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*Short Communication*

# On Representable Linearly Compact Modules*

**Nguyen Tu Cuong and Le Thanh Nhan**
*Institute of Mathematics, P. O. Box 631, Bo Ho, Hanoi, Vietnam*

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## 1. Introduction

The concept of linear compactness was introduced by Lefschetz in [4] for vector spaces of infinitive dimension and extended first to modules by Zelinsky in [11] and further studied by Macdonald in [5], Zöschinger in [12]. Note that every Artinian $R$-module is a representable linearly compact $R$-module, but the converse is not true. Melkersson and Schenzel [9] defined the co-localization $\mathrm{Hom}_R(R_S; M)$ of an Artinian $R$-module $M$ with respect to a multiplicative set $S$ in $R$ and they showed that this construction does not usually give an Artinian $R_S$-module. In general, it even does not have finite Goldie-dimension, but we shall see in Sec. 4 that it is always a representable linearly compact $R$-module. The purpose of this paper is not only to extend the results for Artinian modules, which are presented in [9], to representable linearly compact modules, but also to give an affirmative answer to a question of Melkersson [8] in the case when $M$ is a representable linearly compact module without assumption that $M$ has finite Goldie-dimension.

## 2. Linearly Compact Modules

Let $R$ be a commutative topological ring and $M$ a topological $R$-module. A nucleus of $M$ is a neighborhood of the zero element of $M$ and a nuclear base of $M$ is a base for the nuclei of $M$.

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First we recall the concept of linearly compact modules by using the terminology of Macdonald [5].

**Definition 2.1.**
(i) *$M$ is said to be linearly topologized if $M$ has a nuclear base $\mathcal{M}$ consisting of open submodules which satisfies the condition: Given $x \in M$ and $N \in \mathcal{M}$, there exists a nucleus $U$ of $R$ such that $Ux \subseteq N$.*
(ii) *A Hausdorff linearly topologized $R$-module $M$ is said to be linearly compact if $M$ has the following property: If $\mathcal{F}$ is a family of closed cosets (i.e., the cosets of closed submodules) in $M$ which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection.*

It should be mentioned that any Artinian $R$-module is a linearly compact $R$-module with respect to the discrete topology. But the converse is not true (see Sec. 4).

The following theorem is the key result for our further investigations in the next sections and its proof is mainly based on the results of Jensen in [2].

**Theorem 2.2.** *Let $F$ be a flat $R$-module and $M$ a linearly compact $R$-module. Then*
(i) $\mathrm{Hom}_R(F;M)$ *is linearly compact,*
(ii) $\mathrm{Ext}^i_R(F;M) = 0$, *for all $i > 0$.*

The following immediate consequence of Theorem 2.2, which is a generalization of Proposition 2.4 in [9], is often used in the sequel.

**Corollary 2.3.** *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

*be a short exact sequence of linearly compact $R$-modules and $F$ a flat $R$-module. Then the derived sequence*

$$0 \longrightarrow \mathrm{Hom}_R(F;M') \longrightarrow \mathrm{Hom}_R(F;M) \longrightarrow \mathrm{Hom}_R(F;M'') \longrightarrow 0$$

*is also exact.*

## 3. Representability of $\mathrm{Hom}_R(F;M)$

In this section we need the notion of secondary representation which is due to Macdonald [6]. This concept is in some sense dual to that of primary decomposition. An $R$-module $M \neq 0$ is said to be *secondary* if, for any $x \in R$, the multiplication by $x$ on $M$ is either surjective or nilpotent. The radical of the annihilator of $M$ is then a prime ideal $\mathfrak{p}$ and we say that $M$ is $\mathfrak{p}$-secondary.

Let $M$ be an $R$-module. A *secondary representation* of $M$ is an expression of $M$ as a finite sum of secondary submodules, say,

$$M = M_1 + M_2 + \ldots + M_n. \qquad (*)$$

Suppose that $M_i$ is $\mathfrak{p}_i$-secondary for $i = 1, \dots, n$. Then the representation $(*)$ is said to be *minimal* if (i) the prime ideals $\mathfrak{p}_i$ are all distinct and (ii) none of the summands $M_i$ is redundant. Any secondary representation of $M$ can be refined to a minimal one. The prime ideals $\mathfrak{p}_i$ depend only on $M$, but not on the minimal secondary representation of $M$. So we denote by $\mathrm{Att}_R(M)$ the set $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$. It is called the set of prime ideals attached to $M$. The summand $M_i$, $i = 1, \dots, n$, is called *secondary component* of $M$. If $M$ has a secondary representation, then we say that $M$ is *representable.* For convenience, we stipulate that the zero module is representable.

The theorem below is the main result of this section.

**Theorem 3.1.** *Let $F$ be a flat $R$-module and $M$ a repesentable linearly compact $R$-module. Then* $\mathrm{Hom}_R(F; M)$ *is a representable linearly compact $R$-module.*

To prove this theorem we need two following lemmas.

**Lemma 3.2.** *Let $F$ be a flat $R$-module and $M$ a linearly compact $R$-module. If $M$ is $\mathfrak{p}$-secondary, then* $\mathrm{Hom}_R(F; M)$ *is either* 0 *or $\mathfrak{p}$-secondary.*

**Lemma 3.3.** *Let $M$ be a representable linearly compact $R$-module. Then there exists a minimal secondary representation of $M$ in which all the secondary components are linearly compact submodules.*

## 4. Co-Localization

We recall first the notion of co-localization which is due to Melkersson and Schenzel [9]. Let $M$ be an $R$-module and $S$ a multiplicative subset of $R$. The *co-localization* of $M$ with respect to $S$ is the module $\mathrm{Hom}_R(R_S; M)$. It follows from [9] that the co-localization of an Artinian module is almost never Artinian. It even does not finite Goldie-dimension in general, but it is always a linearly compact $R$-module by Theorem 3.1. This shows that the set of Artinian $R$-modules is a proper subset of representable linearly compact $R$-modules. However, many good properties of Artinian modules can be found in representable linearly compact modules.

We can generalize Melkersson–Schenzel's result [9, Theorem 3.2] for representable linearly compact modules as follows:

**Theorem 4.1.** *Let $S$ be a multiplicative set of $R$ and $M$ a representable linearly compact $R$-module with*

$$M = M_1 + M_2 + \cdots + M_n$$

*a minimal secondary representation in which all $M_i$ are linearly compact. Let $\mathfrak{p}_i = \mathrm{Rad}(\mathrm{Ann}_R M_i)$ for $i = 1, \dots, n$ and assume that $S \cap \mathfrak{p}_i = \emptyset$ for $i = 1, \dots, m$ and $S \cap \mathfrak{p}_i \neq \emptyset$ for $i = m+1, \dots, n$, respectively. Then*

$$\mathrm{Hom}_R(R_S; M) = \mathrm{Hom}_R(R_S; M_1) + \mathrm{Hom}_R(R_S; M_2) + \cdots + \mathrm{Hom}_R(R_S; M_m)$$

*is a minimal secondary representation of* $\mathrm{Hom}_R(R_S; M)$. *In particular, we have*

$$\mathrm{Att}_R(\mathrm{Hom}_R(R_S; M)) = \{\mathfrak{p} \in \mathrm{Att}_R(M) : \mathfrak{p} \cap S = \emptyset\}.$$

To prove this result, we apply Theorem 3.1 and the following lemma.

**Lemma 4.2.** *Let $S$ be a multiplicative subset of $R$ and $M$ a linearly compact $R$-module. Let*

$$\varphi : \mathrm{Hom}_R(R_S; M) \longrightarrow M$$

*be a homomorphism defined by $\varphi(f) = f(1)$, for any $f \in \mathrm{Hom}_R(R_S; M)$. Then*

$$\mathrm{Im}\varphi = \bigcap_{s \in S} sM.$$

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