Vietnam Journal of Mathematics **28**:3 (2000) 263-273

Vietnam Journal
of
MATHEMATICS

Matrix Transformations of Generalized Holomorphic Dirichlet Series in a Bounded ρ-Convex Domain

Trinh Dao Chien
Gia Lai Education and Training Department,
56 Tran Hung Dao Str., Pleiku, Gia Lai, Vietnam

Received November 1, 1999
Revised April 14, 2000

Abstract. This paper deals with matrix transformations of generalized Dirichlet series with complex frequencies that define holomorphic functions in a bounded ρ-convex domain of $\mathbb{C}$.

1. Introduction

The matrix transformation is one of the methods for summing series and sequences using an infinite matrix. Matrix transformations of power series of one complex variable has been studied previously by several authors. Most papers dealt with Nörlund matrices, i.e., triangular matrices of a special form (see, e.g., [7, 8]). For the general case of matrices there seem to be very few articles. In [1], Borwein and Jakimovski considered matrix transformations of power series in the complex plane $\mathbb{C}$ and obtained some results on this direction. Later, Lê Hai Khôi [4, 5] considered cases of the class of multiple Dirichlet series with complex frequencies that define entire functions on $\mathbb{C}^n$ as well as holomorphic functions in bounded convex domains of $\mathbb{C}^n$.

Based on the ideas in [4], in our previous paper [10], we considered matrix transformations of generalized entire Dirichler series with complex frequencies in $\mathbb{C}$.

In this paper, following the methods of [5], we consider matrix transformations of generalized Dirichlet series with complex frequencies that define holomorphic functions in a bounded ρ-convex domain of $\mathbb{C}$.

In Sec. 2 we recall some notions and, by the same method as in [6], prove some auxiliary lemmas which will be used in the sequel. In Sec. 3 we consider matrix transformations.

2. Generalized Holomorphic Dirichlet Series in a Bounded ρ-Convex Domain

First we recall some notions.

Let $0 < \rho < +\infty$. We suppose that the reader already knows the notions of ρ-convex compact set with its ρ-support function (see, e.g., [3, p. 139]). A domain G is called a ρ-convex domain if there exists a sequence of ρ-convex compact sets $\overline{G}_n$ such that $\overline{G} = \bigcup_{n=1}^{\infty} G_n$ and $\overline{G}_n \subset G_{n+1} \subset G$, $n = 1, 2, \dots$, where G_n is the set of interior points of the compact set $\overline{G}_n$. In this case we say that the sequence of compact sets $\overline{G}_n$ is inside convergent to G. Everywhere in what follows concerning the ρ-convex domain (in the case $\rho \neq 1$), we suppose that $0 \in G$. Without loss of generality we can always assume that $0 \in G_n$, $n = 1, 2, \dots$.

Let G be a ρ-convex domain, not necessarily bounded and let $(G_n)_{n=1}^{\infty}$ be a sequence of ρ-convex compact sets with the ρ-support functions $h_n(-\varphi)$, $\varphi \in (-\pi, \pi]$, which converges from inside to G. Then $0 < h_n(\varphi) < h_{n+1}(\varphi)$, $n \geq 1$, $\varphi \in (-\pi, \pi]$, and there exists $h(-\varphi) = \lim_{n\to\infty} h_n(-\varphi)$. As $h_n(-\varphi)$ are ρ-trigonometrically convex functions, the limit function $h(-\varphi)$ belongs to the same class of functions. This limit function is called the ρ-support function of the ρ-convex domain G (see, e.g., [2]). It should be noted that in the case $\rho = 1$ the notions of 1-convexity and 1-support function coincide with the usual notions of convexity and support function.

Furthermore, we denote by $\mathcal{O}(G)$ (G being a ρ-convex domain) the space of holomorphic functions in G, with the topology of uniform convergence on compact subsets of G.

Now let G be a bounded ρ-convex domain ($G \ni 0$) with the ρ-support function $h(-\varphi) > 0$, $\varphi \in (-\pi; \pi]$ and let $(\lambda_k)_{k=1}^{\infty}$ be a sequence of complex numbers in $\mathbb{C}$, $0 < |\lambda_k| \uparrow +\infty$ as $k \to \infty$. Consider a generalized Dirichlet series

$$\sum_{k=1}^{\infty} c_k E_\rho(\lambda_k z), \quad z \in G, \tag{2.1}$$

where coefficients $c_k \in \mathbb{C}$ and $E_\rho(z)$ is the Mittag–Leffler function

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\frac{n}{\rho} + 1)} \quad (\Gamma \text{ being the Gamma function}).$$

First we recall the following estimates which will be used in the sequel (see, e.g., [2]).

Lemma 2.1.

(a) *Let K be an arbitrary compact subset of $G (K \ni 0)$. Then there exists $q \in (0; 1)$ such that $K \subset qG$ and, furthermore, there exists $C = C(\rho) > 0$ such that, for all $k \geq 1$, we have*

$$\sup_{z\in K}|E_\rho(\lambda_k z)| \le \sup_{z\in qG}|E_\rho(\lambda_k z)| \le Ce^{\left(qh(\arg\lambda_k)\right)|\lambda_k|^\rho} \tag{2.2}$$

(so we can assume in addition that $C > 1$).

(b) *For $\theta \in (0, 1)$ and $\theta_1 \in (\theta, 1)$, there exists $C_1 = C_1(\rho, \theta, \theta_1) > 0$ such that, for all $k \ge 1$, we have*

$$\sup_{z\in\theta_1 G}|E_\rho(\lambda_k z)| \ge C_1 e^{\left(\theta h(\arg\lambda_k)\right)|\lambda_k|^\rho} \tag{2.3}$$

(so we can assume in addition that $0 < C_1 < 1$). ∎

The following charaterization [2] of the coefficients of the series (2.1) when it converges in the topology of $\mathcal{O}(G)$ is important and necessary for further study.

Theorem 2.1. *If the series (2.1) converges in the topology of $\mathcal{O}(G)$, then*

$$\limsup_{k\to\infty}\left(\frac{\log|c_k|}{|\lambda_k|^\rho} + h(\arg\lambda_k)\right) \le 0. \tag{2.4}$$

Conversely, if the coefficients of (2.1) satisfy condition (2.4) and if

$$\lim_{k\to\infty}\frac{\log k}{|\lambda_k|^\rho} = 0, \tag{2.5}$$

then the series (2.1) converges absolutely in the topology of $\mathcal{O}(G)$. ∎

In connection with Theorem 2.1, we can associate to the sequence (λ_k) the following class:

$$\mathcal{A}_G = \{c = (c_k) \,:\, (2.4) \text{ holds}\}.$$

It is easy to verify that $\mathcal{A}_G$ is a vector space (with the usual vector addition and scalar multiplication).

Theorem 2.1 then shows that in the compact-open topology of $\mathcal{O}(G)$, the series (2.1) converges if and only if it converges absolutely. In this case this series represents a holomorphic function in the bounded ρ-convex domain G, i.e., an element of the space $\mathcal{O}(G)$. Thus the space $\mathcal{A}_G$ defines the class $\mathcal{A}(\Lambda, G)$ of generalized Dirichlet series with the sequence of frequencies $\Lambda = (\lambda_k)$ that converge locally uniformly in G.

Note that $\mathcal{A}(\Lambda, G) \subset \mathcal{O}(G)$, the equality holds if and only if the system $\left(E_\rho(\lambda_k z)\right)_{k=1}^{\infty}$ is an absolutely representing in the space $\mathcal{O}(G)$ (see, e.g., [2]).

Before going on we recall the following fact [11] which will be used in the sequel: *if (λ_k) satisfies condition (2.5), then*

$$\sum_{k=1}^{\infty} r^{|\lambda_k|^\rho} < +\infty, \quad \forall r \in (0, 1). \tag{2.6}$$

We prove the following:

Lemma 2.2. *For any $c = (c_k) \in \mathcal{A}_G$ and $\ell \in (0, 1)$, we have*

$$\sum_{k=1}^{\infty}|c_k|e^{\left(\ell h(\arg\lambda_k)\right)|\lambda_k|^{\rho}}<+\infty.$$

Proof. Let $c=(c_k)\in\mathcal{A}_G$. Then for some $\varepsilon\in(0;1)$, there exists N such that, for all $k\geq N$, we have

$$\frac{\log|c_k|}{|\lambda_k|^{\rho}}+h(\arg\lambda_k)\leq\varepsilon,$$

which is equivalent to

$$|c_k|\leq e^{\left(\varepsilon-h(\arg\lambda_k)\right)|\lambda_k|^{\rho}}.$$

We put $\xi=\min h(\varphi)>0,\varphi\in(-\pi;\pi]$. Then, by (2.6), we have

$$\begin{aligned}\sum_{k=1}^{\infty}|c_k|e^{\left(\ell h(\arg\lambda_k)\right)|\lambda_k|^{\rho}}&\leq\sum_{k=1}^{\infty}e^{\left(\varepsilon+(\ell-1)h(\arg\lambda_k)\right)|\lambda_k|^{\rho}}\\&\leq\sum_{k=1}^{\infty}e^{\left(\varepsilon+(\ell-1)\xi\right)|\lambda_k|^{\rho}}<+\infty,\end{aligned}$$

by choosing ε such that $0<\varepsilon<(1-\ell)\xi$. The proof is complete. ■

Denote by $\mathcal{A}_G^{\alpha}$ the Köthe dual of the space $\mathcal{A}_G$, i.e.,

$$\mathcal{A}_G^{\alpha}=\Big\{(u_k)\ :\ \sum_{k=1}^{\infty}c_ku_k\ \text{ converges absolutely for all }\ (c_k)\in\mathcal{A}_G\Big\}.$$

Also we consider the following set:

$$\mathcal{A}_G^{\beta}=\Big\{(u_k);\ \sum_{k=1}^{\infty}c_ku_k\ \text{converges for all}\ (c_k)\in\mathcal{A}_G\Big\}.$$

We prove the following:

Lemma 2.3. *If (2.5) holds, then* $(u_k)\in\mathcal{A}_G^{\beta}$ *if and only if* $(u_k)\in\mathcal{A}_G^{\alpha}$, *i.e.,* $\mathcal{A}_G^{\alpha}=\mathcal{A}_G^{\beta}$.

In this case these sequence spaces can be defined as follows:

$$\mathcal{A}_G^{\beta}=\mathcal{A}_G^{\alpha}=\Big\{(u_k)\ :\limsup_{k\to\infty}\Big(\frac{\log|u_k|}{|\lambda_k|^{\rho}}-h(\arg\lambda_k)\Big)<0\Big\}.$$

Proof. Necessity. Let $(u_k)\in\mathcal{A}_G^{\beta}$. Suppose that

$$\limsup_{k\to\infty}\Big(\frac{\log|u_k|}{|\lambda|_k^{\rho}}-h(\arg\lambda_k)\Big)\geq 0.$$

Then, for a sequence $(\varepsilon_p)_{p=1}^{\infty}\downarrow 0$ there exists an increasing sequence $(k_p)_{p=1}^{\infty}$ of positive numbers such that

$$\frac{\log|u_{k_p}|}{|\lambda_{k_p}|^{\rho}}-h(\arg\lambda_{k_p})\geq-\varepsilon_p,\quad\forall p\geq 1,$$

which is equivalent to

$$\log(1/u_{k_p}) \le (\varepsilon_p - h(\arg \lambda_{k_p}))|\lambda_{k_p}|^\rho, \quad \forall p \ge 1.$$

Define a sequence (c_k) as follows:

$$c_k = \begin{cases} 1/|u_{k_p}|, & \text{if } k = k_p,\ p = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \limsup_{k\to\infty} \left(\frac{\log|c_k|}{|\lambda_k|^\rho} + h(\arg \lambda_k) \right) &\le \limsup_{p\to\infty} \left(\frac{\log(1/|u_{k_p}|)}{|\lambda_{k_p}|^\rho} + h(\arg \lambda_{k_p}) \right) \\ &\le \limsup_{p\to\infty} (\varepsilon_p) = 0, \end{aligned}$$

which means that $(c_k) \in \mathcal{A}_G$.

However, since $|c_{k_p} u_{k_p}| = 1$ for $p = 1, 2, \ldots$, it follows that the series $\sum_{k=1}^\infty c_k u_k$ does not converge. We get a contradiction.

Sufficiency. Assume that there exists a constant Q such that

$$\limsup_{k\to\infty} \left(\frac{\log|u_k|}{|\lambda_k|^\rho} - h(\arg \lambda_k) \right) = Q < 0,$$

and also the condition (2.5) is satisfied. Then, for $\varepsilon > 0$ (satisfying $Q + \varepsilon < 0$), there exists N_1 such that, for all $k \ge N_1$, we have

$$\frac{\log|u_k|}{|\lambda_k|^\rho} - h(\arg \lambda_k) \le Q + \frac{\varepsilon}{2},$$

or, equivalently,

$$|u_k| \le e^{\left((Q + \frac{\varepsilon}{2} + h(\arg \lambda_k))|\lambda_k|^\rho\right)}.$$

Now, let $(c_k) \in \mathcal{A}_G$. Then there exists N_2 such that, for all $k \ge N_2$, we have

$$|c_k| \le e^{\left(\frac{\varepsilon}{2} - h(\arg \lambda_k)\right)|\lambda_k|^\rho}.$$

Hence, for all $k \ge N = \max\{N_1, N_2\}$, we have

$$\sum_{k=N}^\infty |c_k u_k| \le \sum_{k=N}^\infty \left(e^{Q+\varepsilon}\right)^{|\lambda_k|^\rho} < +\infty,$$

due to (2.6).

Consequently, the series $\sum_{k=1}^\infty c_k u_k$ converges absolutely. This completes the proof. ■

We prove the following:

Lemma 2.4. *Let (a_k) be a sequence of real numbers. Suppose that*

$$\limsup_{k\to\infty} \left(a_k + \frac{\log|E_\rho(\lambda_k z)|}{|\lambda_k|^\rho h(\arg \lambda_k)} \right) < A < +\infty, \quad \forall z \in G. \tag{2.7}$$

Then

$$\limsup_{k\to\infty} a_k \le A-1.$$

Proof. As the function $\log|E_\rho(\lambda_k z)|$ is subharmonic in G, $k=1,2,...$, and we already have condition (2.7), it is desirable to apply Hartogs' lemma for the sequence

$$\varphi_k(z) = a_k + \frac{\log|E_\rho(\lambda_k z)|}{|\lambda_k|^\rho h(\arg\lambda_k)}, \quad z\in G,\ k=1,2,\dots.$$

Since $|\lambda_k|^\rho h(\arg\lambda_k)>0$ for all $k=1,2,...$, it is clear that the function $\varphi_k(z)$, $k=1,2,...$, is also subharmonic in G.

Now, let K be an arbitrary compact subset of $G(K\ni 0)$. Then, due to Lemma 2.1, there exist $q_1\in(0,1)$ and $C_1=C_1(\rho)>1$ such that, for all $k\ge 1$, we have

$$\begin{aligned}|E_\rho(\lambda_k z)| &\le \sup_{z\in K}|E_\rho(\lambda_k z)| \le \sup_{z\in q_1 G}|E_\rho(\lambda_k z)|\\ &\le C_1 e^{\left(q_1 h(\arg\lambda_k)\right)|\lambda_k|^\rho},\quad \forall z\in K. \end{aligned}\tag{2.8}$$

Hence, by (2.8), we have

$$\frac{\log|E_\rho(\lambda_k z)|}{|\lambda_k|^\rho h(\arg\lambda_k)} \le \frac{\log C_1}{|\lambda_k|^\rho h(\arg\lambda_k)} + q_1 \le \frac{\log C_1}{|\lambda_1|^\rho \xi} + q_1 = M'_K,\quad \forall z\in K. \tag{2.9}$$

Moreover, from (2.7), it follows, in particular, for $z=0$ that

$$\limsup_{k\to\infty} a_k < A < +\infty. \tag{2.10}$$

By (2.9) and (2.10), there exists $M_K>0$ such that

$$\varphi_k(z)\le M_K,\quad \forall z\in K,\ \forall k\ge 1.$$

Now applying Hartogs' lemma (see, e.g., [9]) we obtain that if K is a compact set in G and $\varepsilon>0$, then there exists $k_0\in\mathbb{N}$ such that, for all $k\ge k_0$, we have

$$\varphi_k(z)\le A+\frac{\varepsilon}{2},\quad \forall z\in K,$$

which implies that, for all $k\ge k_0$,

$$\sup_{z\in K}\varphi_k(z)\le A+\frac{\varepsilon}{2}. \tag{2.11}$$

Furthermore, for such an $\varepsilon>0$, we put $q_2=1-\varepsilon/3$ and $q_3=1-\varepsilon/4$. It is clear that $0<q_2<q_3<1$. Then, due to Lemma 2.1, there exists $0<C_2=C_2(\rho,\varepsilon)<1$ such that, for all $k\ge 1$, we have

$$\sup_{z\in q_3\overline{G}}|E_\rho(\lambda_k z)| \ge \sup_{z\in q_3 G}|E_\rho(\lambda_k z)| \ge C_2 e^{\left(q_2 h(\arg\lambda_k)\right)|\lambda_k|^\rho}. \tag{2.12}$$

Furthermore, since $\log C_2<0$, there exists $k_1\in\mathbb{N}$ such that, for all $k\ge k_1$, we have

$$\frac{\log C_2}{|\lambda_k|^\rho h(\arg\lambda_k)} \geq \frac{\log C_2}{|\lambda_k|^\rho \xi} \geq -\frac{\varepsilon}{6}. \tag{2.13}$$

Then, by (2.12) and (2.13), for all $k \geq k_1$, we have

$$\sup_{z\in q_3\overline{G}} \varphi_k(z) = a_k + \frac{\log\left(\sup_{z\in q_3\overline{G}} |E_\rho(\lambda_k z)|\right)}{|\lambda_k|^\rho h(\arg\lambda_k)} \geq a_k + \frac{\log C_2}{|\lambda_k|^\rho h(\arg\lambda_k)} + q_2$$

$$\geq a_k - \frac{\varepsilon}{6} + \left(1 - \frac{\varepsilon}{3}\right) = a_k + 1 - \frac{\varepsilon}{2}. \tag{2.14}$$

Since K is an arbitrary compact subset of G, we can choose $K = q_3\overline{G}$. Then, by (2.11) and (2.14), for all $k \geq N = \max\{k_0, k_1\}$, we have

$$a_k + 1 - \frac{\varepsilon}{2} \leq A + \frac{\varepsilon}{2},$$

which implies that

$$a_k \leq A - 1 + \varepsilon, \ \forall k \geq N.$$

Hence,

$$\limsup_{k\to\infty} a_k \leq A - 1.$$

The proof is complete. ∎

We also recall the following fact which will be used in the sequel: *Let $(c_k)_{k=1}^\infty$ be a sequence of real numbers and $(u_k)_{k=1}^\infty$ be a sequence of positive numbers such that $0 < m \leq u_k \leq M$, for all $k \geq 1$. If $\limsup_{k\to\infty} c_k < 0$, then* $\limsup_{k\to\infty}(u_k c_k) < 0$. ∎

3. Matrix Transformations of Generalized Holomorphic Dirichlet Series

Denote by $\mathcal{A}_G(\mathcal{U})$ the class of all matrices $[u_{jk}]_{j,k=1}^\infty$ having the property that whenever the sequence $c = (c_k) \in \mathcal{A}_G$, the sequence of functions $(f_j(z))_{j=1}^\infty$ given by

$$f_j(z) := \sum_{k=1}^\infty u_{jk} c_k E_\rho(\lambda_k z), \ \ j = 1, 2, ..., \tag{3.1}$$

converges uniformly on every compact subset of G, each generalized Dirichlet series $\sum_{k=1}^\infty u_{jk} c_k E_\rho(\lambda_k z)$ being convergent in G, $j = 1, 2, \ldots$.

We shall study conditions for a given matrix $[u_{jk}]_{j,k=1}^\infty$ to belong to the class $\mathcal{A}_G(\mathcal{U})$.

Theorem 3.1. *If the following conditions hold*

$$\exists \lim_{j\to\infty} u_{jk} = u_k, \ \ k = 1, 2, ..., \tag{3.2}$$

and

$$\limsup_{k\to\infty}\left(\sup_{j\geq 1}\frac{\log|u_{jk}|}{|\lambda_k|^\rho h(\arg\lambda_k)}\right)\leq 0, \tag{3.3}$$

then the matrix $[u_{jk}]$ *belongs to* $\mathcal{A}_G(\mathcal{U})$.

Proof. Assume that conditions (3.2) and (3.3) hold. Let $c=(c_k)\in\mathcal{A}_G$. Take an arbitrary compact subset K of $G(K\ni 0)$. Then, we have $K\subset q_1G$ for some $q_1\in(0,1)$.

Due to condition (3.2), for every $k\in\mathbb{N}$, the sequence $(u_{jk})_{j=1}^{\infty}$ is bounded and therefore,

$$Q_k:=\sup_{j\geq 1}\log|u_{jk}|<+\infty,\quad \forall k\geq 1.$$

Hence,

$$|u_{jk}|\leq e^{Q_k},\quad \forall k\geq 1,\ \forall j\geq 1. \tag{3.4}$$

Furthermore, by condition (3.3), for $q_4=(1-q_1)/2$, there exists $N=N(q_1)$ such that

$$\frac{\log|u_{jk}|}{|\lambda_k|^\rho h(\arg\lambda_k)}\leq q_4,\ \forall k>N,\ \forall j\geq 1,$$

or, equivalently,

$$|u_{jk}|\leq e^{\left(q_4 h(\arg\lambda_k)\right)|\lambda_k|^\rho},\ \forall k>N,\ \ \forall j\geq 1. \tag{3.5}$$

Then, due to Lemma 2.1, Lemma 2.2, and by (3.4), (3.5), for all $j\geq 1$, we have

$$\begin{aligned}
&\sup_{z\in K}\left|\sum_{k=1}^{\infty}u_{jk}c_kE_\rho(\lambda_k z)\right|\leq\sum_{k=1}^{\infty}\left|u_{jk}c_k\right|\sup_{z\in q_1G}\left|E_\rho(\lambda_k z)\right|\\
&\leq C_1\sum_{k=1}^{\infty}|u_{jk}c_k|e^{\left(q_1h(\arg\lambda_k)\right)|\lambda_k|^\rho}\\
&=C_1\left[\sum_{k=1}^{N}|u_{jk}c_k|e^{\left(q_1h(\arg\lambda_k)\right)|\lambda_k|^\rho}+\sum_{k=N+1}^{\infty}|u_{jk}c_k|e^{\left(q_1h(\arg\lambda_k)\right)|\lambda_k|^\rho}\right]\\
&\leq C_1\left[\sum_{k=1}^{N}|c_k|e^{Q_k+\left(q_1h(\arg\lambda_k)\right)|\lambda_k|^\rho}+\sum_{k=N+1}^{\infty}|c_k|e^{\left((q_1+q_4)h(\arg\lambda_k)\right)|\lambda_k|^\rho}\right]\\
&<+\infty.
\end{aligned}$$

Thus, each series

$$\sum_{k=1}^{\infty}u_{jk}c_kE_\rho(\lambda_k z),\quad j=1,2,\ldots,$$

converges absolutely in the topology of the space $\mathcal{O}(G)$ and therefore, represents a holomorphic function $(f_j(z))$ in G.

We now prove that the sequence (f_j) converges uniformly on K.

Let ε be any positive number. Due to Lemma 2.2, we choose $N_1\geq N$ so that

$$\sum_{k=N_1+1}^{\infty} |c_k| e^{\left((q_1+q_4)h(\arg\lambda_k)\right)|\lambda_k|^\rho} < \frac{\varepsilon}{4C_1}. \tag{3.6}$$

Denote

$$C_3(N_1) := \sum_{k=1}^{N_1} |c_k| e^{\left(q_1 h(\arg\lambda_k)\right)|\lambda_k|^\rho}. \tag{3.7}$$

From condition (3.2) it follows that there exists N_2 such that

$$|u_{mk} - u_{nk}| < \frac{\varepsilon}{2C_1C_3(N_1)}, \quad \forall k = 1, 2, ..., N_1, \quad \forall m, n > N_2. \tag{3.8}$$

Furthermore, by (3.5) and (3.6), we have

$$\sum_{k=N_1+1}^{\infty} \Big(|u_{mk}| + |u_{nk}|\Big) |c_k| e^{\left(q_1 h(\arg\lambda_k)\right)|\lambda_k|^\rho}$$

$$= \sum_{k=N_1+1}^{\infty} |u_{mk}||c_k| e^{\left(q_1 h(\arg\lambda_k)\right)|\lambda_k|^\rho} + \sum_{k=N_1+1}^{\infty} |u_{nk}||c_k| e^{\left(q_1 h(\arg\lambda_k)\right)|\lambda_k|^\rho}$$

$$\leq 2 \sum_{k=N_1+1}^{\infty} |c_k| e^{\left((q_1+q_4)h(\arg\lambda_k)\right)|\lambda_k|^\rho}$$

$$< 2\frac{\varepsilon}{4C_1}$$

$$= \frac{\varepsilon}{2C_1}, \quad \forall m, n > N_2. \tag{3.9}$$

Then, due to Lemma 2.1 and by (3.7), (3.8), (3.9), for all $m, n > N_2$, we get

$$\sup_{z\in K} |f_m(z) - f_n(z)| = \sup_{z\in K} \left|\sum_{k=1}^{\infty} (u_{mk} - u_{nk}) c_k E_\rho(\lambda_k z)\right|$$

$$\leq \sum_{k=1}^{\infty} |u_{mk} - u_{nk}||c_k| \sup_{z\in K} |E_\rho(\lambda_k z)| le C_1 \sum_{k=1}^{\infty} |u_{mk} - u_{nk}||c_k| e^{\left(q_1 h(\arg\lambda_k)\right)|\lambda_k|^\rho}$$

$$= C_1 \left[\sum_{k=1}^{N_1} |u_{mk} - u_{nk}||c_k| e^{\left(q_1 h(\arg\lambda_k)\right)|\lambda_k|^\rho} \right.$$

$$\left. + \sum_{k=N_1+1}^{\infty} |u_{mk} - u_{nk}||c_k| e^{\left(q_1 h(\arg\lambda_k)\right)|\lambda_k|^\rho}\right]$$

$$< C_1 \left[\frac{\varepsilon}{2C_1C_3(N_1)} \sum_{k=1}^{N_1} |c_k| e^{\left(q_1 h(\arg\lambda_k)\right)|\lambda_k|^\rho} \right.$$

$$\left. + \sum_{k=N_1+1}^{\infty} \left(|u_{mk}| + |u_{nk}|\right) |c_k| e^{\left(q_1 h(\arg\lambda_k)\right)|\lambda_k|^\rho}\right]$$

$$< C_1 \left[\frac{\varepsilon}{2C_1C_3(N_1)} C_3(N_1) + \frac{\varepsilon}{2C_1}\right] = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The theorem is proved. ∎

Theorem 3.2. *If the matrix $\left[u_{jk}\right]_{j,k=1}^{\infty}$ belongs to $\mathcal{A}_G(\mathcal{U})$, then condition (3.2) and the following condition hold:*

$$\limsup_{k\to\infty}\left(\frac{\log|u_{jk}|}{|\lambda_k|^\rho h(\arg\lambda_k)}\right)\le 0,\quad \forall j=1,2,\dots. \tag{3.10}$$

Proof. Assume that the matrix $\left[u_{jk}\right]$ belongs to $\mathcal{A}_G(\mathcal{U})$. Consider "unit vectors" $a^{(m)}$, $m=1,2,...$, in $\mathcal{A}_G$, with

$$a_k^{(m)}=\begin{cases}1, & \text{if } k=m,\ m=1,2,...,\\ 0, & \text{otherwise.}\end{cases}$$

Obviously, for each "unit vector" $a^{(m)}$ of the space $\mathcal{A}_G$, the sequence $\left(f_j^{(m)}(z)\right)_{j=1}^{\infty}$, given by

$$f_j^{(m)}(z):=\sum_{k=1}^{\infty}u_{jk}a_k^{(m)}E_\rho(\lambda_k z),\quad j=1,2,\dots,$$

is well defined. Furthermore, from the convergence of the sequence $\left(f_j^{(m)}(0)\right)_{j=1}^{\infty}$, which in this case has the form $(u_{jm})_{j=1}^{\infty}$, it follows that $u_m=\lim_{j\to\infty}u_{jm}$, $m\in\mathbb{N}$, exists. Thus condition (3.2) is satisfied. Now let $c=(c_k)\in\mathcal{A}_G$. Then the series

$$\sum_{k=1}^{\infty}u_{jk}c_kE_\rho(\lambda_k z),\quad j=1,2,...,$$

converges in G. This implies that

$$\left(u_{jk}E_\rho(\lambda_k z)\right)_{k=1}^{\infty}\in\mathcal{A}_G^{\alpha},\ \forall z\in G,\ \forall j\ge 1.$$

Due to Lemma 2.3 we have

$$\limsup_{k\to\infty}\left(\frac{\log|u_{jk}E_\rho(\lambda_k z)|}{|\lambda_k|^\rho}-h(\arg\lambda_k)\right)<0,\quad \forall z\in G,\ j=1,2,\dots. \tag{3.11}$$

Put $\nu=\max h(\varphi)$, $\varphi\in(-\pi;\pi]$. Then we have

$$0<\frac{1}{\nu}\le\frac{1}{h(\arg\lambda_k)}\le\frac{1}{\xi},\quad \forall k\ge 1. \tag{3.12}$$

By (3.11) and (3.12), we have

$$\limsup_{k\to\infty}\left(\frac{\log|u_{jk}|}{|\lambda_k|^\rho h(\arg\lambda_k)}+\frac{\log|E_\rho(\lambda_k z)|}{|\lambda_k|^\rho h(\arg\lambda_k)}-1\right)<0,\forall z\in G,\ j=1,2,\dots.$$

Hence,

$$\limsup_{k\to\infty}\Big(\frac{\log|u_{jk}|}{|\lambda_k|^\rho h(\arg\lambda_k)}+\frac{\log|E_\rho(\lambda_k z)|}{|\lambda_k|^\rho h(\arg\lambda_k)}\Big)<1,\ \ \forall z\in G,\ \ j=1,2,\dots.$$

Applying Lemma 2.4 gives

$$\limsup_{k\to\infty}\frac{\log|u_{jk}|}{|\lambda_k|^\rho h(\arg\lambda_k)}\le 0,\ \ j=1,2,\dots.$$

The proof is complete. ∎

Acknowledgements. I would like to express my deep gratitude to Professor Nguyen Van Mau and Dr. Lê Hai Khôi for helpful suggestions in the preparation of this paper.

References

1. D. Borwein and A. Jakimovski, Matrix transformations of power series, *Proc. Amer. Math. Soc.* **122** (1994) 511-523.
2. Yu. F. Korobeinik, Representing systems, *Uspekhi Mat. Nauk* **36** (1) (1981) 73-126 (Russian); *Russian Mat. Surveys* **36** (1) (1981) 75-137 (English).
3. A. F. Leont'ev, Representation of functions by Dirichlet generalized series, *Uspekhi Mat. Nauk* **24** (2) (1969) 97-164 (Russian).
4. Lê Hai Khôi, Matrix transformations of entire Dirichlet series, *Vietnam J. Math.* **24** (1996) 109-112.
5. Lê Hai Khôi, A note on matrix transformations of holomorphic Dirichlet series, *Portugaliae Mathematica* **56** (1999) 195-203.
6. Lê Hai Khôi, Holomorphic Dirichlet series in several variables, *Math. Scand.* **77** (1995) 85-107.
7. A. Peyerimhoff, Lectures on summability, Lecture Notes in Mathematics, Vol. 107, Springer-Verlag, 1969.
8. K. Stadtmüller, Summability of power series by non-regular Nörlund methods, *J. Approx. Theory* **68** (1) (1992) 33-44.
9. B. V. Sabat, *Introduction to Complex Analysis*, Part I, Nauka, 1976 (Russian).
10. Trinh Dao Chien, Matrix transformation of generalized entire Dirichlet series, *VNU, Journal of Science, Nat. Sci.* (to appear).
11. Trinh Dao Chien, Sequence spaces of coefficients of generalized holomorphic Dirichlet series in a bounded ρ-convex domain, Scientific Bulletin of Universities, Ministry of Education and Training Vietnam (to appear) (Vietnamese).