater. Therefore, if $\gamma < r/(1-k)$ and $|x_0 - \tilde{T}x_0| > \gamma$, then the sequence $(x_i)$ cannot approach the set $I_\gamma$ of $\gamma$-invariant points but moves further away from this set, although $I_\gamma$ is non-empty for $\gamma \geq r/2$. This is what Remark 2.1 says.

An interesting fact appearing in this example is that, for an arbitrary starting point $x_0 \in \mathbb{R}$, the sequence $(x_i)$ has exactly two cluster points $\pm r/(2(1-k))$, and they are $\gamma$-invariant points with $\gamma = r/(1-k)$.

As mentioned in the introduction of this paper, because of errors, many $k$-contractive mappings only appear during practical computing as $r$-roughly $k$-contractive ones. Therefore, by using the iteration (2.1), in general, one cannot approximate the proper fixed point but only some $\gamma$-invariant points with $\gamma \geq r/(1-k)$.

Example 2.1 shows that, in general, we cannot expect each $r$-roughly $k$-contractive mapping $T : M \to M$ to admit $\gamma$-invariant points with $\gamma < r/(1-k)$. For $k$ near 1, this bound is rather large, e.g. $r/(1-k) = 100r$ if $k = 0.99$. The situation changes essentially if $M$ is assumed to be convex, as we will see in the next section.

## 3. Roughly Contractive Mappings on Convex Domains

Throughout this section we assume $M$ is convex, and show under some assumptions that $r$-roughly $k$-contractive mappings of $M$ into $M$ admit $\gamma$-invariant points with $\gamma$ near $r$.

For $\dim M = \infty$, we use the following result of Klee [9].

**Theorem 3.1.** [9] *Suppose $M$ is a compact convex subset of a normed linear space, $T$ is an $\bar{r}$-continuous mapping of $M$ into $M$, and $\gamma > \bar{r} > 0$. Then some point of $M$ is $\gamma$-invariant under $T$.*

Recall that $T : M \to M$ is called $\bar{r}$-*continuous* provided each $x$ of $M$ admits a neighborhood $U_x$ such that the diameter of the set $TU_x$ is at most $\bar{r}$ (see [9]).

Let $T : M \to M$ be an $r$-roughly $k$-contractive mapping, $\bar{r}$ an arbitrary fixed number satisfying $\bar{r} > r$, and $x \in M$. Choose $U_x$ as the (open) ball $B(x, (\bar{r} - r)/2k)$ (with center $x$ and radius $(\bar{r} - r)/2k$). Then, due to definition, we have

$$\|x' - x''\| \le \|x' - x\| + \|x - x''\| < 2\frac{\bar{r} - r}{2k} = \frac{\bar{r} - r}{k}, \quad \forall x', x'' \in U_x,$$

and consequently,

$$\|Tx' - Tx''\| \le k\,\|x' - x''\| + r < k\frac{\bar{r} - r}{k} + r = \bar{r}, \quad \forall x', x'' \in U_x,$$

which yields $\operatorname{diam} TU_x \le \bar{r}$. Hence, $T$ is $\bar{r}$-continuous for all $\bar{r} > r$. Therefore, Theorem 3.1 implies

**Theorem 3.2.** *Suppose $M$ is a compact convex subset of a normed linear space, $T$ is an $r$-roughly $k$-contractive mapping of $M$ into $M$, and $\gamma > r > 0$. Then some point of $M$ is $\gamma$-invariant under $T$.*

For $\dim M < \infty$, Cromme and Diener [4] obtained the following result.

**Theorem 3.3.** [4] *Assume $M$ is a compact and convex subset of some normed space $(\mathbb{R}^n, \|\cdot\|)$. Let $T : M \to M$ be any mapping and*

$$\delta'(T) := \sup_{x \in M} \limsup_{\sigma \to 0} \sup_{y,z \in B(x,\sigma)\setminus\{x\}} \|Ty - Tz\|. \tag{3.1}$$

*Then for all $\varepsilon > 0$ there exists a point $x^* \in M$ such that*

$$\|x^* - Tx^*\| \le \delta'(T) + \varepsilon. \tag{3.2}$$

Cromme and Diener [4, p. 261] remarked that, in (3.2), $\delta'(T)$ can be replaced by $(n/(n+1))\delta'(T)$. But they omitted the proof. Since it is not obvious, we like to show it now. To do it, we need an assertion based on Kakutani's fixed-point theorem [8], which was used in [4] for proving Theorem 3.3, namely

**Proposition 3.4.** [4] *Let $M$ be a compact and convex subset of $\mathbb{R}^n$ which contains more than one point. Let $T : M \to M$ be any mapping. We define a set-valued map $H_T$ on $M$ by*

$$H_T(x) := \{y \in M : \exists \text{ sequence } x_i \to x,\ x_i \ne x \text{ such that } Tx_i \to y\}. \tag{3.3}$$

*Then there exists a point $x^* \in M$ with $x^* \in \operatorname{conv} H_T(x^*)$.*

Before stating the next tool, let us mention the definition of the *diameter* $D(S)$ and the *radius* $R(S)$ of the circumscribed ball of a bounded subset $S$ in some normed space $(\mathbb{R}^n, \|\cdot\|)$:

$$D(S) := \sup_{x,y \in S} \|x - y\|, \qquad R(S) := \inf_{x \in \mathbb{R}^n} \sup_{y \in S} \|x - y\|. \tag{3.4}$$

Since $g(x) := \sup_{y\in S}\|x-y\|$ is a convex function on $\mathbb{R}^n$ (as the supremum of a family of convex functions), it is continuous. Moreover, if $S$ is bounded and $\alpha := g(z)$ for some arbitrary fixed $z \in \mathbb{R}^n$, then the level set $\mathcal{L}_\alpha(g)$ of $g$ is compact. Therefore, there exist points $c \in \mathbb{R}^n$ satisfying

$$\sup_{y\in S}\|c-y\| = g(c) = \inf_{x\in\mathbb{R}^n} g(x) = \inf_{x\in\mathbb{R}^n}\sup_{y\in S}\|x-y\| = R(S), \tag{3.5}$$

which are called *centers* of the circumscribed ball of a bounded subset $S$. In general, there may be several centers, even outside of clconv$S$. For instance, if $S = \{(0,-1),(0,1)\} \subset \mathbb{R}^2$ and $\|\cdot\|$ is the maximum norm, then $R(S) = 1$ and

$$\sup_{y\in S}\|c-y\| = 1 \quad \text{for all} \quad c \in \{(\xi,0)\in\mathbb{R}^2 : |\xi| \le 1\},$$

under which only $c = (0,0) \in$ clconv$S$. But if $(\mathbb{R}^n, \|\cdot\|)$ is strictly convex (i.e., if the closed ball $\bar{B}(0,1) = \{x \in \mathbb{R}^n : \|x\| \le 1\}$ is strictly convex), then this center is unique and belongs to clconv$S$.

**Proposition 3.5.** *Let $S$ be a bounded subset of some normed space $(\mathbb{R}^n, \|\cdot\|)$ and $z \in convS \setminus S$. Then there exists $s \in S$ such that*

$$\|z-s\| \le \frac{n}{n+1} D(S). \tag{3.6}$$

*Proof.* By Carathéodory's theorem [3], there is a $k$-dimensional simplex

$$S^k := \text{conv}\{x_1, x_2, \ldots, x_{k+1}\}$$

with $k \le n$, $\{x_1, x_2, \ldots, x_{k+1}\} \subset S$, and $z \in S^k$. Consider the function

$$f(x) := \sum_{i=1}^{k+1}\|x-x_i\|.$$

Obviously,

$$f(x_j) = \sum_{i=1}^{k+1}\|x_j-x_i\| \le k \max_{1\le i\le k+1}\|x_j-x_i\| \le k\,D(S) \quad \text{for } j = 1,2,\ldots,k+1.$$

Since $f(.)$ is convex and $z \in$ conv$S$, we have

$$(k+1)\min_{1\le i\le k+1}\|z-x_i\| \le f(z) \le \max_{1\le j\le k+1} f(x_j) \le k\,D(S),$$

which implies

$$\min_{1\le i\le k+1}\|z-x_i\| \le \frac{k}{k+1}D(S) \le \frac{n}{n+1}D(S),$$

i.e., (3.6) holds for $s \in S$ satisfying $\|z-s\| = \min_{1\le i\le k+1}\|z-x_i\|$. ∎

We are now in a position to improve the assertion of Theorem 3.3.

**Theorem 3.6.** *Assume $M$ is a compact and convex subset of some normed space $(\mathbb{R}^n, \|\cdot\|)$. Let $T : M \to M$ be any mapping with the measure of discontinuity $\delta'(T) < \infty$ defined in* (3.1). *Then for all $\varepsilon > 0$ there exists a point $x^* \in M$ such that*

$$\|x^* - Tx^*\| \le \frac{n}{n+1}\,\delta'(T) + \varepsilon. \tag{3.7}$$

*Proof.* Consider the set-valued map $H_T$ defined by (3.3). Due to Proposition 3.4, there exists a point $s^* \in M$ with $s^* \in \mathrm{conv}H_T(s^*)$. Formula (3.1) implies $D(H_T(s^*)) \le \delta'(T)$. Therefore, by Proposition 3.5, there is an $s \in H_T(s^*)$ such that

$$\|s^* - s\| \le \frac{n}{n+1}\,\delta'(T). \tag{3.8}$$

Due to definition (3.3), for any $\varepsilon > 0$, there exists an $x^* \in M$ with $\|x^* - s^*\| \le \varepsilon/2$ and $\|Tx^* - s\| \le \varepsilon/2$. Consequently,

$$\|x^* - Tx^*\| \le \|x^* - s^*\| + \|s^* - s\| + \|s - Tx^*\| \le \frac{n}{n+1}\,\delta'(T) + \varepsilon, \tag{3.9}$$

i.e., (3.7) holds true. ∎

For Euclidean spaces we obtain a better result, namely,

**Theorem 3.7.** *Assume $M$ is a compact and convex subset of the Euclidean space $\mathbb{R}^n$. Let $T : M \to M$ be any mapping with the measure of discontinuity $\delta'(T) < \infty$. Then for all $\varepsilon > 0$ there exists a point $x^* \in M$ such that*

$$\|x^* - Tx^*\| \le \sqrt{\frac{n}{2(n+1)}}\,\delta'(T) + \varepsilon.$$

In order to prove this theorem, we use the following relation found by Jung [7].

**Proposition 3.8.** [7] *For any bounded closed subset $S$ of the Euclidean space $\mathbb{R}^n$, the following inequality holds between the diameter $D(S)$ of $S$ and the radius $R(S)$ of the circumscribed ball:*

$$R(S) \le \sqrt{\frac{n}{2(n+1)}}\,D(S). \tag{3.10}$$

*The inequality* (3.10) *becomes an equality if $S$ is a regular simplex.*

Note that (3.10) is also known as Young's inequality [6, p. 414]. This leads to the following assertion which is similar to Proposition 3.5.

**Proposition 3.9.** *Let $S$ be a bounded subset of some normed space $(\mathbb{R}^n, \|\cdot\|)$ and $z \in \mathrm{conv}S \setminus S$. Then there exists $s \in S$ such that $\|z - s\| \le R(S)$. In particular, if $\|\cdot\|$ is the Euclidean norm, then*

$$\|z - s\| \leq \sqrt{\frac{n}{2(n+1)}}\, D(S). \tag{3.11}$$

*Proof.* We prove by induction that if $k \leq n$ and

$$z = \sum_{i=1}^{k+1} \lambda_i x_i \ \text{ for } x_i \in S,\ \lambda_i > 0,\ 1 \leq i \leq k+1, \text{ and } \sum_{i=1}^{k+1} \lambda_i = 1, \tag{3.12}$$

then

$$\min_{1 \leq i \leq k+1} \|z - x_i\| \leq R(\{x_1, x_2, \ldots, x_{k+1}\}). \tag{3.13}$$

It is obvious for $k = 1$. Assume that this assertion is true for $1 \leq k \leq l < n$. We now consider the case $k = l + 1$.

Let $c$ be a minimizer of the convex function $g(x) := \sup_{1 \leq i \leq l+1} \|x - x_i\|$ on the linear hull $L := \mathrm{span}\{x_1, x_2, \ldots, x_{l+1}\}$, i.e.,

$$g(c) = \sup_{1 \leq i \leq l+1} \|c - x_i\| = \inf_{x \in L} \sup_{1 \leq i \leq l+1} \|x - x_i\| = R(\{x_1, x_2, \ldots, x_{l+1}\}). \tag{3.14}$$

Then the ray from $c$ through $z$ cuts the boundary of $\mathrm{conv}\{x_1, x_2, \ldots, x_{l+1}\}$ at a point $z'$, i.e., $z$ lies in the segment $[c, z']$ and

$$z' \in \mathrm{conv}\{x_{i_1}, x_{i_2}, \ldots, x_{i_l}\} \text{ for some } \{x_{i_1}, x_{i_2}, \ldots, x_{i_l}\} \subset \{x_1, x_2, \ldots, x_{l+1}\}.$$

By induction assumption, there is some $s \in \{x_{i_1}, x_{i_2}, \ldots, x_{i_l}\} \subset S$ such that

$$\|z' - s\| \leq R(\{x_{i_1}, x_{i_2}, \ldots, x_{i_l}\}) \leq R(\{x_1, x_2, \ldots, x_{l+1}\}).$$

Therefore, (3.14) and $z \in [c, z']$ imply

$$\begin{aligned}\min_{1 \leq i \leq l+1} \|z - x_i\| &\leq \|z - s\| \leq \max\{\|c - s\|, \|z' - s\|\} \\ &\leq R(\{x_1, x_2, \ldots, x_{l+1}\}),\end{aligned}$$

i.e., (3.13) holds for $k = l + 1$, too. Hence, (3.12) implies (3.13) for all $k \leq n$.

If $z \in \mathrm{conv}S \backslash S$, then, by Carathéodory's theorem [3], $z$ can be represented as in (3.12), which yields (3.13). Consequently, for $s \in \{x_1, x_2, \ldots, x_{k+1}\}$ satisfying

$$\|z - s\| = \min_{1 \leq i \leq k+1} \|z - x_i\|,$$

we have

$$\|z - s\| \leq R(\{x_1, x_2, \ldots, x_{k+1}\}) \leq R(S).$$

In particular, if $\|\cdot\|$ is the Euclidean norm, then (3.10) implies (3.11). ∎

*Proof of Theorem 3.7.* We only have to do the same as in the proof of Theorem 3.6. The main change is that Proposition 3.9 is used instead of Proposition 3.5, and in (3.8)–(3.9), the factor $n/(n+1)$ must be replaced by $(n/2(n+1))^{1/2}$. ∎

*Remark 3.1.* Bohnenblust (1938) found an analogy to Jung's inequality in Minkowski geometry. Due to [2], for any bounded closed subset $S$ of some normed space $(\mathbb{R}^n, \|\cdot\|)$, the following inequality holds between the diameter $D(S)$ of $S$ and the radius $R(S)$ of the circumscribed ball:

$$R(S) \leq \frac{n}{n+1} D(S). \tag{3.15}$$

(Leichtweiss [10] showed that $n/(n+1)$ is the best factor by showing cases where the equality holds true.) This inequality and the first part of Proposition 3.9 imply immediately Proposition 3.5. Nevertheless, we prefer to choose a direct proof for Proposition 3.5 as it has been done.

*Remark 3.2.* If $T : M \to M$ is an $r$-roughly $k$-contractive mapping on a compact and convex subset $M$ of some normed space $(\mathbb{R}^n, \|\cdot\|)$ (for $r > 0$ and $k \in (0,1)$), then (1.2) and (3.1) imply that $\delta'(T) \leq r$. Therefore, Theorems 3.6 and 3.7 yield that, for all $\varepsilon > 0$, there exists a point $x^* \in M$ such that

$$\|x^* - Tx^*\| \leq \frac{n}{n+1} r + \varepsilon.$$

In particular, if $\|\cdot\|$ is the Euclidean norm, then

$$\|x^* - Tx^*\| \leq \sqrt{\frac{n}{2(n+1)}}\, r + \varepsilon.$$

Next, we like to show that it remains true if $M$ is not necessarily bounded. Moreover, the term $\varepsilon$ in these inequalities can be eliminated. For this elimination, we need the following.

**Proposition 3.10.** *Suppose that $x_1, x_2, \ldots, x_k$ belong to some strictly convex normed space $(\mathbb{R}^n, \|\cdot\|)$ $(n \geq 1)$, and $z \in conv\{x_1, x_2, \ldots, x_k\}$. Then*

$$\min_{1 \leq i \leq k} \|z - x_i\| \geq R(\{x_1, x_2, \ldots, x_k\}) \tag{3.16}$$

*implies*

$$\|z - x_i\| = R(\{x_1, x_2, \ldots, x_k\}) \;\forall i = 1, 2, \ldots, k. \tag{3.17}$$

*Proof.* Let us prove by induction. It is clear that the above implication is true for $k = 2$. Assume that the assertion is true for $2 \leq k \leq l$. We have to prove it for $k = l + 1$.

Let (3.16) hold true for some set $\{x_1, x_2, \ldots, x_{l+1}\}$ $(k = l + 1)$, and let $c$ be the center of its circumscribed ball (see the remark after the definition (3.4) for the existence of this center). Next, we show that the case $z \neq c$ is impossible. For this purpose, we use the following property of a strictly convex normed space which can be verified easily:

$$w \in (v, y) \Rightarrow \|w - x\| < \max\{\|v - x\|, \|y - x\|\}, \tag{3.18}$$

where $v$, $w$, $x$, and $y$ are any points of this space, and $(v, y)$ denotes the open segment connected $v$ and $y$.

Assume the contrary that $z \neq c$. There are only two cases:

$$z \in \mathrm{int}(\mathrm{conv}\{x_1, x_2, \dots, x_{l+1}\}) \tag{3.19}$$

and

$$z \in \mathrm{conv}\{x_{i_1}, x_{i_2}, \dots, x_{i_l}\} \text{ for some } \{x_{i_1}, x_{i_2}, \dots, x_{i_l}\} \subset \{x_1, x_2, \dots, x_{l+1}\}. \tag{3.20}$$

If (3.19) holds, denote by $z'$ the cutting point between the ray from $c$ through $z$ and the boundary of $\mathrm{conv}\{x_1, x_2, \dots, x_{l+1}\}$, i.e.,

$$z' \in \mathrm{conv}\{x_{i_1}, x_{i_2}, \dots, x_{i_l}\} \text{ for some } \{x_{i_1}, x_{i_2}, \dots, x_{i_l}\} \subset \{x_1, x_2, \dots, x_{l+1}\}$$

and $z$ lies in the open segment $(c, z')$. Then (3.4), (3.16), and (3.18) imply

$$\|z' - x_i\| > \|z - x_i\| \geq R(\{x_1, x_2, \dots, x_{l+1}\}) \geq R(\{x_{i_1}, x_{i_2}, \dots, x_{i_l}\})$$

for all $i \in \{i_1, i_2, \dots, i_l\}$, which conflicts with the assumption that (3.16) yields (3.17) for $k \leq l$.

If (3.20) is true, then it follows from

$$\|z - x_i\| \geq R(\{x_1, x_2, \dots, x_{l+1}\}) \geq R(\{x_{i_1}, x_{i_2}, \dots, x_{i_l}\}) \ \forall i \in \{i_1, i_2, \dots, i_l\},$$

and the induction assumption that

$$\|z - x_i\| = R(\{x_{i_1}, x_{i_2}, \dots, x_{i_l}\}) = R(\{x_1, x_2, \dots, x_{l+1}\}) \geq \|c - x_i\|$$

for all $i \in \{i_1, i_2, \dots, i_l\}$. Consequently, (3.18) implies

$$\max_{i \in \{i_1, i_2, \dots, i_l\}} \left\| \frac{1}{2}(z + c) - x_i \right\| < R(\{x_{i_1}, x_{i_2}, \dots, x_{i_l}\}),$$

which conflicts with (3.4).

Hence, only $z = c$ is possible. For this case, (3.4) and (3.16) imply (3.17). ∎

*Remark 3.3.* The assumption of strict convexity in Proposition 3.10 can be neglected if $k = 2$. For $k \geq 3$, this assumption is really needed, as the following example shows. Let $\|\cdot\|$ be the maximum norm, and $\{(0,0), (0,2), (2,0)\} \subset \mathbb{R}^2$. Then

$$\begin{aligned} R(\{(0,0), (0,2), (2,0)\}) &= \|(1,1) - (0,0)\| = \|(1,1) - (0,2)\| \\ &= \|(1,1) - (2,0)\| = 1 \end{aligned}$$

and each $z = (t, 1)$, $0 \leq t < 1$, satisfies

$$\|(t,1) - (0,0)\| = \|(t,1) - (0,2)\| = 1, \text{ but } \|(t,1) - (2,0)\| = 2 - t > 1,$$

i.e., (3.16) is true, but (3.17) does not hold.

One of the main results of this paper is

**Theorem 3.11.** *Let $T : M \to M$ be an $r$-roughly $k$-contractive mapping on a closed and convex subset $M$ of some normed space $(\mathbb{R}^n, \|\cdot\|)$ (for $r > 0$ and $k \in (0,1)$). Then, for all $\varepsilon > 0$, there exists a point $x^* \in M$ such that*

$$\|x^* - Tx^*\| \leq \frac{n}{n+1} r + \varepsilon. \tag{3.21}$$

*If the normed space $(\mathbb{R}^n, \|\cdot\|)$ is strictly convex, then (3.21) even holds true for $\varepsilon = 0$. In particular, if $\|\cdot\|$ is the Euclidean norm, then there exists a point $x^* \in M$ such that*

$$\|x^* - Tx^*\| \leq \sqrt{\frac{n}{2(n+1)}}\, r. \tag{3.22}$$

*Proof.* Let $x_0$ be an arbitrary point of $M$. Denote by $\hat{B}$ the closed ball with center $x_0$ and radius $\bar{r} := (r + \|x_0 - Tx_0\|)/(1-k)$. For $x \in \hat{B}$, (1.2) implies

$$\|Tx - x_0\| - \|x_0 - Tx_0\| \leq \|Tx - Tx_0\| \leq k\|x - x_0\| + r.$$

Therefore,

$$\|Tx - x_0\| \leq k\|x - x_0\| + r + \|x_0 - Tx_0\| \leq k\bar{r} + (1-k)\bar{r} = \bar{r},$$

i.e., $Tx \in \hat{B}$. Hence, $T$ maps the compact and convex subset $M \cap \hat{B}$ into itself. Therefore, due to Remark 3.2, for all $\varepsilon > 0$, there exists a point $x^* \in M$ such that (3.21) holds.

Assume now that the normed space $(\mathbb{R}^n, \|\cdot\|)$ is strictly convex. Applying Proposition 3.4 for $T$ as a mapping from $M \cap \hat{B}$ into itself, we obtain the existence of $s \in M \cap B$ satisfying $s \in \mathrm{conv} H_T(s)$, which of course yields

$$s \in \mathrm{conv} S \quad \text{with} \quad S := H_T(s) \cup \{Ts\}.$$

Assume $s \in \mathrm{conv} S \setminus S$ (the case $s \in S$ is obvious). It is easy to see that there are $x_1, x_2, ..., x_k \in H_T(s)$ such that

$$s \in \mathrm{conv}\{Ts, x_1, x_2, \ldots, x_k\}.$$

Proposition 3.10 yields that at least one of the following cases must appear:
(a) $\|s - Ts\| \leq R(\{Ts, x_1, x_2, \ldots, x_k\})$,
(b) $\|s - x_i\| < R(\{Ts, x_1, x_2, \ldots, x_k\})$ for some $i \in \{1, 2, \ldots, k\}$.
In the case (b), (3.3) implies for

$$\rho := R(\{Ts, x_1, x_2, \ldots, x_k\}) - \|s - x_i\| > 0$$

that there exists $\tilde{s} \in M$ with $\|s - \tilde{s}\| \leq \rho/2$ and $\|x_i - T\tilde{s}\| \leq \rho/2$. Consequently,

$$\|\tilde{s} - T\tilde{s}\| \leq \|\tilde{s} - s\| + \|s - x_i\| + \|x_i - T\tilde{s}\| \leq R(\{Ts, x_1, x_2, \ldots, x_k\}).$$

Hence, we can always say that there is some $x^* \in M$ such that

$$\|x^* - Tx^*\| \leq R(\{Ts, x_1, x_2, \ldots, x_k\}). \tag{3.23}$$

Since

$$D(\{Ts, x_1, x_2, \ldots, x_k\}) \leq r \tag{3.24}$$

follows from the definition of $r$-roughly $k$-contractive mappings in (1.2) and the definition of $H_T(.)$ in (3.3), (3.23) and Bohnenblust's inequality (3.15) yield that (3.21) even holds true for $\varepsilon = 0$.

If $\|\cdot\|$ is the Euclidean norm, then (3.23)–(3.24) and Jung's inequality (3.10) imply (3.22). ■

For $n = 1$, Theorem 3.11 yields that each $r$-roughly $k$-contractive mapping $T : M \to M$ on some closed interval $M \subset \mathbb{R}$ admits at least one $\gamma$-invariant point with $\gamma = r/2$. For instance, since the mapping $\tilde{T} : \mathbb{R} \to \mathbb{R}$ defined by (2.6) is $r$-roughly $k$-contractive, $\mathbb{R}$ is closed and convex. Therefore, it must possess a $\gamma$-invariant point with $\gamma = r/2$. In fact,

$$\tilde{T}0 - 0 = \frac{r}{2} - 0 = \frac{r}{2},$$

i.e., 0 is a $\gamma$-invariant point of $\tilde{T}$ with $\gamma = r/2$.

## 4. Concluding Remarks

As pointed out in Sec. 2, the iteration (2.1) can be used for seeking $\gamma$-invariant points of $r$-roughly $k$-contractive mappings if $\gamma \geq r/(1-k)$. For smaller $\gamma$, this scheme is no longer suitable. One can modify some methods described in [12] for finding $\gamma$-invariant points with $\gamma > r$, as discussed in [5]. For $r$-roughly $k$-contractive mappings on some bounded and closed interval of $\mathbb{R}^1$, an algorithm is given in [11] and [13] for determining $\gamma$-invariant points with $\gamma > r/2$.

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## References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, *Fund. Math.* **3** (1922) 133-181.
2. F. Bohnenblust, Convex regions and projections in Minkowski spaces, *Ann. of Math.* **39** (1938) 301-308.
3. C. Carathéodory, Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen, *Rend. Circ. Mat. Palermo* **32** (1911) 193-217.

4. L. J. Cromme and I. Diener, Fixed point theorems for discontinuous mapping, *Math. Programming* **51** (1991) 257-267.
5. L. J. Cromme, Fixed point theorems for discontinuous functions and applications, *Nonlinear Analysis, Theory, Method & Applications* **30** (1997) 1527-1534.
6. A. D. Ioffe and V. M. Tihomirov, *Theory of Extremal Problems*, North-Holland Publishing Company, Amsterdam, 1979.
7. H. Jung, Über die kleinste Kugel, die eine räumliche Figur einschliesst, *J. Reine Angew. Math.* **123** (1901) 241-257.
8. S. Kakutani, A generalization of Brouwer's fixed point theorem, *Duke Math. J.* **8** (1941) 457-459.
9. V. Klee, Stability of the fixed-point property, *Coll. Math.* **8** (1961) 43-46.
10. K. Leichtweiss, Zwei Extremalprobleme der Minkowski-Geometrie, *Math. Z.* **62** (1955) 37-49.
11. H. X. Phu and T. V. Truong, Some results on approximately fixed points for the class of roughly contractive mappings, Preprint 99/8, Institute of Mathematics, National Centre for Sciences and Technology of Vietnam, 1999.
12. M. J. Todd, *The Computation of Fixed Points and Applications*, Springer-Verlag, New York, 1976.
13. T. V. Truong, *Some Problems on the Fixed-Points Theory*, M. Sc. Thesis, Institute of Mathematics, National Centre for Natural Sciences and Technology of Vietnam, 1998.
14. E. Zeidler, *Nonlinear Functional Analysis and Its Applications I: Fixed-Point Theorems*, Springer-Verlag, New York, 1986.