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Sufficient Optimality Conditions Under Invexity Hypotheses

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Abstract. First-order sufficient optimality conditions are given when the objective function of a minimization problem is pseudo-invex and second-order sufficiency optimality conditions are established when the constraint mappings are invex.

1. Introduction

In the last three decades, theory of sufficient optimality conditions for optimization problems has been the subject of much development. The classical second-order sufficiency conditions for mathematical programs can be found in the works of Hestenes [9] and McCormick [14]. These results have been extended to Banach space by Ioffe [10], Ioffe and Tikhomirov [11], Maurer and Zowe [13], etc. Second-order sufficient optimality conditions for the mathematical programs comprised of locally Lipschitz functions were studied by Chaney [1].

To meet the demand of the theory of extremum problems, theory of invex functions came into being in 1981 (see, e.g., [3,15]). Under invexity hypotheses, Craven and Luu [4 - 6] established optimality conditions for constrained minimax. Note that in sufficiency conditions given by Craven [3], Reiland [15], Craven and Luu [4 - 6], the objective and constraint functions are invex all together.

In this paper, sufficient optimality conditions will be derived under invexity hypotheses only on the objective function or only on the constraint functions.

The paper is organized as follows. After the introduction, Sec. 2 gives some preliminaries on invex functions and the various cones studied by Dubovitsky and Milyutin [7], together with some related results. Section 3 deals with a sufficient optimality condition for problems comprising pseudo-invex objective functions and convex constraint sets. Section 4 is devoted to the study of the problems with invex constraint mappings. If the constraint mapping is K-invex, we shall show that the corresponding feasible set can be approximated in the sense of Maurer and Zowe [13]. Moreover, a constraint mapping g satisfying a stability condition of Robinson type at a point $\bar{x}$ will be K-invex at $\bar{x}$ with respect to the scale mapping $\omega(x,\bar{x}) = x - \bar{x} + o(\|x - \bar{x}\|)$. From these results, second-order sufficient optimality conditions are derived for minimization problems with invex constraint mappings.

2. Preliminaries

Let X be a Banach space. Let $f : X \to R$ be a function which is Fréchet differentiable at $u \in X$.

Following [3] the function f will be called *invex* with respect to a scale mapping $\eta : X \times X \to X$ if

$$f(x) - f(u) \geq f'(u)\eta(x,u) \quad (\forall x, u \in X). \tag{1}$$

If u is fixed, then f will be called invex at u.

Note that if f is convex , then f is invex with the scale function $\eta(x,u) := x - u$. Indeed, with the chosen function η, (1) holds automatically.

Let Y be a Banach space and K a convex cone in Y. A Fréchet differentiable mapping $F : X \to Y$ will be called K-invex at $u \in X$ with respect to a scale mapping $\eta : X \times X \to X$ if

$$F(x) - F(u) - F'(u)\eta(x,u) \in K \quad (\forall x \in X). \tag{2}$$

If (2) holds whenever $\|x - u\|$ is sufficiently small, then f will be called K-invex with respect to a scale mapping η in a neighborhood of u. If (2) holds for all x in a subset A containing u, then f is called K-invex at u on A.

In the case where f and F are only assumed to be directionally differentiable, (1) and (2) are replaced, respectively, by the following (1') and (2'):

$$f(x) - f(u) \geq f'(u;\eta(x,u)) \quad (\forall x, u \in X), \tag{1'}$$

$$F(x) - F(u) - F'(u;\eta(x,u)) \in K \quad (\forall x \in X), \tag{2'}$$

where $f'(u;\eta)$ and $F'(u;\eta)$ stand for the directional derivatives of f and F at u in direction η, respectively.

Suppose now that f has directional derivatives at $u \in Q \subset X$. The function f is said to be *pseudo-invex* at u on Q if there exists a scale mapping $\eta : X \times X \to X$ such that, for all $x \in Q$,

$$f'(u;\eta(x,u)) \geq 0 \Longrightarrow f(x) - f(u) \geq 0. \tag{3}$$

The function f is said to be *quasi-invex* at u on Q if there exists a scale mapping $\eta : X \times X \to X$ such that for all $x \in Q$,

$$f(x) - f(u) \leq 0 \Longrightarrow f'(u;\eta(x,u)) \leq 0. \tag{4}$$

Taking $\eta(x,u) = x - u$, (3) and (4) become the definitions of pseudo-convex and quasi-convex functions, respectively.

From the definition it follows that f is pseudo-invex at $x_0 \in Q$ if and only if there is a scale mapping $\eta : X \times X \to X$ such that, for all $x \in Q$,

$$f(x) < f(x_0) \implies f'(x_0; \eta(x, x_0)) < 0.$$

If f is invex, f is pseudo-invex. Thus, if f is convex, it is invex, hence it is pseudo-invex .

Following [7], a vector $v \in V$ will be called a *decreasing direction* of f at x_0 if there are a neighborhood U of v, and numbers $\alpha < 0$ and $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$ and $u \in U$,

$$f(x_0 + \epsilon u) \leq f(x_0) + \epsilon\alpha.$$

The set of all the decreasing directions of f at x_0 is an open cone with vertex at the origin. The function f is called *regularly decreasing* at x_0 if the cone of decreasing directions at x_0 is convex .

Let Q_1 be a constraint of the inequality-type of an optimization problem. A vector $v \in X$ is said to be an *admissible direction* of Q_1 at x_0 if there are a neighborhood U of v and a number $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$ and $u \in U$,

$$x_0 + \epsilon u \in Q_1 .$$

Note that the cone of admissible directions at x_0 is open with vertex at the origin. The constraint Q_1 of the inequality-type is called *regular* at x_0, if the cone of admissible directions is convex.

Let Q_2 be a constraint of the equality-type (or the inequality-type) of an optimization problem. A vector $v \in X$ will be called a *tangent direction* of Q_2 at x_0, if there is $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$, there exists $x_\epsilon \in Q_2$ so that

$$x_\epsilon = x_0 + \epsilon v + r(\epsilon),$$

where $r(\epsilon) \in X$ is such that, for every neighborhood U of the origin, $r(\epsilon)/\epsilon \in U$ for all sufficiently small $\epsilon > 0$.

Note that the cone of tangent directions at x_0 may fail to be open or closed. The constraint Q_2 of the equality-type is said to be *regular* at x_0 if the cone of tangent directions is convex.

It should be noted here that any admissible direction is also a tangent direction as well. Indeed, if v is an admissible direction of the inequality-type constraint Q_1 at x_0, by definition, there is $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$,

$$x_\epsilon := x_0 + \epsilon v \in Q_1,$$

which means that v is a tangent direction of Q_1 at x_0 (with $r(\epsilon) \equiv 0$).

We recall some results from [7, 8, 13] which are needed for the next few sections.

Proposition 2.1. [8] *Let f be a real-valued function defined on X. Assume that f is locally Lipschitz at $\bar{x}$; f has directional derivative $f'(\bar{x}; v)$ at $\bar{x}$ in the direction $v \in X$ and $f'(\bar{x}; v) < 0$. Then v is a decreasing direction of f at $\bar{x}$.*

Proposition 2.2. [7] *Assume that $K_1, \ldots, K_n, K_{n+1}$ are convex cones with vertices at the origin. Suppose, furthermore, that the cones $K_1, \ldots, K_n$ are open. Then, $\bigcap_{i=1}^{n+1} K_i = \emptyset$ if and only if there exist $\xi_i \in K_i^*$ $(i = 1, \ldots, n+1)$, not all zero, such that*

$$\xi_1 + \cdots + \xi_{n+1} = 0,$$

where K_i^* is the dual cone of K_i:

$$K_i^* = \{\xi \in X^* : \langle \xi, x \rangle \geq 0, \forall x \in K_i\}.$$

Proposition 2.3. [13] *Let T be a continuous linear mapping from X into Y. Assume that*

$$TX + K = Y.$$

Then there exists a number $\alpha > 0$ such that

$$B_Y(0,1) \subset \alpha(TB_X(0,1) + (K \cap B_Y(0,1)))$$

where $B_X(0,1)$ and $B_Y(0,1)$ are unit balls of X and Y, respectively; K is a closed convex cone in Y.

3. Optimization Problems with Pseudo-Invex Objective Functions

Let X be a Banach space and let f be a function defined on X. Let $Q_1, \ldots, Q_{n+1}$ be subsets of X. In this section we are concerned with the following problem:

$$\text{(P1)} \qquad \begin{cases} \text{minimize } f(x), \\ x \in Q \end{cases}$$

where $Q = \bigcap_{i=1}^{n+1} Q_i$.

This problem was studied by Dubovitsky and Milyutin [7].

Let $\bar{x} \in Q$. Denote by K_0 the cone of decreasing directions of f at $\bar{x}$; denote by $K_1, \ldots, K_n$ the cones of admissible directions of the inequality-type constraints $Q_1, \ldots, Q_n$ at $\bar{x}$, respectively; denote by K_{n+1} the cone of tangent directions of the equality-type constraint Q_{n+1} at $\bar{x}$.

A sufficient condition for optimality can be stated as follows:

Theorem 3.1. *Assume that*

(a) *The function f is locally Lipschitz at $\bar{x}$ and it has directional derivative at $\bar{x}$ in any directions; f is pseudo-invex at $\bar{x}$ on Q with the scale function:*

$$\omega(x, \bar{x}) = x - \bar{x} + r(x, \bar{x}),$$

where

$$\|r(x, \bar{x})\| / \|x - \bar{x}\| \longrightarrow 0 \text{ whenever } \|x - \bar{x}\| \to 0;$$

(b) *$Q_1, \ldots, Q_{n+1}$ are convex sets such that there exists*

$$\hat{x} \in \bigcap_{i=1}^{n} (\operatorname{int} Q_i) \cap Q_{n+1},$$

where $\operatorname{int} Q_i$ *denotes the interior of the set* Q_i $(i = 1, \ldots, n)$;

(c) *There exist* $\xi_i \in K_i^*$ $(i = 0, 1, \ldots, n+1)$, *not all zero, such that*

$$\xi_0 + \xi_1 + \ldots + \xi_{n+1} = 0.$$

Then $\bar{x}$ *is a local minimum of* f *over* Q.

Proof. Suppose that $\bar{x}$ is not a local minimum of the function f over Q. Then, for every neighborhood B of $\bar{x}$, there exists $x_1 \in Q \cap B$ such that

$$f(x_1) < f(\bar{x}). \tag{5}$$

Especially, taking $B = X$, the inequality (5) holds for $x_1 \in Q$.

For $\lambda \in (0,1)$ we denote $x_\lambda := \lambda\hat{x} + (1-\lambda)x_1$. For $i = 1, \ldots, n+1$, since Q_i is a convex set, $\hat{x} \in Q_i$ and $x_1 \in Q_i$, it follows that $x_\lambda \in Q_i$. Hence, $x_\lambda \in Q$.

Since $\hat{x} \in \operatorname{int} Q_i$, it follows that $x_\lambda \in \operatorname{int} Q_i$ $(i = 1, \ldots, n)$ for any $\lambda \in (0,1)$. In view of the continuity of f, for sufficiently small $\lambda > 0$,

$$f(x_\lambda) < f(\bar{x}). \tag{6}$$

According to the hypotheses, f is a pseudo-invex function at $\bar{x} \in Q$ over Q, with the scale function

$$\omega(x, \bar{x}) = x - \bar{x} + r(x, \bar{x}),$$

in which $\|r(x,\bar{x})\| / \|x - \bar{x}\| \longrightarrow 0$ as $x \to \bar{x}$.

We now prove that, for sufficiently small $\lambda > 0$, $\omega(x_\lambda, \bar{x}) = x_\lambda - \bar{x} + r(x_\lambda, \bar{x}) \in K_0$.

Assume the contrary that $\omega(x_\lambda, \bar{x}) \in K_0$, i.e., $\omega(x_\lambda, \bar{x})$ is not a decreasing direction of f at $\bar{x}$. Then, by virtue of Proposition 2.1, we get

$$f'(\bar{x}; \omega(x_\lambda, \bar{x})) \geq 0.$$

By the definition of pseudo-invexity,

$$f(x_\lambda) \geq f(\bar{x}),$$

which conflicts with (6).

We now prove that $\omega(x_\lambda, \bar{x}) \in K_i$ $(i = 1, \ldots, n+1)$. For $\epsilon \in (0,1)$ we have $\bar{x} + \epsilon(x_\lambda - \bar{x}) \in Q_{n+1}$, and

$$\bar{x} + \epsilon\omega(x_\lambda, \bar{x}) = \bar{x} + \epsilon(x_\lambda - \bar{x}) + \epsilon r(x_\lambda, \bar{x}).$$

Hence,

$$\bar{x} + \epsilon\omega(x_\lambda, \bar{x}) + \epsilon(-r(x_\lambda, \bar{x})) \in Q_{n+1},$$

where $r(x_\lambda, \bar{x}) = o(\|x_\lambda - \bar{x}\|)$.

Consequently, $\omega(x_\lambda, \bar{x})$ is a tangent direction of Q_{n+1} at $\bar{x}$, i.e., $\omega(x_\lambda, \bar{x}) \in K_{n+1}$.

Moreover, by the aforementioned proof, for each $i = 1, \dots, n$, one has $x_\lambda \in \operatorname{int} Q_i$ for sufficiently small $\lambda > 0$. Therefore,

$$\bar{x} + \epsilon\omega(x_\lambda, \bar{x}) = \bar{x} + \epsilon(x_\lambda - \bar{x}) + \epsilon r(x_\lambda, \bar{x}) \in \operatorname{int} Q_i,$$

for sufficiently small $\epsilon > 0$. Hence,

$$\omega(x_\lambda, \bar{x}) \in \frac{1}{\epsilon}(\operatorname{int} Q_i - \bar{x}) \ (i = 1, \dots, n).$$

Since Q_i is convex, by Theorem 8.2 of [8], we have

$$\frac{1}{\epsilon}(\operatorname{int} Q_i - \bar{x}) \subset K_i \ (i = 1, \dots, n).$$

So we get

$$\omega(x_\lambda, \bar{x}) \in K_i \ (i = 1, \dots, n).$$

Hence,

$$\omega(x_\lambda, \bar{x}) \in \bigcap_{i=0}^{n+1} K_i,$$

or

$$\bigcap_{i=0}^{n+1} K_i \neq \emptyset.$$

By virtue of Proposition 2.2, one cannot find $\xi_i \in K_i^*$ $(i = 0, 1, \dots, n+1)$, not all zero, such that

$$\xi_0 + \xi_1 + \ldots + \xi_{n+1} = 0.$$

This contradicts Assumption (c). The proof is complete. ∎

Remark. It is worth noting that a convex function is invex, and hence it is pseudo-invex. So Theorem 3.1 is valid for problems with convex objective functions.

4. Optimization Problems with Invex Constraint Functions

Let f be a real-valued function defined on a Banach space X. Let g be a mapping from X into another Banach space Y and let K be a closed convex cone in Y. In this section we shall deal with the following problem:

$$\text{(P2)} \qquad \begin{cases} \text{minimize } f(x), \\ \text{subject to } g(x) \in -K. \end{cases}$$

Denote by M the feasible set of (P2), i.e.,

$$M := \{x \in X : \ g(x) \in -K\}.$$

Assume that f and g are Fréchet differentiable of first and second-order at $\bar{x} \in M$ with first-order Fréchet derivatives $f'(\bar{x}), g'(\bar{x})$ and second-order ones $f''(\bar{x}), g''(\bar{x})$.

Denote by $T_M(\bar{x})$ the sequential tangent cone of M at $\bar{x}$:

$$T_M(\bar{x}) := \{v \in X : v = \lim_{t_n \downarrow 0} \frac{x_n - \bar{x}}{t_n},\ x_n \in M\},$$

and denote by $L_M(\bar{x})$ the linearizing cone of M at $\bar{x}$:

$$L_M(\bar{x}) := \{v \in X : g'(\bar{x})v \in -K_{g(\bar{x})}\},$$

where

$$K_{g(\bar{x})} := K + \{\lambda g(\bar{x}) : \lambda \in R\}.$$

Note that if $K_{g(\bar{x})}$ is closed, then

$$T_M(\bar{x}) \subset L_M(\bar{x})$$

(see [13]).

Following [13], the feasible set M is said to be approximated at $\bar{x} \in M$ by $L_M(\bar{x})$, if there exists a mapping $\xi : M \to L_M(\bar{x})$ such that, for every $x \in M$,

$$\|\xi(x) - (x - \bar{x})\| = o(\|x - \bar{x}\|).$$

Theorem 4.1. *Assume that the mapping g is K-invex at $\bar{x}$ on M, with respect to a scale mapping ω satisfying:*

$$\omega(x, \bar{x}) = x - \bar{x} + r(x, \bar{x}),$$

where $\|r(x, \bar{x})\| / \|x - \bar{x}\| \longrightarrow 0$ as $x \to \bar{x}$. Then M is approximated at $\bar{x}$ by $L_M(\bar{x})$.

Proof. According to the hypothesis, the mapping g is K-invex at $\bar{x}$ on M, that is,

$$g(x) - g(\bar{x}) - g'(\bar{x})\omega(x, \bar{x}) \in K \quad (\forall x \in M)$$

which implies that, for every $x \in M$,

$$\begin{aligned} g(\bar{x}) + g'(\bar{x})\omega(x, \bar{x}) &\in -K + g(x) \\ &\subset -K - K \subset -K, \end{aligned}$$

as K is a closed convex cone.

Hence, for every $x \in M$,

$$\begin{aligned} g'(\bar{x})\omega(x, \bar{x}) &\in -K - g(\bar{x}) \\ &\subset -K - \{\lambda g(\bar{x}) : \lambda \in R\} = -K_{g(\bar{x})}, \end{aligned}$$

whence

$$\omega(x, \bar{x}) \in L_M(\bar{x}).$$

So we get a mapping $\omega(., \bar{x}) : M \to L_M(\bar{x})$ with

$$\omega(x, \bar{x}) = x - \bar{x} + r(x, \bar{x}).$$

By hypothesis

$$\|\omega(x,\bar{x})-(x-\bar{x})\|=\|r(x,\bar{x})\|=o(\|x-\bar{x}\|),$$

hence, M is approximated by $L_M(\bar{x})$ at $\bar{x}$. ■

A sufficient condition for K-invexity can be stated as follows.

Theorem 4.2. *Assume that the following stability condition of Robinson-type is fulfilled:*

$$0 \in \operatorname{int}\{g'(\bar{x})X+K\}. \tag{7}$$

Then the mapping g is K-invex at $\bar{x}$ on M with a scale mapping of the form:

$$\omega(x,\bar{x})=x-\bar{x}+o(\|x-\bar{x}\|).$$

Proof. It is easy to see that condition (7) is equivalent to the condition

$$g'(\bar{x})X+K=Y. \tag{8}$$

Since g is Fréchet differentiable at $\bar{x}$, we have

$$g(x)-g(\bar{x})=g'(\bar{x})(x-\bar{x})+r(x,\bar{x}),$$

where $\|r(x,\bar{x})\|/\|x-\bar{x}\| \longrightarrow 0$ as $x\to\bar{x}$ or, the same, $r(x,\bar{x})=o(\|x-\bar{x}\|)$.

By Proposition 2.3, there exists $\alpha>0$ such that, for every $x\in M$, there exist elements $y=y(x)\in -K$ and $z=z(x)\in\alpha\|r(x,\bar{x})\|B(0,1)$ such that

$$r(x,\bar{x})=g'(\bar{x})z-y.$$

Putting

$$\omega(x,\bar{x})=x-\bar{x}+z,$$

one gets

$$\|\omega(x,\bar{x})-(x-\bar{x})\|\leq\alpha\|r(x,\bar{x})\|.$$

This implies $\|\omega(x,\bar{x})-(x-\bar{x})\|=o(\|x-\bar{x}\|)$.

Moreover,

$$\begin{aligned}g'(\bar{x})\omega(x,\bar{x})&=g'(\bar{x})(x-\bar{x})+g'(\bar{x})z\\&=g'(\bar{x})(x-\bar{x})+r(x,\bar{x})+y\\&=g(x)-g(\bar{x})+y.\end{aligned}$$

Hence,

$$g(x)-g(\bar{x})-g'(\bar{x})\omega(x,\bar{x})=-y\in K.$$

Thus g is K-invex at $\bar{x}$ with the scale mapping

$$\omega(x,\bar{x})=x-\bar{x}+r(x,\bar{x}).$$ ■

Denote by $L(x,y^*,\lambda)$, the Lagrange function for problem (P2):

$$L(x, y^*, \lambda) = \lambda f(x) + \langle y^*, g(x) \rangle.$$

We shall need the following auxiliary result.

Theorem 4.3. [13] *Let $\bar{x} \in M$. Assume that the following conditions are fulfilled:*
(a) *The feasible set M is approximated by $L_M(\bar{x})$ at $\bar{x}$;*
(b) *There exists $\bar{y}^* \in Y^*$ such that*

$$L'_x(\bar{x}, \bar{y}^*, 1) = 0,$$
$$\langle \bar{y}^*, g(\bar{x}) \rangle = 0;$$

(c) *There are numbers $\delta > 0$ and $\beta > 0$ such that*

$$L''_{xx}(\bar{x}, \bar{y}^*, 1)(v, v) \geq \delta \|v\|^2$$
$$(\forall v \in L_M(\bar{x}) \cap \{v : \langle \bar{y}^*, g'(\bar{x})v \rangle \leq \beta \|v\|\}).$$

Then, there exist numbers $\alpha > 0$ and $\rho > 0$ such that

$$f(x) \geq f(\bar{x}) + \alpha \|x - \bar{x}\| \quad (\forall x \in M \cap B(\bar{x}; \rho)),$$

which means $\bar{x}$ is a strictly local minimum of Problem(P2)*, where $B(\bar{x}; \rho)$ stands for the closed ball around $\bar{x}$ with radius $\rho > 0$.*

Now we can formulate second-order sufficient optimality condition for minimization problems with invex constraint functions.

Theorem 4.4. *Let $\bar{x} \in M$. Assume that Assumptions* (b), (c) *of Theorem* 4.3 *hold. Suppose, in addition, that the mapping g is K-invex at $\bar{x}$ on M with a scale mapping ω satisfying*

$$\omega(x, \bar{x}) = x - \bar{x} + r(x, \bar{x}) \quad (\forall x \in M),$$

where $\|r(x, \bar{x})\| / \|x - \bar{x}\| \longrightarrow 0$ as $x \to \bar{x}$. Then there exist numbers $\alpha > 0$ and $\rho > 0$ such that

$$f(x) \geq f(\bar{x}) + \alpha \|x - \bar{x}\| \quad (\forall x \in M \cap B(\bar{x}; \rho)).$$

Proof. Since the mapping g is K-invex at $\bar{x}$ on M, it follows from Theorem 4.1 that M is approximated at $\bar{x}$ by $L_M(\bar{x})$. Then all the hypotheses of Theorem 4.3 are satisfied. Applying this theorem, the conclusion follows. ∎

Theorem 4.5. *Let $\bar{x} \in M$. Assume that Assumptions* (b) *and* (c) *of Theorem* 4.3 *hold. Suppose, furthermore, that the following condition is fulfilled:*

$$0 \in \operatorname{int} \{g'(\bar{x})X + K\}.$$

Then there exist numbers $\alpha > 0$ and $\rho > 0$ such that

$$f(x) \geq f(\bar{x}) + \alpha \|x - \bar{x}\| \quad (\forall x \in M \cap B(\bar{x}; \rho)).$$

Proof. Due to Theorem 4.2 the mapping g is K-invex at $\bar{x}$ on M with a scale mapping $\omega(x,\bar{x}) = x - \bar{x} + o(\|x - \bar{x}\|)$. So all the hypotheses of Theorem 4.4 are fulfilled. Applying this theorem the conclusion follows. ∎

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