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Existence and Stabilization of a Degenerate Equation with Nonlinear Boundary Conditions

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Abstract. In this work we study a hyperbolic-parabolic equation involving a non-linear boundary feedback $(|y'|^\rho+1)y'$ and other non-linear boundary term $|y|^\gamma y$. For $\rho = \gamma$ we prove the exponential and algebraic decay and for $\gamma \neq \rho$ we obtain the algebraic one.

1. Introduction

Let Ω be a bounded domain of R^n with C^2 boundary Γ. Let Γ_0 and Γ_1 be nonempty sets of Γ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, where $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ (note that this assumption excludes the simply connected regions). Let ν be the unit normal vector pointing toward the exterior of Ω and let $\partial/\partial\nu$ be the normal derivative in the direction ν. We consider the following hyperbolic-parabolic problem:

$$\begin{cases} K(x,t)\dfrac{\partial^2 y}{\partial t^2} + \alpha(x,t)\dfrac{\partial y}{\partial t} - \Delta y = 0 \text{ in } Q = \Omega\times(0,\infty) \\ y = 0 \text{ on } \Sigma_1 = \Gamma_1 \times (0,\infty) \\ \dfrac{\partial y}{\partial \nu} + \mathcal{F}\left(x,t,y,\dfrac{\partial y}{\partial t}\right) = 0 \text{ on } \Sigma_0 = \Gamma_0\times(0,\infty) \\ y(x,0) = y^0;\, \dfrac{\partial y}{\partial t}(x,0) = y^1 \text{ in } \Omega, \end{cases} \tag{1.1}$$

where

$$\mathcal{F}\left(x,t,y,\frac{\partial y}{\partial t}\right) = \frac{\partial y}{\partial t} + |y|^\gamma y + \left|\frac{\partial y}{\partial t}\right|^\rho \frac{\partial y}{\partial t}.$$

Here

$$\rho,\gamma \in (0, \frac{2}{n-2}] \text{ if } n \geq 3 \text{ and } \rho,\gamma > 0 \text{ if } n = 1,2. \tag{1.2}$$

In this paper we study solvability and decay rates of strong solutions to problem (1.1) for the degenerate equation, that is, when $K \geq 0$.

Hyperbolic-parabolic equations are very interesting since they have several applications in Mechanics. We can mention, for instance, the models of transonic Karman equation

$$y_t\, y_{tt} - \Delta y = 0$$

which describes a compressive gas flowing in a transonic region where the velocity of the gas varies from subsonic values to supersonic ones. In the supersonic region, where $y_t \geq 0$, this equation is hyperbolic-parabolic as in the present case.

In recent years, important progress has been obtained in boundary stabilization for distributed systems with non-linear feedback. Among the various works in this direction we can mention the following ones: Cipolatti, Machtyngier, and San Pedro Siqueira [3]; Lasiecka [8, 9]; Favini, Horn, Lasiecka, and Tataru [5]; Lagnese and Leugering [6]; Lasiecka and Tataru [10]; Zuazua [12].

As far as we are concerned, semigroup theory is not suitable to prove the existence of degenerate problems, that is why we use Galerkin procedure. But, as we are looking for strong solutions, Galerkin method offers us some technical difficulties which led us to transform the problem (1.1) into an equivalent one with zero initial data.

Under adequate assumptions on the coefficients $K(x,t)$ and $\alpha(x,t)$, when $\rho = \gamma$ we obtain the exponential and algebraic decay and when $\rho \neq \gamma$ we obtain the algebraic one. In both cases the derivative $E'(t)$ of the energy

$$E(t) = \frac{1}{2}\int_\Omega K(x,t)|y'|^2 dx + \frac{1}{2}\int_\Omega |\nabla y|^2 dx + \frac{1}{\gamma+2}\int_{\Gamma_0} |y|^{\gamma+2} d\Gamma \tag{1.3}$$

is non-positive.

We use the perturbed energy method to obtain the above-mentioned decay rates. Our paper is divided into five sections where in Sec. 2 we give notations and state the principal results. In Sec. 3 we prove solvability of (1.1) while in Sec. 4 we prove the decay rates of the solutions obtained in Sec. 3.

2. Notations and Main Results

In what follows we are going to consider

$$(u(t), v(t)) = \int_\Omega u(x,t)v(x,t)dx, \quad |u(t)|^2 = (u(t), u(t))$$

and

$$\|u\|_\infty = \operatorname{ess\,sup}_{t\geq 0} \|u(t)\|_{L^\infty(\Omega)},$$

where $u = u(x,t)$.

In addition, if $z, w \in L^2(\Gamma_0)$ we are going to denote

$$(z, \omega)_{\Gamma_0} = \int_{\Gamma_0} z(x)\omega(x) d\Gamma.$$

We define

$$V = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_1\}, \tag{2.1}$$

which is a Hilbert space equipped with the topology induced by the inner product $(\nabla ., \nabla .)$.

Let p be a real number such that

$$p = \max\{2(2\rho + 1), 2(2\gamma + 1)\}. \tag{2.2}$$

Since Ω has a smooth boundary Γ we are able to construct a trace operator

$$\gamma_0 : H^1(\Omega) \cap L^p(\Omega) \to L^{\frac{p+2}{2}}(\Gamma) \tag{2.3}$$

and, therefore, as we have $0 < (p+2)/2 \leq 2n/(n-2)$, $n \geq 3$, we deduce

$$\|\gamma_0 u\|_{L^{\frac{p+2}{2}}(\Gamma)} \leq C\|u\|_{H^1(\Omega)}, \forall u \in H^1(\Omega) \cap L^p(\Omega). \tag{2.4}$$

The construction of the above trace operator is given in the appendix of this paper.

Remark 1. We recall the existence of the trace operator $\gamma_0 : H^1(\Omega) \to L^q(\Gamma)$ for $1 \leq q \leq (2n-2)/(n-2)$ if $n \geq 3$. Then, in order to obtain $H^1(\Omega) \hookrightarrow L^{2(\gamma+1)}(\Gamma)$ (analogously $H^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Gamma)$) we could consider $\gamma, \rho \in (0, 1/(n-2)]$. But this assumption is more restrictive than those considered in (1.2).

In order to establish our results, we make the following assumptions:

(A1) **Assumptions on the Coefficients**

We consider

$$K, \alpha \in W^{1,\infty}(0, \infty; L^\infty(\Omega)), \tag{2.5}$$

and that there exists $\delta > 0$ satisfying

$$K \geq 0 \text{ in } Q; \ \alpha - \frac{1}{2}|K_t| \geq \delta > 0 \text{ in } Q. \tag{2.6}$$

The assumption (2.6) is widely used in degenerate problems. We refer the reader to the works of Lar'kin et al. [7] and Cavalcanti et al. [1, 2].

(A2) **Assumptions on the Initial Data**

The initial data y^0 and y^1 are chosen in $V \cap H^{3/2}(\Omega) \cap L^p(\Omega)$ satisfying the compatibility hypothesis

$$\begin{cases} \alpha(x,0)y^1 - \Delta y^0 = 0 \text{ in } \Omega \\ y^0 = 0 \text{ on } \Gamma_1 \\ \dfrac{\partial y^0}{\partial \nu} + y^1 + |y^0|^\gamma y^0 + |y^1|^\rho y^1 = 0 \text{ on } \Gamma_0. \end{cases} \tag{2.7}$$

Remark 2. The assumptions (A2) are required to obtain an estimate to $y''(0)$ term and will be clear in Sec. 3. We also note that, for any fixed $y^1 \in V \cap$

$H^{3/2}(\Omega) \cap L^p(\Omega)$, the elliptic problem (2.7) possesses a unique solution $y^0 \in V \cap L^p(\Omega)$. Now, since $\partial y^0/\partial\nu = -y^1 - |y^0|^\gamma y^0 - |y^1|^\rho y^1 \in L^2(\Gamma_0)$, it follows that $y^0 \in H^{3/2}(\Omega)$.

Now, we are in a position to state our main result.

Theorem 2.1. *Under the assumptions* (A1)–(A2), *problem* (1.1) *possesses a unique strong solution* $y : \Omega \times (0,\infty) \to \mathbb{R}$ *satisfying*

$$y \in L^\infty(0,\infty;V) \tag{2.8}$$

$$y' \in L^\infty(0,\infty;V) \tag{2.9}$$

$$\sqrt{K}y'' \in L^\infty(0,\infty;L^2(\Omega)) \text{ and } y'' \in L^2(0,\infty;L^2(\Omega)) \tag{2.10}$$

$$Ky'' + \alpha y' - \Delta y = 0 \text{ in } Q$$

$$\frac{\partial y}{\partial\nu} + \mathcal{F}(x,t,y,y') = 0 \text{ on } \Gamma_0$$

$$y(0) = y^0; y'(0) = y^1 \text{ on } \Omega$$

Moreover
(A) *If* $\lambda = \rho$, *there exist constants* $C > 0$ *and* $\theta > 0$ *such that*

$$E(t) \le CE(0)\exp(-\theta t), \quad t \ge 0, \tag{2.11}$$

and also there exists a constant $M > 0$ *such that*

$$E(t) \le 2(Mt + [E(0)]^{-\frac{\rho}{2}})^{-\frac{2}{\rho}}, \quad t \ge 0. \tag{2.12}$$

(B) *If* $\lambda \neq \rho$, *we obtain the decay rate given in* (2.12).

3. Strong Solutions

In this section we are going to prove the existence and uniqueness of strong solutions to problem (1.1). For this end we transform the boundary value problem (1.1) into an equivalent one with zero initial data, using the change of variables

$$v(x,t) = y(x,t) - \phi(x,t), \tag{3.1}$$

where

$$\phi(x,t) = y^0(x) + ty^1(x); (x,t) \in \Omega \times (0,T). \tag{3.2}$$

Considering (3.1) and (3.2) we obtain the equivalent problem for v:

$$\begin{cases} K(x,t)v'' + \alpha(x,t)v' - \Delta v = F \text{ in } \Omega \times (0,T) \\ v = 0 \text{ on } \Gamma_1 \times (0,T) \\ \dfrac{\partial v}{\partial\nu} + \mathcal{F}(x,t,v+\phi,v'+\phi') = G \text{ on } \Gamma_0 \times (0,T) \\ v(0) = v'(0) = 0 \text{ in } \Omega \end{cases} \tag{3.3}$$

where

$$F = -\alpha\phi' + \Delta\phi, \tag{3.4}$$

$$G = -\frac{\partial\phi}{\partial\nu} - \phi', \tag{3.5}$$

and

$$\mathcal{F}(x,t,v+\phi,v'+\phi') = v' + |v+\phi|^\gamma(v+\phi) + |v'+\phi'|^\rho(v'+\phi'). \tag{3.6}$$

In the sequel we are able to prove that, for every $T > 0$, $\exists C > 0$ such that

$$|\Delta v(t)|^2 + |\nabla v'(t)|^2 \le C; \forall t \in (0,T). \tag{3.7}$$

We observe that if (3.7) is obtained we get the same inequality for the solution y provided that (3.1) and (3.2) hold. Therefore we can extend y to the whole interval $(0,\infty)$ using standard arguments. Hence, it is sufficient to prove that (3.3) has a solution in $(0,T)$, which will be done by the Galerkin method.

We represent by $(\omega_\nu)_{\nu\in\mathbb{N}}$ a basis in $V \cap L^p(\Omega)$ which is orthonormal in $L^2(\Omega)$, by V_m the subspace of V generated by the m first vectors $\omega_1,\ldots,\omega_m$, and we define for each $\varepsilon > 0$

$$K_\varepsilon = K + \varepsilon \text{ and } v_{\varepsilon m}(t) = \sum_{i=1}^m g_{\varepsilon i m}(t)\omega_i,$$

where $v_{\varepsilon m}(t)$ is the solution to the following Cauchy problem:

$$\begin{aligned}&(K_\varepsilon(t)v''_{\varepsilon m}(t),\omega_j) + (\alpha(t)v'_{\varepsilon m}(t),\omega_j) + (\nabla v_{\varepsilon m}(t),\nabla\omega_j) + (v'_{\varepsilon m}(t),\omega_j)_{\Gamma_0}\\ &\quad + \int_{\Gamma_0}\{|v_{\varepsilon m}+\phi|^\gamma(v_{\varepsilon m}+\phi) + |v'_{\varepsilon m}+\phi'|^\rho(v'_{\varepsilon m}+\phi')\}\omega_j d\Gamma\\ &\quad = (F(t),\omega_j) + (G(t),\omega_j)_{\Gamma_0};\ j=1,\ldots,m,\ v_{\varepsilon m}(0) = v'_{\varepsilon m}(0) = 0. \end{aligned}\tag{3.8}$$

The approximate system (3.8) is a normal one of ordinary differential equations which has a solution in $[0,t_{\varepsilon m})$. The existence of a solution in the whole interval $(0,T)$ is a consequence of the first estimate.

A Priori Estimates

The First Estimate. Multiplying both sides of (3.8) by $g'_{\varepsilon jm}(t)$ and summing over $1 \le j \le n$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega K_\varepsilon|v'_{\varepsilon m}|^2dx + \int_\Omega|\nabla v_{\varepsilon m}|^2dx\right\} + \int_\Omega(\alpha - \frac{K_t}{2})|v'_{\varepsilon m}|^2dx\\ &\quad + \int_{\Gamma_0}|v'_{\varepsilon m}|^2d\Gamma + \int_{\Gamma_0}\{|v_{\varepsilon m}+\phi|^\gamma(v_{\varepsilon m}+\phi) + |v'_{\varepsilon m}+\phi'|^\rho(v'_{\varepsilon m}+\phi')\}v'_{\varepsilon m}d\Gamma\\ &= (F(t),v'_{\varepsilon m}(t)) + \frac{d}{dt}(G(t),v_{\varepsilon m}(t))_{\Gamma_0} - (G'(t),v_{\varepsilon m}(t))_{\Gamma_0}. \end{aligned}\tag{3.9}$$

Analysis of $I_1 = \int_{\Gamma_0} \{|v_{\varepsilon m} + \phi|^\gamma (v_{\varepsilon m} + \phi) + |v'_{\varepsilon m} + \phi'|^\rho (v'_{\varepsilon m} + \phi')\} v'_{\varepsilon m} d\Gamma$.

We have

$$I_1 = \int_{\Gamma_0} \{|v_{\varepsilon m} + \phi|^\gamma (v_{\varepsilon m} + \phi) + |v'_{\varepsilon m} + \phi'|^\rho (v'_{\varepsilon m} + \phi')\}(v'_{\varepsilon m} + \phi')d\Gamma$$
$$- \int_{\Gamma_0} \{|v_{\varepsilon m} + \phi|^\gamma (v_{\varepsilon m} + \phi) + |v'_{\varepsilon m} + \phi'|^\rho (v'_{\varepsilon m} + \phi')\}\phi' d\Gamma \tag{3.10}$$

and therefore, from (3.9), (3.10), and (2.6), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega K_\varepsilon |v'_{\varepsilon m}|^2 dx + \int_\Omega |\nabla v_{\varepsilon m}|^2 dx + \frac{2}{\gamma+2}\int_{\Gamma_0} |v_{\varepsilon m} + \phi|^{\gamma+2} d\Gamma\right\}\\
&\quad + \delta \int_\Omega |v'_{\varepsilon m}|^2 dx + \int_{\Gamma_0} |v'_{\varepsilon m}|^2 d\Gamma + \int_{\Gamma_0} |v'_{\varepsilon m} + \phi'|^{\rho+2} d\Gamma\\
&\le (F(t), v'_{\varepsilon m}(t)) + \frac{d}{dt}(G(t), v_{\varepsilon m}(t))_{\Gamma_0} - (G'(t), v_{\varepsilon m}(t))_{\Gamma_0}\\
&\quad + \int_{\Gamma_0} |v_{\varepsilon m} + \phi|^\gamma (v_{\varepsilon m} + \phi)\phi' d\Gamma\\
&\quad + \int_{\Gamma_0} |v'_{\varepsilon m} + \phi'|^\rho (v'_{\varepsilon m} + \phi')\phi' d\Gamma.
\end{aligned} \tag{3.11}$$

Analysis of $I_2 = \int_{\Gamma_0} |v_{\varepsilon m} + \phi|^\gamma (v_{\varepsilon m} + \phi)\,\phi'\, d\Gamma$ and $I_3 = \int_{\Gamma_0} |v'_{\varepsilon m} + \phi'|^\rho (v'_{\varepsilon m} + \phi')\,\phi'\, d\Gamma$.

Using the Young inequality and since $(\gamma+1)/(\gamma+2) + 1/(\gamma+2) = 1$, we have

$$I_2 \le k_1 \int_{\Gamma_0} |v_{\varepsilon m} + \phi|^{\gamma+2} d\Gamma + k_2 \int_{\Gamma_0} |\phi'|^{\gamma+2} d\Gamma. \tag{3.12}$$

Now observing that $(\rho+1)/(\rho+2) + 1/(\rho+2) = 1$, we obtain for an arbitrary $\eta > 0$

$$I_3 \le \eta \int_{\Gamma_0} |v'_{\varepsilon m} + \phi'|^{\rho+2} d\Gamma + C_1(\eta) \int_{\Gamma_0} |\phi'|^{\rho+2} d\Gamma. \tag{3.13}$$

Also, for an arbitrary $\eta > 0$, it follows that

$$(F(t), v'_{\varepsilon m}(t)) \le C_2(\eta)|F(t)|^2 + \eta |v'_{\varepsilon m}(t)|^2. \tag{3.14}$$

Thus, from (3.11)–(3.14), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega K_\varepsilon |v'_{\varepsilon m}|^2 dx + \int_\Omega |\nabla v_{\varepsilon m}|^2 dx + \frac{2}{\gamma+2}\int_{\Gamma_0} |v_{\varepsilon m} + \phi|^{\gamma+2} d\Gamma\right\}\\
&\quad + (\delta - \eta)\int_\Omega |v'_{\varepsilon m}|^2 dx + \int_{\Gamma_0} |v'_{\varepsilon m}|^2 d\Gamma + (1-\eta)\int_{\Gamma_0} |v'_{\varepsilon m} + \phi'|^{\rho+2} d\Gamma\\
&\le C_2(\eta)|F(t)|^2 + \frac{d}{dt}(G(t), v_{\varepsilon m}(t))_{\Gamma_0} + C_4 |\nabla v_{\varepsilon m}(t)|^2\\
&\quad + k_1 \int_{\Gamma_0} |v_{\varepsilon m} + \phi|^{\gamma+2} d\Gamma + k_2 \int_{\Gamma_0} |\phi'|^{\gamma+2} d\Gamma + C_1(\eta)\int_{\Gamma_0} |\phi'|^{\rho+2} d\Gamma.
\end{aligned}$$

Integrating the above inequality over $[0,t]$, $t \in (0, t_{\varepsilon m})$ and noting that $v_{\varepsilon m}(0) = v'_{\varepsilon m}(0) = 0$, we get

$$\begin{aligned}
&\frac{1}{2}\Big\{\big|\sqrt{K_\varepsilon(t)}v'_{\varepsilon m}(t)\big|^2 + \big|\nabla v_{\varepsilon m}(t)\big|^2 + \frac{2}{\gamma+2}\|v_{\varepsilon m}(t) + \phi(t)\|^{\gamma+2}_{L^{\gamma+2}(\Gamma_0)}\Big\}\\
&\quad + (\delta - \eta)\int_0^t |v'_{\varepsilon m}(s)|^2 ds\\
&\quad + \int_0^t |v'_{\varepsilon m}(s)|^2_{L^2(\Gamma_0)} ds + (1-\eta)\int_0^t \|v'_{\varepsilon m}(s) + \phi'(s)\|^{\rho+2}_{L^{\rho+2}(\Gamma_0)} ds\\
&\le C_5 + C_2(\eta)\int_0^t |F(s)|^2 ds + (G(t), v_{\varepsilon m}(t))_{\Gamma_0} + C_4\int_0^t |\nabla v_{\varepsilon m}(s)|^2 ds\\
&\quad + k_1\int_0^t \|v_{\varepsilon m}(s) + \phi(s)\|^{\gamma+2}_{L^{\gamma+2}(\Gamma_0)} ds + k_2\int_0^t \|\phi'(s)\|^{\gamma+2}_{L^{\gamma+2}(\Gamma_0)} ds\\
&\quad + C_1(\eta)\int_0^t \|\phi'(s)\|^{\rho+2}_{L^{\rho+2}(\Gamma_0)} ds. \qquad (3.15)
\end{aligned}$$

We observe that for an arbitrary $\eta > 0$

$$(G(t), v_{\varepsilon m}(t))_{\Gamma_0} \le C_6(\eta)|G(t)|^2 + \eta|\nabla v_{\varepsilon m}(t)|^2. \qquad (3.16)$$

From (3.15), (3.16), choosing $\eta > 0$ small enough and employing Gronwall's lemma, we obtain the first estimate

$$\begin{aligned}
&\big|\sqrt{K_\varepsilon(t)}v'_{\varepsilon m}(t)\big|^2 + \big|\nabla v_{\varepsilon m}(t)\big|^2 + \|v_{\varepsilon m}(t) + \phi(t)\|^{\gamma+2}_{L^{\gamma+2}(\Gamma_0)}\\
&\quad + \int_0^t \big|v'_{\varepsilon m}(s)\big|^2 ds + \int_0^t \big|v'_{\varepsilon m}(s)\big|^2_{L^2(\Gamma_0)} ds\\
&\quad + \int_0^t \|v'_{\varepsilon m}(s) + \phi'(s)\|^{\rho+2}_{L^{\rho+2}(\Gamma_0)} ds \le C, \qquad (3.17)
\end{aligned}$$

where C is a positive constant which is independent of $\varepsilon > 0$, $m \in N$ and $t \in [0, T]$.

The Second Estimate. First of all we are going to estimate $v''_{\varepsilon m}(0)$ in $L^2(\Omega)$ norm. Taking $t = 0$ in (3.8) and noting that $\phi(0) = y^0$, $\phi'(0) = y^1$, $v_{\varepsilon m}(0) = v'_{\varepsilon m}(0) = 0$, we have

$$\begin{aligned}
&(K_\varepsilon(0)v''_{\varepsilon m}(0), \omega_j) + \int_{\Gamma_0}\{|y^0|^\gamma y^0 + |y^1|^\rho y^1\}\omega_j d\Gamma\\
&\quad = (F(0), \omega_j) + (G(0), \omega_j)_{\Gamma_0}, \quad j = 1, \dots, m. \qquad (3.18)
\end{aligned}$$

On the other hand, from (2.7), (3.4), (3.5) and (3.18) we obtain

$$(K_\varepsilon(0)v''_{\varepsilon m}(0), \omega_j) = 0 \text{ for all } j = 1, \dots, m \text{ and for all } \varepsilon > 0. \qquad (3.19)$$

Multiplying both sides of (3.19) by $g''_{j\varepsilon m}(0)$ and adding over $1 \le j \le m$, we have

$$|\sqrt{K_\varepsilon(0)}v''_{\varepsilon m}(0)| = 0 \ \text{ for all } \varepsilon > 0 \text{ and } m \in N. \tag{3.20}$$

Now, taking the derivative of (3.8) with respect to t and observing that $\phi''(t) = 0$, we conclude that

$$\begin{aligned}
&(K_t(t)v''_{\varepsilon m}(t), \omega_j) + (K_\varepsilon(t)v'''_{\varepsilon m}(t), \omega_j) + (\alpha_t(t)v'_{\varepsilon m}(t), \omega_j) \\
&\quad + (\alpha(t)v''_{\varepsilon m}(t), \omega_j) + (\nabla v'_{\varepsilon m}(t), \nabla \omega_j) \\
&\quad + (v''_{\varepsilon m}(t), \omega_j)_{\Gamma_0} + (\gamma+1)\int_{\Gamma_0} |v_{\varepsilon m} + \phi|^\gamma (v'_{\varepsilon m} + \phi')\omega_j d\Gamma \\
&\quad + (\rho+1)\int_{\Gamma_0} |v'_{\varepsilon m} + \phi'|^\rho v''_{\varepsilon m}\omega_j d\Gamma \\
&= (F'(t), \omega_j) + (G'(t), \omega_j)_{\Gamma_0}.
\end{aligned} \tag{3.21}$$

Multiplying (3.21) by $g''_{\varepsilon jm}(t)$ and summing over $0 \le j \le m$, from (3.5) it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega K_\varepsilon |v''_{\varepsilon m}|^2 dx + \int_\Omega |\nabla v'_{\varepsilon m}|^2 dx\right\} \\
&\quad + \int_\Omega (\alpha + \frac{K_t}{2})|v''_{\varepsilon m}|^2 dx\, (\alpha_t(t)v'_{\varepsilon m}(t), v''_{\varepsilon m}(t)) \\
&\quad + \int_{\Gamma_0} |v''_{\varepsilon m}|^2 d\Gamma + (\gamma+1)\int_{\Gamma_0} |v_{\varepsilon m} + \phi|^\gamma (v'_{\varepsilon m} + \phi')v''_{\varepsilon m} d\Gamma \\
&\quad + (\rho+1)\int_{\Gamma_0} |v'_{\varepsilon m} + \phi'|^\rho |v''_{\varepsilon m}|^2 d\Gamma \\
&= (F'(t), v''_{\varepsilon m}(t)) - \frac{d}{dt}\left(\frac{\partial y^1}{\partial \nu}, v'_{\varepsilon m}(t)\right)_{L^2(\Gamma_0)}.
\end{aligned} \tag{3.22}$$

Analysis of $I_4 = (\gamma+1)\int_{\Gamma_0} |v_{\varepsilon m} + \phi|^\gamma (v'_{\varepsilon m} + \phi')v''_{\varepsilon m} d\Gamma$.

First we observe that $(v_{\varepsilon m} + \phi) \in L^{(p+2)/2}(\Gamma_0)$ and therefore $(v_{\varepsilon m} + \phi)^\gamma \in L^{2(\gamma+1)/\gamma}(\Gamma_0)$. From the continuity of the trace operator given in (2.3) we conclude that

$$\begin{aligned}
\|(v_{\varepsilon m}(t) + \phi(t))^\gamma\|_{L^{\frac{2(\gamma+1)}{\gamma}}(\Gamma_0)} &\le C_1 \|v_{\varepsilon m}(t) + \phi(t)\|^\gamma_{L^{\frac{p+2}{2}}(\Gamma_0)} \\
&\le C_2 |\nabla v_{\varepsilon m}(t) + \nabla\phi(t)|^\gamma.
\end{aligned} \tag{3.23}$$

On the other hand, if we define $q_1 = 2(\gamma+1)/\gamma$, then $1/q_1 + 1/q_2 = 1/2$ if and only if $q_2 = 2(\gamma+1)$. Then, from this equality, (2.2) and (2.4) we deduce

$$\begin{aligned}
\|v'_{\varepsilon m}(t) + \phi'(t)\|_{L^{q_2}(\Gamma_0)} &= \|v'_{\varepsilon m}(t) + \phi'(t)\|_{L^{2(\gamma+1)}(\Gamma_0)} \\
&\le C'_1 \|v'_{\varepsilon m}(t) + \phi'(t)\|_{L^{\frac{p+2}{2}}(\Gamma_0)} \\
&\le C'_2 |\nabla v'_{\varepsilon m}(t) + \nabla\phi'(t)|.
\end{aligned} \tag{3.24}$$

Thus, from (3.23), (3.24) and using the generalized Hölder inequality we obtain

$$|I_4| \le C_3 |\nabla v_{\varepsilon m}(t) + \nabla\phi(t)|^\gamma |\nabla v'_{\varepsilon m}(t) + \nabla\phi'(t)|\, |v''_{\varepsilon m}(t)|_{L^2(\Gamma_0)}.$$

From the above inequality, (3.17) and for an arbitrary $\eta > 0$ it follows that

$$\begin{aligned} |I_4| &\le k_1(1 + |\nabla v'_{\varepsilon m}(t)|)|v''_{\varepsilon m}(t)|_{L^2(\Gamma_0)} \\ &\le k_2(\eta) + k_2(\eta)|\nabla v'_{\varepsilon m}(t)|^2 + \eta |v''_{\varepsilon m}(t)|^2_{L^2(\Gamma_0)}. \end{aligned} \tag{3.25}$$

Combining (2.6), (3.22), and (3.25), we conclude that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{|\sqrt{K_\varepsilon(t)}v''_{\varepsilon m}(t)|^2 + |\nabla v'_{\varepsilon m}(t)|^2\right\} + \delta|v''_{\varepsilon m}(t)|^2 + (1-\eta)|v''_{\varepsilon m}(t)|^2_{L^2(\Gamma_0)} \\ &\le k_2(\eta) + k_2(\eta)|\nabla v'_{\varepsilon m}(t)|^2 + (F'(t), v''_{\varepsilon m}(t)) - \frac{d}{dt}\left(\frac{\partial y^1}{\partial\nu}, v'_{\varepsilon m}(t)\right)_{L^2(\Gamma_0)} \\ &\quad + \|\alpha_t\|_\infty |v'_{\varepsilon m}(t)||v''_{\varepsilon m}(t)|. \end{aligned} \tag{3.26}$$

We also have for an arbitrary $\eta > 0$

$$(F'(t), v''_{\varepsilon m}(t)) \le k_3(\eta)|F'(t)|^2 + \eta|v''_{\varepsilon m}(t)|^2 \tag{3.27}$$

and

$$\|\alpha_t\|_\infty |v'_{\varepsilon m}(t)||v''_{\varepsilon m}(t)| \le k_4(\eta)\|\alpha_t\|_\infty |v'_{\varepsilon m}(t)|^2 + \eta|v''_{\varepsilon m}(t)|^2. \tag{3.28}$$

Integrating (3.26) over $[0,t]$ it results from (3.20), (3.27) and (3.28) that

$$\begin{aligned} &|\sqrt{K_\varepsilon(t)}v''_{\varepsilon m}(t)|^2 + |\nabla v'_{\varepsilon m}(t)|^2 + (\delta - 2\eta)\int_0^t |v''_{\varepsilon m}(s)|^2 ds \\ &\quad + (1-\eta)\int_0^t |v''_{\varepsilon m}(s)|^2_{L^2(\Gamma_0)} ds \\ &\le k_2(\eta)T + k_2(\eta)\int_0^t |\nabla v'_{\varepsilon m}(s)|^2 ds + k_3(\eta)\int_0^t |F'(s)|^2 ds \\ &\quad + k_4(\eta)\|\alpha_t\|_\infty \int_0^t |v'_{\varepsilon m}(s)|^2 ds + \left|\left(\frac{\partial y^1}{\partial\nu}, v'_{\varepsilon m}(t)\right)_{\Gamma_0}\right|. \end{aligned} \tag{3.29}$$

Finally, from the inequality

$$\left|\left(\frac{\partial y^1}{\partial\nu}, v'_{\varepsilon m}(t)\right)_{L^2(\Gamma_0)}\right| \le C'_3\left|\frac{\partial y^1}{\partial\nu}\right|_{L^2(\Gamma_0)} |\nabla v'_{\varepsilon m}(t)| \le k_5(\eta) + \eta|\nabla v'_{\varepsilon m}(t)|^2,$$

choosing $\eta > 0$ sufficiently small, we obtain from (3.29) and employing Gronwall's lemma the second estimate

$$|\sqrt{K_\varepsilon(t)}v''_{\varepsilon m}(t)|^2 + |\nabla v'_{\varepsilon m}(t)|^2 + \int_0^t |v''_{\varepsilon m}(s)|^2 ds + \int_0^t |v''_{\varepsilon m}(s)|^2_{L^2(\Gamma_0)} ds \le C, \tag{3.30}$$

where C is a positive constant which is independent of $\varepsilon > 0$, $m \in N$, and $t \in [0, t_{\varepsilon m}]$.

Due to estimates (3.17) and (3.30) we can extract a subsequence $(v_{\varepsilon\mu})$ of $(v_{\varepsilon m})$ such that

$$v_{\varepsilon\mu} \rightharpoonup v_\varepsilon \text{ weak star in } L^\infty(0,T;V), \tag{3.31}$$

$$v'_{\varepsilon\mu} \rightharpoonup v'_\varepsilon \text{ weak star in } L^\infty(0,T;V), \tag{3.32}$$

$$\sqrt{K_\varepsilon}\, v''_{\varepsilon\mu} \rightharpoonup \sqrt{K_\varepsilon}\, v''_\varepsilon \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \tag{3.33}$$

$$v''_{\varepsilon\mu} \rightharpoonup v''_\varepsilon \text{ weak in } L^2(0,T;L^2(\Omega)), \tag{3.34}$$

$$v_{\varepsilon\mu} \rightharpoonup v_\varepsilon \text{ weak star in } L^\infty(0,T;L^{\gamma+2}(\Gamma_0)), \tag{3.35}$$

$$v'_{\varepsilon\mu} \rightharpoonup v'_\varepsilon \text{ weak in } L^{\rho+2}(0,T;L^{\rho+2}(\Gamma_0)), \tag{3.36}$$

$$v'_{\varepsilon\mu} \rightharpoonup v'_\varepsilon \text{ weak in } L^2(0,T;L^2(\Gamma_0)), \tag{3.37}$$

$$v''_{\varepsilon\mu} \rightharpoonup v''_\varepsilon \text{ weak in } L^2(0,T;L^2(\Gamma_0)). \tag{3.38}$$

The convergences (3.31), (3.32), (3.33) and (3.37) are sufficient to pass to the limit in the linear terms of the approximate equation given in (3.8).

Analysis of the Nonlinear Terms

From (3.17) and taking into account the continuity of the trace operator $\gamma_0 : H^1(\Omega) \mapsto H^{1/2}(\Gamma)$ we have that

$$(v_{\varepsilon\mu}) \text{ is bounded in } L^2(0,T;H^{1/2}(\Gamma_0)), \tag{3.39}$$

$$(v'_{\varepsilon\mu}) \text{ is bounded in } L^2(0,T;L^2(\Gamma_0)). \tag{3.40}$$

From (3.39), (3.40), taking into consideration that the immersion $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is continuous and compact and using Aubin–Lions theorem, we can extract a subsequence, still represented by $(v_{\varepsilon\mu})$, such that

$$v_{\varepsilon\mu} \to v_\varepsilon \text{ strong in } L^2(0,T;L^2(\Gamma_0))$$

and therefore,

$$v_{\varepsilon\mu} \to v_\varepsilon \text{ a.e. on } \Sigma_0. \tag{3.41}$$

Using analogous arguments, from (3.30) we obtain a subsequence $(v'_{\varepsilon\mu})$ such that

$$v'_{\varepsilon\mu} \to v'_\varepsilon \text{ a.e. on } \Sigma_0. \tag{3.42}$$

Consequently, from (3.41) and (3.42), we get

$$|v_{\varepsilon\mu}|^\gamma v_{\varepsilon\mu} \to |v_\varepsilon|^\gamma v_\varepsilon \text{ and } |v'_{\varepsilon\mu}|^\rho v'_{\varepsilon\mu} \to |v'_\varepsilon|^\rho v'_\varepsilon \text{ a.e. on } \Sigma_0. \tag{3.43}$$

On the other hand, by the first and second estimates, one has

$$(|v_{\varepsilon\mu}|^\gamma v_{\varepsilon\mu}) \text{ is bounded in } L^2(\Sigma_0) \tag{3.44}$$

$$(|v'_{\varepsilon\mu}|^\rho v'_{\varepsilon\mu}) \text{ is bounded in } L^2(\Sigma_0). \tag{3.45}$$

Thus, combining (3.43), (3.44), and (3.45), we obtain

$$|v_{\varepsilon\mu}|^\gamma v_{\varepsilon\mu} \rightharpoonup |v_\varepsilon|^\gamma v_\varepsilon \text{ weak in } L^2(\Sigma_0)$$

$$|v'_{\varepsilon\mu}|^\rho v'_{\varepsilon\mu} \rightharpoonup |v'_\varepsilon|^\rho v'_\varepsilon \text{ weak in } L^2(\Sigma_0).$$

The above convergences are sufficient to pass to the limit in the non-linear terms of (3.8).

Uniqueness. Suppose y and $\hat{y}$ are solutions to problem (1.1). Then $z = y - \hat{y}$ satisfies

$$\begin{aligned}
&(K(t)z''(t), w) + (\alpha(t)z'(t), w) + (\nabla z(t), \nabla w) + (z'(t), w)_{\Gamma_0} \\
&\quad + \int_{\Gamma_0} (|y|^\gamma y - |\hat{y}|^\gamma \hat{y}) w d\Gamma + \int_{\Gamma_0} (|y'|^\rho y' - |\hat{y}'|^\rho \hat{y}') w d\Gamma \\
&\leq C(\gamma) \int_{\Gamma_0} (|y|^\gamma + |\hat{y}|^\gamma)|z|\,|z'| d\Gamma + C(\rho) \int_{\Gamma_0} (|y'|^\rho + |\hat{y}'|^\rho)|z'|^2 d\Gamma
\end{aligned} \tag{3.46}$$

(for all $\omega \in V \cap L^P(\Omega)$).

Using the same arguments considered in the above estimates and Gronwall's lemma we obtain from (3.46) that $|\nabla z| = |z'| = 0$. This completes the proof of the first part of Theorem 2.1. ■

4. Asymptotic Behavior

The derivative of the energy defined in (1.3) is

$$E'(t) = -\int_\Omega \left(\alpha(x,t) - \frac{1}{2}K_t(x,t)\right)|y'|^2 dx - \int_{\Gamma_0} |y'|^{\rho+2} d\Gamma - \int_{\Gamma_0} |y'|^2 d\Gamma. \tag{4.1}$$

Let λ be a positive constant such that

$$|v|^2 \leq \lambda |\nabla v|^2; \forall v \in V. \tag{4.2}$$

We are going to divide our proof into two cases.

A. Exponential Decay

For an arbitrary $\varepsilon > 0$, we define the perturbed energy

$$E_\varepsilon(t) = E(t) + \varepsilon\psi(t), \tag{4.3}$$

where

$$\psi(t) = \int_\Omega K(x,t) y' y dx. \tag{4.4}$$

Proposition 4.1. *There exists $C_1 > 0$ such that*

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon C_1 E(t), \forall t \geq 0, \forall \varepsilon > 0. \tag{4.5}$$

Proof. From (4.4) we obtain

$$|\psi(t)| \leq \|K\|_\infty^{1/2} |y(t)|\, |\sqrt{K} y'(t)| \leq \frac{1}{2}\|K\|_\infty |y(t)|^2 + \frac{1}{2}|\sqrt{K} y'(t)|^2.$$

From the above inequality and (4.2) we have

$$|\psi(t)| \leq (\|K\|_\infty \lambda + 1) E(t). \tag{4.6}$$

If we define $C_1 = (\|K\|_\infty \lambda + 1)$, then from (4.3) and (4.6), we get

$$|E_\varepsilon(t) - E(t)| = \varepsilon |\psi(t)| \leq \varepsilon C_1 E(t)$$

which concludes the proof. ■

Proposition 4.2. *Suppose that $\rho = \gamma$. Then there exist $C_2 > 0$ and $\varepsilon_1 > 0$ such that*

$$E_\varepsilon'(t) \leq -\varepsilon C_2 E(t), \forall t \geq 0 \text{ and } \forall \varepsilon \in (0, \varepsilon_1].$$

Proof. Differentiating (4.4) and replacing Ky'' by $-\alpha y' + \Delta y$ in the obtained expression, we get

$$\psi'(t) = \int_\Omega K_t(x,t) y' y dx - \int_\Omega \alpha(x,t) y' y dx + \int_\Omega \Delta y\, y dx + \int_\Omega K(x,t)|y'|^2 dx. \tag{4.7}$$

By the generalized Green formula and taking into account that $\partial y/\partial \nu = -(y' + |y|^\gamma y + |y'|^\rho y')$ on Σ_0, it comes that

$$\int_\Omega \Delta y y dx = -\int_\Omega |\nabla y|^2 dx - \int_{\Gamma_0} (y' y + |y|^{\gamma+2} + |y'|^\rho y' y) d\Gamma. \tag{4.8}$$

Substituting (4.8) in (4.7), adding the term $-\int_\Omega K|y'|^2 dx$ on both sides of the equality (4.7) and defining $L = \min\{2, \gamma + 2\}$, we obtain

$$\psi'(t) \leq -LE(t) + 2\int_\Omega K(x,t)|y'|^2 dx + \int_\Omega K_t(x,t) y' y dx - \int_\Omega \alpha(x,t) y' y dx - \int_{\Gamma_0} |y'|^\rho y' y d\Gamma - \int_{\Gamma_0} y' y d\Gamma. \tag{4.9}$$

Now we are going to analyse the term $I = \int_{\Gamma_0} |y'|^\rho y' y d\Gamma$. From Young's inequality, we have for an arbitrary $\eta > 0$

$$|I| \leq \theta_1(\eta) \int_{\Gamma_0} |y'|^{\rho+2} d\Gamma + \eta \int_{\Gamma_0} |y|^{\rho+2} d\Gamma. \tag{4.10}$$

From (2.6), (4.1), (4.3) (4.9), and (4.10), we obtain

$$
\begin{aligned}
E_\varepsilon'(t) &= E'(t) + \varepsilon\psi'(t) \\
&\le -\delta \int_\Omega |y'|^2 dx - \int_{\Gamma_0} |y'|^{\rho+2} d\Gamma - \int_{\Gamma_0} |y'|^2 d\Gamma - L\varepsilon E(t) \\
&\quad + 2\varepsilon \int_\Omega K(x,t)|y'|^2 dx + \varepsilon \int_\Omega K_t(x,t) y' y dx - \varepsilon \int_\Omega \alpha(x,t) y' y dx \\
&\quad - \varepsilon \int_{\Gamma_0} y' y d\Gamma + \varepsilon\theta_1(\eta) \int_{\Gamma_0} |y'|^{\rho+2} d\Gamma + \varepsilon\eta \int_{\Gamma_0} |y|^{\rho+2} d\Gamma. \qquad (4.11)
\end{aligned}
$$

Let us consider $\mu > 0$ such that

$$
\int_{\Gamma_0} |u|^2 d\Gamma \le \mu \int_\Omega |\nabla u|^2 dx, \ \forall u \in V. \qquad (4.12)
$$

Then, from (4.11), (4.12) and making use of Schwarz inequality, we get for an arbitrary $\eta > 0$

$$
\begin{aligned}
E_\varepsilon'(t) \le &- \int_\Omega (\delta - \varepsilon M_1(\eta))|y'|^2 dx - \int_{\Gamma_0} (1 - \varepsilon\theta_1(\eta))|y'|^{\rho+2} d\Gamma \\
&- \int_{\Gamma_0} (1 - \varepsilon\theta_4(\eta))|y'|^2 d\Gamma - \varepsilon E(t)[L - \eta(\|K_t\|_\infty \lambda + \|\alpha\|_\infty \lambda + 2\mu)] \\
&+ \varepsilon\eta \int_{\Gamma_0} |y|^{\rho+2} d\Gamma, \qquad (4.13)
\end{aligned}
$$

where

$$
M_1(\eta) = 2\|K\|_\infty + \|K_t\|_\infty \theta_2(\eta) + \|\alpha\|_\infty \theta_3(\eta).
$$

Since $\gamma = \rho$, from (4.13), we obtain

$$
\begin{aligned}
E_\varepsilon'(t) \le &- \int_\Omega (\delta - \varepsilon M_1(\eta))|y'|^2 dx - \int_{\Gamma_0} (1 - \varepsilon\theta_1(\eta))|y'|^{\rho+2} d\Gamma \\
&- \int_{\Gamma_0} (1 - \varepsilon\theta_4(\eta))|y'|^2 d\Gamma - \varepsilon E(t)(L - \eta M_2),
\end{aligned}
$$

where

$$
M_2 = (\|K_t\|_\infty + \|\alpha\|_\infty)\lambda + 2\mu + (\gamma + 2).
$$

Choosing $\eta > 0$ small enough in order to obtain $C_2 = L - \eta M_2 > 0$, it comes from the last inequality that

$$
\begin{aligned}
E_\varepsilon'(t) \le &- \int_\Omega (\delta - \varepsilon M_1(\eta))|y'|^2 dx - \int_{\Gamma_0} (1 - \varepsilon\theta_1(\eta))|y'|^{\rho+2} d\Gamma \\
&- \int_{\Gamma_0} (1 - \varepsilon\theta_4(\eta))|y'|^2 d\Gamma - \varepsilon C_2 E(t).
\end{aligned}
$$

For the chosen η we define $\varepsilon_1 = \min\{\delta/M_1, 1/\theta_1, 1/\theta_4\}$. Thus, if $\varepsilon \in (0, \varepsilon_1]$ from the above inequality we conclude that there exists $C_2 > 0$ which verifies

$$
E_\varepsilon'(t) \le -\varepsilon C_2 E(t). \qquad (4.14)
$$

The proof of Proposition 4.2 is complete. ∎

Defining

$$\varepsilon_0 = \min\left\{\frac{1}{2C_1}, \varepsilon_1\right\},$$

for all $\varepsilon \in (0, \varepsilon_0]$ we obtain from Proposition 4.1

$$(1 - C_1\varepsilon)E(t) \le E_\varepsilon(t) \le (1 + C_1\varepsilon)E(t),\ \forall t \ge 0 \tag{4.15}$$

and consequently,

$$\frac{1}{2}E(t) \le E_\varepsilon(t) \le \frac{3}{2}E(t) \le 2E(t),\ \forall t \ge 0. \tag{4.16}$$

From the above inequality, we get

$$-2\varepsilon C_2 E(t) \le -\varepsilon C_2 E_\varepsilon(t), \tag{4.17}$$

where $C_2 > 0$ is the constant obtained in Proposition 4.2. Hence, from (4.17) and Proposition 4.2, we obtain

$$E_\varepsilon'(t) \le -\frac{\varepsilon}{2}C_2 E_\varepsilon(t),$$

that is,

$$\frac{d}{dt}\left(E_\varepsilon(t)\exp\left\{\frac{\varepsilon}{2}C_2 t\right\}\right) \le 0.$$

Integrating the above inequality over $[0, t]$, we get

$$E_\varepsilon(t) \le E_\varepsilon(0)\exp\left\{-\frac{\varepsilon}{2}C_2 t\right\}. \tag{4.18}$$

Combining (4.16) and (4.18) we conclude the exponential decay, that is,

$$E(t) \le 3\exp\left\{-\frac{\varepsilon}{2}C_2 t\right\}E(0).$$ ∎

B. Algebraic Decay

Let us define

$$\psi(t) = [E(t)]^{\rho/2}\int_\Omega K(x,t)y'y\,dx. \tag{4.19}$$

Taking the derivative of ψ with respect to t, we obtain

$$\psi'(t) = \frac{\rho}{2}[E(t)]^{\frac{\rho-2}{2}}E'(t)\int_\Omega Ky'y\,dx + [E(t)]^{\frac{\rho}{2}}\int_\Omega (K_t y'y + Ky''y + K|y'|^2)\,dx. \tag{4.20}$$

Substituting $Ky'' = -\alpha y' + \Delta y$ in (4.20) and taking into account that

$$\int_{\Omega} \Delta y y dx = -\int_{\Omega} |\nabla y|^2 dx - \int_{\Gamma_0} (|y|^{\gamma} y + |y'|^{\rho} y' + y') y d\Gamma,$$

we have

$$\begin{aligned} \psi'(t) = {} & \frac{\rho}{2}[E(t)]^{\frac{\rho-2}{2}} E'(t) \int_{\Omega} K y' y dx \\ & + [E(t)]^{\frac{\rho}{2}} \Big\{ \int_{\Omega} K_t y' y dx + \int_{\Omega} K |y'|^2 dx - \int_{\Omega} \alpha y' y dx \\ & - \int_{\Omega} |\nabla y|^2 dx - \int_{\Gamma_0} y' y d\Gamma - \int_{\Gamma_0} |y|^{\gamma+2} d\Gamma - \int_{\Gamma_0} |y'|^{\rho} y' y d\Gamma \Big\}. \end{aligned} \tag{4.21}$$

On the other hand, using Schwarz's inequality, considering (4.2) and observing that $E'(t) \leq 0$, it follows that

$$\Big| \int_{\Omega} K y' y dx \Big| \leq \|K\|_{\infty}^{1/2} \lambda^{1/2} E(t) \leq \|K\|_{\infty}^{1/2} \lambda^{1/2} E(0).$$

Then we infer

$$-\frac{\rho}{2}[E(t)]^{\frac{\rho-2}{2}} \int_{\Omega} K y' y dx \leq C \frac{\rho}{2} [E(0)]^{\frac{\rho}{2}},$$

where $C = \|K\|_{\infty}^{1/2} \lambda^{1/2}$.

Defining $C_1 = C\frac{\rho}{2}[E(0)]^{\frac{\rho}{2}}$, from the above inequality, we conclude

$$\frac{\rho}{2}[E(t)]^{\frac{\rho-2}{2}} E'(t) \int_{\Omega} K y' y dx \leq -C_1 E'(t). \tag{4.22}$$

From (4.21) and (4.22) it results that

$$\begin{aligned} \psi'(t) \leq & - C_1 E'(t) + [E(t)]^{\frac{\rho}{2}} \Big\{ \int_{\Omega} K_t y' y dx + \int_{\Omega} K |y'|^2 dx \\ & - \int_{\Omega} \alpha y' y dx - \int_{\Omega} |\nabla y|^2 dx - \int_{\Gamma_0} y' y d\Gamma \\ & - \int_{\Gamma_0} |y|^{\gamma+2} d\Gamma - \int_{\Gamma_0} |y'|^{\rho} y' y d\Gamma \Big\}. \end{aligned} \tag{4.23}$$

Considering that $(\rho+1)/(\rho+2) + 1/(\rho+2) = 1$ and that (2.4) holds from Hölder and Young inequalities, we obtain, for an arbitrary $\eta > 0$,

$$\begin{aligned} \Big| \int_{\Gamma_0} |y'|^{\rho} y' y d\Gamma \Big| & \leq \Big(\int_{\Gamma_0} |y'|^{\rho+2} d\Gamma \Big)^{\frac{\rho+1}{\rho+2}} \|y(t)\|_{L^{\rho+2}(\Gamma_0)} \\ & \leq C_2 \Big(\int_{\Gamma_0} |y'|^{\rho+2} d\Gamma \Big)^{\frac{\rho+1}{\rho+2}} |\nabla y(t)| \\ & \leq C_3(\eta) \int_{\Gamma_0} |y'|^{\rho+2} d\Gamma + C_2 \eta |\nabla y(t)|^{\rho+2}. \end{aligned} \tag{4.24}$$

Thus, combining (4.23) and (4.24), we have

$$\begin{aligned}\psi'(t) \leq & - C_1 E'(t) + [E(t)]^{\rho/2}\Big\{ \int_\Omega K_t y' y dx + \int_\Omega K|y'|^2 dx \\ & - \int_\Omega \alpha y' y dx - \int_\Omega |\nabla y|^2 dx - \int_{\Gamma_0} y' y d\Gamma - \int_{\Gamma_0} |y|^{\gamma+2} d\Gamma \Big\} \\ & + C_3(\eta)[E(t)]^{\rho/2} \int_{\Gamma_0} |y'|^{\rho+2} d\Gamma + C_2 \eta [E(t)]^{\rho/2} |\nabla y(t)|^{\rho+2}. \end{aligned} \tag{4.25}$$

Applying Hölder's inequality, we get from (4.2), (4.12), and (4.25), for an arbitrary $\eta > 0$

$$\begin{aligned}\psi'(t) \leq & - C_1 E'(t) + [E(t)]^{\rho/2}\Big\{ \|K\|_\infty \int_\Omega |y'|^2 dx \\ & + \|K_t\|_\infty C_4(\eta) \int_\Omega |y'|^2 dx + \|\alpha\|_\infty C_5(\eta) \int_\Omega |y'|^2 dx \\ & + C_6(\eta) \int_{\Gamma_0} |y'|^2 d\Gamma + 2\eta\lambda \int_\Omega |\nabla y|^2 dx \\ & + \eta\mu \int_\Omega |\nabla y|^2 dx - \int_\Omega |\nabla y|^2 dx - \int_{\Gamma_0} |y|^{\gamma+2} d\Gamma \\ & + C_3(\eta) \int_{\Gamma_0} |y'|^{\rho+2} d\Gamma + C_2 \eta |\nabla y(t)|^{\rho+2} \Big\}. \end{aligned} \tag{4.26}$$

In addition, we have that

$$|\nabla y(t)|^{\rho+2}_{L^2(\Omega)} \leq 2^{\rho/2}[E(t)]^{\rho/2} |\nabla y(t)|^2 \leq 2^{\rho/2}[E(0)]^{\rho/2} |\nabla y(t)|^2_{L^2(\Omega)}. \tag{4.27}$$

Combining (4.26) and (4.27), we conclude that

$$\begin{aligned}\psi'(t) \leq & - C_1 E'(t) + [E(t)]^{\rho/2} M_1(\eta) \int_\Omega |y'|^2 dx + [E(t)]^{\rho/2} \eta M_2 \int_\Omega |\nabla y|^2 dx \\ & - [E(t)]^{\rho/2} \int_\Omega |\nabla y|^2 dx - [E(t)]^{\rho/2} \int_{\Gamma_0} |y|^{\gamma+2} d\Gamma \\ & + [E(t)]^{\rho/2} C_3(\eta) \int_{\Gamma_0} |y'|^{\rho+2} d\Gamma + C_6(\eta)[E(t)]^{\rho/2} \int_{\Gamma_0} |y'|^2 d\Gamma, \end{aligned} \tag{4.28}$$

where

$$M_1(\eta) = \|K\|_\infty + C_4(\eta)\|K_t\|_\infty + C_5(\eta)\|\alpha\|_\infty,$$
$$M_2 = 2\lambda + 2^{\rho/2} C_2 [E(0]^{\rho/2} + \mu.$$

Choosing $\eta = 1/2M_2$ from (4.28) it results that

$$\begin{aligned}\psi'(t) \leq & - C_1 E'(t) + M_1 [E(t)]^{\rho/2} \int_\Omega |y'|^2 dx \\ & - \frac{1}{2}[E(t)]^{\rho/2} \int_\Omega |\nabla y|^2 dx - [E(t)]^{\rho/2} \int_{\Gamma_0} |y|^{\gamma+2} d\Gamma \\ & + [E(t)]^{\rho/2} C_3 \int_{\Gamma_0} |y'|^{\rho+2} d\Gamma + C_6 [E(t)]^{\rho/2} \int_{\Gamma_0} |y'|^2 d\Gamma. \end{aligned} \tag{4.29}$$

However, taking into account (2.7), (4.1) and considering

$$C_7 = \max\{\delta^{-1}M_1, C_3, C_6\}[E(0)]^{\rho/2},$$

we obtain from (4.29)

$$\psi'(t) \leq -(C_1 + C_7)E'(t) - \frac{1}{2}\big[E(t)\big]^{\rho/2}\int_\Omega |\nabla y|^2 dx - \big[E(t)\big]^{\rho/2}\int_{\Gamma_0} |y|^{\gamma+2} d\Gamma. \tag{4.30}$$

Defining the perturbed energy

$$E_\varepsilon(t) = [1 + \varepsilon C_8]E(t) + \varepsilon\psi(t), \tag{4.31}$$

where $C_8 = C_1 + C_7$, we have from (4.2) and (4.19)

$$\begin{aligned}|E_\varepsilon(t) - E(t)| &\leq \varepsilon C_8 E(t) + \varepsilon|\psi(t)| \\ &\leq \varepsilon(C_8 + \big[E(0)\big]^{\rho/2}\|K\|_\infty^{1/2}\lambda^{1/2})E(t).\end{aligned} \tag{4.32}$$

Considering $C_1' = C_8 + [E(0)]^{\rho/2}\|K\|_\infty^{1/2}\lambda^{1/2}$ and $\varepsilon \leq (2C_1')^{-1}$, from (4.32) we have

$$\frac{1}{2}E(t) \leq E_\varepsilon(t) \leq 2E(t)$$

and therefore,

$$\begin{aligned}\frac{1}{2^{\frac{\rho+2}{2}}}\Big[E(t)\Big]^{\frac{\rho+2}{2}} &\leq \Big[E_\varepsilon(t)\Big]^{\frac{\rho+2}{2}} \\ &\leq 2^{\frac{\rho+2}{2}}\Big[E(t)\Big]^{\frac{\rho+2}{2}}\ \forall t \geq 0\,,\ \forall\varepsilon \in (0, (2C_1')^{-1}.\end{aligned} \tag{4.33}$$

On the other hand, from (4.30) and (4.31), it follows that

$$E_\varepsilon'(t) \leq E'(t) - \frac{\varepsilon}{2}\big[E(t)\big]^{\rho/2}\int_\Omega |\nabla y|^2 dx - \varepsilon[E(t)]^{\rho/2}\int_{\Gamma_0} |y|^{\gamma+2} d\Gamma. \tag{4.34}$$

Now, from (1.2), we infer

$$-\frac{1}{2}\int_\Omega |\nabla y|^2 dx = \frac{1}{2}\int_\Omega K(x,t)|y'|^2 dx + \frac{1}{\gamma+2}\int_{\Gamma_0} |y|^{\gamma+2} d\Gamma - E(t)$$

and consequently, from (2.5), (4.1), and (4.34), we have

$$\begin{aligned}E_\varepsilon'(t) \leq &-\delta\int_\Omega |y'|^2 dx - \varepsilon[E(t)]^{\rho/2}E(t) + \frac{\varepsilon}{2}[E(t)]^{\rho/2}\int_\Omega K|y'|^2 dx \\ &+ \frac{\varepsilon}{(\gamma+2)}[E(t)]^{\rho/2}\int_{\Gamma_0} |y|^{\gamma+2} d\Gamma - \varepsilon[E(t)]^{\rho/2}\int_{\Gamma_0} |y|^{\gamma+2} d\Gamma.\end{aligned}$$

Since $\gamma > 0$, the above inequality becomes

$$
\begin{aligned}
E_\varepsilon'(t) \leq & -\delta \int_\Omega |y'|^2 dx - \varepsilon \big[E(t)\big]^{\rho/2} E(t) \\
& + \frac{\varepsilon}{2} \big[E(0)\big]^{\rho/2} \|K\|_\infty \int_\Omega |y'|^2 dx \\
\leq & -\delta \int_\Omega |y'|^2 dx - \varepsilon \big[E(t)\big]^{(\rho+2)/2} E(t) \\
& + \frac{\varepsilon}{2} \big[E(0)\big]^{\rho/2} (\|K\|_\infty + 1) \int_\Omega |y'|^2 dx.
\end{aligned} \tag{4.35}
$$

Choosing $\varepsilon \leq \varepsilon_0 = \min \left\{ \frac{2\delta}{(\|K\|_\infty + 1)\big[E(0)\big]^{\rho/2}}, (2C_1')^{-1} \right\}$, we conclude from (4.33) and (4.35) that

$$
E_\varepsilon'(t) \leq -\frac{\varepsilon}{2^{(\rho+2)/2}} \big[E_\varepsilon(t)\big]^{(\rho+2)/2},
$$

that is,

$$
E_\varepsilon'(t) \big[E_\varepsilon(t)\big]^{(-\rho-2)/2} \leq -\frac{\varepsilon}{2^{(\rho+2)/2}}. \tag{4.36}
$$

But since

$$
\frac{d}{dt} \big[E_\varepsilon(t)\big]^{-\rho/2} = -\frac{\rho}{2} \big[E_\varepsilon(t)\big]^{(-\rho-2)/2} E_\varepsilon'(t),
$$

it results from (4.36) that

$$
\frac{d}{dt} \big[E_\varepsilon(t)\big]^{-\rho/2} \geq \frac{\varepsilon\rho}{2^{(\rho+4)/2}}.
$$

Integrating the above inequality, it follows that

$$
\big[E_\varepsilon(t)\big]^{-\rho/2} \geq \big[E_\varepsilon(0)\big]^{-\rho/2} + \frac{\varepsilon\rho}{2^{(\rho+4)/2}} t. \tag{4.37}
$$

Finally, from (4.37), we obtain

$$
\begin{aligned}
E_\varepsilon(t) & \leq \left\{ \big[E_\varepsilon(0)\big]^{-\rho/2} + \frac{\varepsilon\rho}{2^{(\rho+4)/2}} t \right\}^{-2/\rho} \\
& \leq \left\{ 2^{-\rho/2} \big[E(0)\big]^{-\rho/2} + \frac{\varepsilon\rho}{2^{(\rho+4)/2}} t \right\}^{-2/\rho} \\
& \leq 2 \left\{ 2^{-(\rho+2)} \varepsilon\rho t + \big[E(0)\big]^{-\rho/2} \right\}^{-2/\rho}
\end{aligned}
$$

which concludes the algebraic decay and consequently the proof of Theorem 2.1. ■

5. Appendix

Lemma. *There exists a continuous linear map*

$$\gamma_0 : H^1(\Omega) \cap L^p(\Omega) \mapsto L^{\frac{p+2}{2}}(\Gamma) \text{ such that } \gamma_0\varphi = \varphi|_\Gamma, \ \forall \varphi \in C^2(\overline{\Omega}).^1$$

Proof. It is sufficient to prove the above lemma when $\Omega = R^n_+ = \{(x', x_n), x' \in R^{n-1}$ and $x_n \in R_+\}$, since by local chart we have the same result when Ω has smooth boundary.

Indeed, first of all we are going to prove that

$$\|\varphi\|_{L^{\frac{p+2}{2}}(R^{n-1})} \le C\|\varphi\|^{p/p+2}_{L^p(\Omega)} \|\varphi\|^{2/p+2}_{H^1(R^n_+)}, \ \forall \varphi \in C^2_0(R^n). \tag{5.1}$$

For this end, let $G(t) = |t|^{p/2}t$ and consider $\varphi \in C^2_0(R^n)$. We have

$$\begin{aligned} G(\varphi(x',0)) &= -\int_0^\infty \frac{\partial}{\partial x_n} G(\varphi(x',x_n))dx_n \\ &= -\int_0^\infty G'(\varphi(x',x_n))\frac{\partial \varphi}{\partial x_n}(x',x_n)dx_n \end{aligned}$$

and therefore,

$$|\varphi(x',0)|^{\frac{p+2}{2}} = |G(\varphi(x',0))| \le \frac{p+2}{2}\int_0^\infty |\varphi(x',x_n)|^{p/2}|\frac{\partial \varphi}{\partial x_n}(x',x_n)|dx_n.$$

Hence,

$$\begin{aligned} \|\varphi\|^{\frac{p+2}{2}}_{L^{\frac{p+2}{2}}(R^{n-1})} &= \int_{R^{n-1}} |\varphi(x',0)|^{\frac{p+2}{2}} dx' \le \frac{p+2}{2}\int_{R^n_+} |\varphi|^{p/2}\left|\frac{\partial \varphi}{\partial x_n}\right| dx \\ &\le \frac{p+2}{2}\left(\int_{R^n_+} |\varphi|^p dx\right)^{1/2}\left(\sum_{i=1}^n \int_{R^n_+}\left|\frac{\partial \varphi}{\partial x_i}\right|^2\right)^{1/2}, \end{aligned}$$

which proves (5.1).

Observing that $p/(p+2) + 2/(p+2) = 1$ we obtain from Young's inequality that

$$\|\varphi\|_{L^{\frac{p+2}{2}}(R^{n-1})} \le C_0\|\varphi\|^{p/p+2}_{L^p(R^n_+)}\|\varphi\|^{2/p+2}_{H^1(R^n_+)} \le C(\|\varphi\|_{L^p(R^n_+)} + \|\varphi\|_{H^1(R^n_+)}).$$

Considering the above inequality we obtain by a density argument the desired result which is proved by marking use of a density argument. ■

Remark. Note that in our manuscript we consider the Laplace operator but without loss of generality we can consider the elliptic one given by

$$A(t) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right),$$

[1] Here $C^2(\overline{\Omega})$ means $\{\varphi|_{\overline{\Omega}}, \ \varphi \in C^2_0(R^n)\}$.

where

$$a_{ij} = a_{ji} \text{ and } \exists a_0 > 0 \text{ such that } \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq a_0|\xi|^2, \text{ for all } \xi \in R^n > 0.$$

This conclusion comes from the fact that if we define

$$a(t,u,v) = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx, \text{ for all } u,v \in V,$$

provided that the functions $a_{ij}(x,t)$ satisfy some appropriate hypotheses, we have

$$a_0|\nabla u|^2 \leq a(t,u,u) \leq a_1|\nabla u|^2, \text{ for all } u \in V.$$

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