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Non-Smooth Solutions for a Class of Infinitely Degenerate Elliptic Differential Equations

Nguyen Minh Tri

Institute of Mathematics, P. O. Box 631, Bo Ho 10000, Hanoi, Vietnam

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Abstract. We construct explicit global non-smooth solutions for a class of infinitely degenerate elliptic differential operators, for which the hypoellipticity failes at discrete values of a complex parameter.

1. Introduction

It is well known that the operator

$$X_0 + \sum_{j=1}^{m} X_j^2, \tag{1}$$

where $X_0, X_1, \ldots, X_m$ are real vector fields in $\Omega \subset \mathbb{R}^n$, is hypoelliptic in Ω if $X_0, X_1, \ldots, X_m$ and their repeated commutators of at most k times (k is some positive integer) span the the tangent space $T_x(\Omega)$ at any point $x \in \Omega$ (see [6]). The simplest example of degenerate elliptic differential operators that satisfy the above hypothesis may be the following operator in $\mathbb{R}^2$:

$$\frac{\partial^2}{\partial x^2} + a(x)\frac{\partial^2}{\partial y^2} + b(x,y)\frac{\partial}{\partial x} + c(x,y)\frac{\partial}{\partial y},$$

where $a(x) = x^{2k}$, and $b(x,y), c(x,y)$ are real smooth functions in $\mathbb{R}^2$. However, the situation changes drastically if $b(x,y), c(x,y)$ are complex-valued functions. Grushin [5] showed that the operator

$$\frac{\partial^2}{\partial x^2} + x^{2k}\frac{\partial^2}{\partial y^2} + \mu x^{k-1}\frac{\partial}{\partial y}, \mu \in \mathbb{C}, k \geq 1 \tag{2}$$

is hypoelliptic if and only if μ avoids some discrete set of $\mathbb{C}$. Unlike operators of principal type (see [2]), the hypoellipticity of operators with multiple characteristics depends much on lower order terms, and the discrete phenomena like above are common for them. These phenomena were obtained in many works later. Another generalization of Hörmander's theorem is to investigate the hypoellipticity of (1) when the "finite type" condition violates, starting from Fedii [3]. (Systematic study is then in [8, 9, 10, 12, 14, 16]; see also [11] for more complete references). Finally, the discrete phenomena for infinitely degenerate elliptic operators have been observed recently in [7, 13]. The aim of this paper is to give explicit global non-smooth solutions of the following homogeneous equation:

$$G_{\phi,\lambda}u(x,y) = \frac{\partial^2 u(x,y)}{\partial x^2} + x^{-4}e^{2i\phi-\frac{2}{|x|}}\frac{\partial^2 u(x,y)}{\partial y^2} + i\lambda(x)x^{-4}e^{i\phi-\frac{1}{|x|}}\frac{\partial u(x,y)}{\partial y} = 0,$$

where $(x,y)\in\mathbb{R}^2, i=\sqrt{-1}, -\pi/2<\phi<\pi/2$, and the function $\lambda(x)$ satisfies

$$\lambda(x) = \begin{Bmatrix} \lambda_+ \in \mathbb{C} & \text{if } x>0, \\ \lambda_- \in \mathbb{C} & \text{if } x<0 \end{Bmatrix} =: \lambda.$$

We will construct the solution $u(x,y)$ as a composition of powers of two complex flows which annul each other at a discrete set of values of the parameter λ. Such a special kind of solutions was first obtained in [4] for the Kohn–Laplacian on the Heisenberg group, and recently in [15] for Eq. (2). The operator $G_{\phi,\lambda}$ was treated in [7] when $\phi=0$ and $\lambda_+=\lambda_-$ or $\lambda_+=-\lambda_-$. For the general case we refer to the paper [13]. The paper is organized as follows. In Sec. 2 we give some definitions of notations used in the paper, and establish some auxiliary lemmas. In Sec. 3, we state and prove the main results.

2. Auxiliary Lemmas

We will use the following notation

$$(z,m) = z(z+1)\cdots(z+m-1) = \frac{\Gamma(z+m)}{\Gamma(z)} \quad \text{for } z\in\mathbb{C},\ m\in\mathbb{N}.$$

We denote by C a general constant which may vary from place to place. For two complex numbers $z_1, z_2 \in \mathbb{C}$, we define $z_1^{z_2}$ as $e^{z_2 \ln z_1}$ and if $z_1 = re^{i\varphi}, -\pi < \varphi \le \pi$, then $\ln z_1 = \ln r + i\varphi$. Now let us recall the following lemma from [15].

Lemma 1. *Assume that* $\omega_1, \omega_3 \in \mathbb{C}, \operatorname{Re}\omega_3 > -1$. *Then we have*

$$\int_0^\pi (\sin\theta + i\cos\theta)^{\omega_1}\sin^{\omega_3}\theta\, d\theta = \frac{2^{-\omega_3}\pi\Gamma(\omega_3+1)}{\Gamma(1+\frac{\omega_3-\omega_1}{2})\Gamma(1+\frac{\omega_3+\omega_1}{2})}. \tag{3}$$

Corollary 1. *Assume that* $\omega_1, \omega_3 \in \mathbb{C}, \operatorname{Re}\omega_3 > -1$. *Then we have*

$$\int_0^\pi \ln(\sin\theta)(\sin\theta + i\cos\theta)^{\omega_1}\sin^{\omega_3}\theta\, d\theta = \frac{\pi\left(2\Gamma(\omega_3+1)\ln 2 + 2\Gamma'(\omega_3+1) - \Gamma(\omega_3+1)\left(\psi\left(1+\frac{\omega_3-\omega_1}{2}\right)+\psi\left(1+\frac{\omega_3+\omega_1}{2}\right)\right)\right)}{2^{\omega_3+1}\Gamma(1+\frac{\omega_3-\omega_1}{2})\Gamma(1+\frac{\omega_3+\omega_1}{2})},$$

where ψ is the Psi function of Gauss $\psi(z) = \Gamma'(z)/\Gamma(z)$.

Proof. Differentiating in ω_3 both parts of (3) yields the desired result. ∎

Lemma 2. *Assume that $\omega_1, \omega_3 \in \mathbb{C}, \operatorname{Re}\omega_3 > -1$. Then we have*

$$\begin{aligned}&\int_0^\pi \ln(\sin\theta)(\sin\theta + i\cos\theta)^{\omega_1}\sin^{\omega_3}\theta\cos\theta\,d\theta\\ &= -\frac{i\omega_1}{(\omega_3+1)^2}\int_0^\pi (\sin\theta + i\cos\theta)^{\omega_1}\sin^{\omega_3+1}\theta\,d\theta\\ &\quad+\frac{i\omega_1}{\omega_3+1}\int_0^\pi \ln(\sin\theta)(\sin\theta + i\cos\theta)^{\omega_1}\sin^{\omega_3+1}\theta\,d\theta.\end{aligned}$$

Proof. Two times differentiating by parts gives the desired result. ∎

Lemma 3. *Assume that $\omega_1, \omega_2, \omega_3 \in \mathbb{C}, \operatorname{Re} w_1, \operatorname{Re} w_2 > 0, \operatorname{Re}\omega_3 > -1$. Then*

$$\begin{aligned}&\int_0^\pi (w_1\sin\theta + i\cos\theta)^{\omega_1}(w_2\sin\theta - i\cos\theta)^{\omega_2}\sin^{\omega_3}\theta\,d\theta\\ &= \frac{\pi\Gamma(\omega_3+1)F_2\left(1+\omega_3, -\omega_1, -\omega_2, 1+\frac{\omega_3-\omega_1+\omega_2}{2}, 1+\frac{\omega_3+\omega_1-\omega_2}{2}, \frac{1-w_1}{2}, \frac{1-w_2}{2}\right)}{2^{\omega_3}\Gamma(1+\frac{\omega_3+\omega_1-\omega_2}{2})\Gamma(1+\frac{\omega_3-\omega_1+\omega_2}{2})},\end{aligned} \tag{4}$$

where $F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y)$ is the two-variable hypergeometric function of Appel [1] *defined as*

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) = \sum \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m)(\gamma', n)(1, m)(1, n)} x^m y^n.$$

Proof. Define the left side of (4) by $F(\omega_1, \omega_2, \omega_3, w_1, w_2)$. It is clear that F is an analytic function of (w_1, w_2) when $\operatorname{Re} w_1, \operatorname{Re} w_2 > 0$. We have

$$\begin{aligned}&\frac{\partial^{m+n}F(\omega_1, \omega_2, \omega_3, w_1, w_2)}{\partial w_1^m \partial w_2^n}\\ &= (-1)^{m+n}\int_0^\pi (-\omega_1, m)(-\omega_2, n)(w_1\sin\theta + i\cos\theta)^{\omega_1-m}\\ &\qquad (w_2\sin\theta - i\cos\theta)^{\omega_2-n}\sin^{\omega_3+m+n}\theta\,d\theta\\ &= (-1)^{m+n}(-\omega_1, m)(-\omega_2, n)F(\omega_1-m, \omega_2-n, \omega_3+m+n, w_1, w_2).\end{aligned}$$

By using Lemma 1, we deduce that

$$(-1)^{m+n}\frac{\partial^{m+n}F(\omega_1,\omega_2,\omega_3,w_1,w_2)}{\partial w_1^m\partial w_2^n}\bigg|_{(w_1,w_2)=(1,1)}$$

$$= (-\omega_1,m)(-\omega_2,n)F(\omega_1-m,\omega_2-n,\omega_3+m+n,1,1)$$

$$= (-\omega_1,m)(-\omega_2,n)\int_0^\pi(\sin\theta+i\cos\theta)^{\omega_1-m}(\sin\theta-i\cos\theta)^{\omega_2-n}\sin^{\omega_3+m+n}\theta\,d\theta$$

$$= \int_0^\pi(-\omega_1,m)(-\omega_2,n)(\sin\theta+i\cos\theta)^{\omega_1-m-\omega_2+n}\sin^{\omega_3+m+n}\theta\,d\theta$$

$$= \frac{\pi 2^{-(\omega_3+m+n)}(-\omega_1,m)(-\omega_2,n)\Gamma(\omega_3+1+m+n)}{\Gamma(1+n+\frac{\omega_3+\omega_1-\omega_2}{2})\Gamma(1+m+\frac{\omega_3-\omega_1+\omega_2}{2})}.$$

Hence, the desired formula follows. ■

Corollary 2. *Under the assumptions of Lemma 3, if* $\omega_1+\omega_2+\omega_3=-2$, *then*

$$\int_0^\pi(w_1\sin\theta+i\cos\theta)^{\omega_1}(w_2\sin\theta-i\cos\theta)^{\omega_2}\sin^{\omega_3}\theta\,d\theta$$

$$= \frac{2\pi\Gamma(\omega_3+1)}{(w_1+w_2)^{\omega_3+1}\Gamma(1+\frac{\omega_3+\omega_1-\omega_2}{2})\Gamma(1+\frac{\omega_3-\omega_1+\omega_2}{2})}.$$

Proof. By Lemma 3, we have

$$\int_0^\pi(w_1\sin\theta+i\cos\theta)^{\omega_1}(w_2\sin\theta-i\cos\theta)^{\omega_2}\sin^{\omega_3}\theta\,d\theta$$

$$= \frac{\pi\Gamma(\omega_3+1)F_2\left(1+\omega_3,-\omega_1,-\omega_2,-\omega_1,-\omega_2,\frac{1-w_1}{2},\frac{1-w_2}{2}\right)}{2^{\omega_3}\Gamma(1+\frac{\omega_3+\omega_1-\omega_2}{2})\Gamma(1+\frac{\omega_3-\omega_1+\omega_2}{2})}.$$

Now, using the following relation (see [1, p. 15])

$$F_2(\alpha,\beta,\beta',\beta,\beta',x,y)=(1-x-y)^{-\alpha}$$

we get the desired result. ■

Corollary 3. *Under the assumptions of Corollary 2 but with* $\mathrm{Re}\,\omega_3>-1$ *replaced by* $\mathrm{Re}\,\omega_3>0$, *we have*

$$\int_0^\pi(w_1\sin\theta+i\cos\theta)^{\omega_1}(w_2\sin\theta-i\cos\theta)^{\omega_2}\sin^{\omega_3-1}\theta\cos\theta\,d\theta$$

$$= \frac{i\left((w_1+w_2)(\omega_1-\omega_2)+(w_1-w_2)\omega_3\right)\pi\Gamma(\omega_3)}{(w_1+w_2)^{\omega_3+1}\Gamma(1+\frac{\omega_3+\omega_1-\omega_2}{2})\Gamma(1+\frac{\omega_3-\omega_1+\omega_2}{2})}.$$

Proof. Note that

$$\int_0^\pi(w_1\sin\theta+i\cos\theta)^{\omega_1}(w_2\sin\theta-i\cos\theta)^{\omega_2}\sin^{\omega_3-1}\theta\cos\theta\,d\theta$$

$$= \frac{w_2}{i(w_1+w_2)}\int_0^\pi(w_1\sin\theta+i\cos\theta)^{\omega_1+1}(w_2\sin\theta-i\cos\theta)^{\omega_2}\sin^{\omega_3-1}\theta\,d\theta$$

$$- \frac{w_1}{i(w_1+w_2)}\int_0^\pi(w_1\sin\theta+i\cos\theta)^{\omega_1}(w_2\sin\theta-i\cos\theta)^{\omega_2+1}\sin^{\omega_3-1}\theta\,d\theta.$$

Applying Corollary 2 yields the desired result. ∎

3. Main Results

We try to find $u(x,y)$ in the following form:

$$u(x,y) = F_{\phi,\lambda}(x,y) \sim x\left(e^{i\phi-\frac{1}{|x|}}+iy\right)^{-\frac{1+\lambda}{2}}\left(e^{i\phi-\frac{1}{|x|}}-iy\right)^{-\frac{1-\lambda}{2}},$$

which are C^∞ away from the origin. The following conditions guarantee that $F_{\phi,\lambda}(x,y) \in C^\infty\left(\mathbb{R}^2\backslash(0,0)\right)$

$$\lambda_+ - \lambda_- = 4k, k \in \mathbb{Z} \text{ and}$$

$$F_{\phi,\lambda}(x,y) = \begin{cases} x\left(e^{i\phi-\frac{1}{|x|}}+iy\right)^{-\frac{1+\lambda_+}{2}}\left(e^{i\phi-\frac{1}{|x|}}-iy\right)^{-\frac{1-\lambda_+}{2}}, x \geq 0, \\ x\left(e^{i\phi-\frac{1}{|x|}}+iy\right)^{-\frac{1+\lambda_-}{2}}\left(e^{i\phi-\frac{1}{|x|}}-iy\right)^{-\frac{1-\lambda_-}{2}}, x \leq 0, \end{cases}$$

$$\text{or } \lambda_+ - \lambda_- = 4k+2, k \in \mathbb{Z} \text{ and}$$

$$F_{\phi,\lambda}(x,y) = \begin{cases} x\left(e^{i\phi-\frac{1}{|x|}}+iy\right)^{-\frac{1+\lambda_+}{2}}\left(e^{i\phi-\frac{1}{|x|}}-iy\right)^{-\frac{1-\lambda_+}{2}}, x \geq 0, \\ -x\left(e^{i\phi-\frac{1}{|x|}}+iy\right)^{-\frac{1+\lambda_-}{2}}\left(e^{i\phi-\frac{1}{|x|}}-iy\right)^{-\frac{1-\lambda_-}{2}}, x \leq 0. \end{cases}$$

In future, we will consider only the above cases. Therefore, we can simplify our writing as follows (in both cases):

$$F_{\phi,\lambda}(x,y) = \begin{cases} x\left(e^{i\phi-\frac{1}{|x|}}+iy\right)^{-\frac{1+\lambda_+}{2}}\left(e^{i\phi-\frac{1}{|x|}}-iy\right)^{-\frac{1-\lambda_+}{2}}, x \geq 0, \\ (-1)^{\frac{\lambda_+-\lambda_-}{2}}x\left(e^{i\phi-\frac{1}{|x|}}+iy\right)^{-\frac{1+\lambda_-}{2}}\left(e^{i\phi-\frac{1}{|x|}}-iy\right)^{-\frac{1-\lambda_-}{2}}, x \leq 0. \end{cases}$$

Let us rewrite $G_{\phi,\lambda} = X_2X_1 + i\left((\lambda+1)x^{-4} - 2|x|^{-3}\right)e^{i\phi-1/|x|}\,\partial/\partial y$, where $X_1 = \partial/\partial x - i\,\mathrm{sign}(x)x^{-2}e^{i\phi-1/|x|}\,\partial/\partial y$, $X_2 = \partial/\partial x + i\,\mathrm{sign}(x)x^{-2}e^{i\phi-1/|x|}\,\partial/\partial y$. Note that

$$X_1\left(e^{i\phi-\frac{1}{|x|}}+iy\right) = 2\mathrm{sign}(x)x^{-2}e^{i\phi-\frac{1}{|x|}}, X_1\left(e^{i\phi-\frac{1}{|x|}}-iy\right) = 0,$$
$$X_2\left(e^{i\phi-\frac{1}{|x|}}+iy\right) = 0, X_2\left(e^{i\phi-\frac{1}{|x|}}-iy\right) = 2\mathrm{sign}(x)x^{-2}e^{i\phi-\frac{1}{|x|}}.$$

Therefore, we have $G_{\phi,\lambda}F_{\phi,\lambda}(x,y) = 0$ in $\mathbb{R}^2\backslash(0,0)$. Since $F_{\phi,\lambda}(x,y) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ (see the proof of Theorem 1), it follows from a theorem of Schwartz that $G_{\phi,\lambda}F_{\phi,\lambda}$ is a finite sum of δ-Dirac function and its derivatives.

Theorem 1. *Assume that* $-\pi/2 < \phi < \pi/2$, $\lambda_+, \lambda_- \in \mathbb{C}$, $\lambda_+ - \lambda_- \in 2\mathbb{Z}$. *Then*

$$G_{\phi,\lambda}F_{\phi,\lambda}(x,y) = T_{\phi,\lambda}\delta(x,y) + 4\cos\frac{\pi\lambda_+}{2}\frac{\partial\delta(x,y)}{\partial x}, \tag{5}$$

where

$$T_{\phi,\lambda} = \int_0^{\pi} A_{+,\phi}^{-\frac{1+\lambda_+}{2}} A_{-,\phi}^{-\frac{1-\lambda_+}{2}} \sin\theta\, d\theta + (-1)^{\frac{2+\lambda_+-\lambda_-}{2}} \int_{-\pi}^{0} A_{+,\phi}^{-\frac{1+\lambda_-}{2}} A_{-,\phi}^{-\frac{1-\lambda_-}{2}} |\sin\theta|\, d\theta$$
$$+ e^{i\phi} \int_0^{\pi} \ln(\sin\theta) A_{+,\phi}^{-\frac{3+\lambda_+}{2}} A_{-,\phi}^{-\frac{3-\lambda_+}{2}} \left(e^{i\phi}\sin\theta - i\lambda_+\cos\theta\right) d\theta +$$
$$+ (-1)^{\frac{2+\lambda_+-\lambda_-}{2}} e^{i\phi} \int_{-\pi}^{0} \ln|\sin\theta| A_{+,\phi}^{-\frac{3+\lambda_-}{2}} A_{-,\phi}^{-\frac{3-\lambda_-}{2}} \left(e^{i\phi}|\sin\theta| - i\lambda_-\cos\theta\right) d\theta,$$

where we have used $A_{+,\phi} = e^{i\phi}|\sin\theta| + i\cos\theta, A_{-,\phi} = e^{i\phi}|\sin\theta| - i\cos\theta.$

Proof. We begin by noting that if $-\pi/2 < \phi < \pi/2$ then $\left(e^{i\phi-1/|x|} + iy\right)^{\alpha}$ and $\left(e^{i\phi-1/|x|} - iy\right)^{\beta} \in C^{\infty}\left(\mathbb{R}^2\backslash(0,0)\right)$ for every α and β in $\mathbb{C}$. Let us introduce the following "polar coordinate":

$$x = -\frac{\operatorname{sign}(\sin\theta)}{\ln(\rho|\sin\theta|)}, y = \rho\cos\theta, dxdy = \frac{d\rho d\theta}{|\sin\theta|\left(\ln\rho|\sin\theta|\right)^2}; \rho < 1, -\pi < \theta < \pi.$$

Note that the map $(x,y) \mapsto (\rho,\theta)$ is not a diffeomorphism along the line $x = 0$. But it is good enough for us because in future we will use it only for integration, and if necessary we can take integrals as a limit. It is easy to verify that $\rho^2 = e^{-2/|x|} + y^2$. First we prove that $F_{\phi,\lambda}(x,y) \in L^1_{loc}(\mathbb{R}^2)$. Indeed, since $F_{\phi,\lambda}(x,y) \in C^{\infty}\left(\mathbb{R}^2\backslash(0,0)\right)$ it suffices to prove that $F_{\phi,\lambda}(x,y) \in L^1(B_\varepsilon)$, where $B_\varepsilon = \{(x,y)|\rho(x,y) < \varepsilon\}$, and ε is small enough. We have

$$\int_{B_\varepsilon} |F_{\phi,\lambda}(x,y)|\, dxdy \le 2\int_{-\pi}^{\pi}\int_0^{\varepsilon} \frac{d\rho d\theta}{\rho|\sin\theta|\left|\ln\rho|\sin\theta|\right|^3}$$
$$= 8\int_0^{\varepsilon} \rho^{-1}\, d\rho \int_0^{\pi/2} \frac{d\theta}{\left|\ln\rho|\sin\theta|\right|^3 |\sin\theta|}.$$

For a small fixed ρ let us make use of the following change of variables $\xi = \ln(\sin\theta)/\ln\rho$. If $\rho \le e^{-1}$, we have

$$\int_0^{\pi/2} \frac{d\theta}{\left|\ln\rho|\sin\theta|\right|^3|\sin\theta|} = \frac{1}{(\ln\rho)^2}\int_0^{\infty} \frac{d\xi}{\left(1 - e^{2\xi\ln\rho}\right)^{1/2}|1+\xi|^3}$$
$$\le \frac{1}{(\ln\rho)^2}\int_0^{\infty} \frac{d\xi}{\left(1 - e^{-2\xi}\right)^{1/2}|1+\xi|^3} \le \frac{C}{(\ln\rho)^2}.$$

Thus,

$$\int_{B_\varepsilon} |F_{\phi,\lambda}(x,y)|\, dxdy \le C\int_0^{\varepsilon} \frac{d\rho}{\rho(\ln\rho)^2} < \infty.$$

Note that $F_{\phi,\lambda}(x,y) \notin L^p_{loc}(\mathbb{R}^2)$ for any $p \in (1,\infty]$. Now, let $\mathbb{R}^2_\varepsilon = \{(x,y) \in \mathbb{R}^2 | \rho(x,y) \geq \varepsilon\}$. By applying Green's formula, we have

$$\int_{\mathbb{R}^2_\varepsilon} f(x,y) G_{\phi,-\lambda} v(x,y)\,dxdy = \int_{\mathbb{R}^2_\varepsilon} v(x,y) G_{\phi,\lambda} f(x,y)\,dxdy$$

$$- \int_{\rho=\varepsilon} v(x,y)\Big\{\nu_1 \frac{\partial f(x,y)}{\partial x} + \nu_2\, x^{-4} e^{2i\phi - \frac{2}{|x|}} \frac{\partial f(x,y)}{\partial y} + i\lambda\,\nu_2\, x^{-4} e^{i\phi - \frac{1}{|x|}} f(x,y)\Big\} ds$$

$$+ \int_{\rho=\varepsilon} f(x,y)\Big\{\nu_1 \frac{\partial v(x,y)}{\partial x} + \nu_2\, x^{-4} e^{2i\phi - \frac{2}{|x|}} \frac{\partial v(x,y)}{\partial y}\Big\} ds$$

$$:= \int_{\mathbb{R}^2_\varepsilon} V(f,v,\phi,\lambda)\,dxdy - \int_{\rho=\varepsilon} v(x,y) B_1(f,\phi,\lambda)\,ds + \int_{\rho=\varepsilon} f(x,y) B_2(v,\phi)\,ds \qquad (6)$$

for every $v(x,y) \in C_0^\infty(\mathbb{R}^2), f(x,y) \in C^\infty(\mathbb{R}^2 \backslash (0,0))$, where $\nu = (\nu_1, \nu_2)$ is the unit outward normal to $\partial\mathbb{R}^2_\varepsilon$. Replacing $f(x,y)$ in (6) by $F_{\phi,\lambda}(x,y)$, we obtain

$$\int_{\mathbb{R}^2_\varepsilon} F_{\phi,\lambda}(x,y) G_{\phi,-\lambda} v(x,y)\,dxdy$$

$$= \int_{\mathbb{R}^2_\varepsilon} V(F_{\phi,\lambda}, v, \phi, \lambda)\,dxdy - \int_{\rho=\varepsilon} v(x,y) B_1(F_{\phi,\lambda},\phi,\lambda)\,ds$$

$$+ \int_{\rho=\varepsilon} F_{\phi,\lambda}(x,y) B_2(v,\phi)\,ds. \qquad (7)$$

The first integral in the right side of (7) vanishes. We now compute the third integral in the right side of (7). It is easy to check that

$$ds\Big|_{\partial B_\varepsilon} = \Big((\ln\varepsilon|\sin\theta|)^{-4}\sin^{-2}\theta\cos^2\theta + \varepsilon^2\sin^2\theta\Big)^{1/2}\,d\theta$$

and

$$\nu = (\nu_1,\nu_2) = -\left(\frac{\operatorname{sign}(x) x^{-2} e^{-2/|x|}}{(x^{-4}e^{-4/|x|} + y^2)^{1/2}}, \frac{y}{(x^{-4}e^{-4/|x|} + y^2)^{1/2}}\right).$$

Hence,

$$\Big(x^{-4}e^{-4/|x|} + y^2\Big)^{-1/2} ds\Big|_{\partial B_\varepsilon} = \frac{d\theta}{\varepsilon|\sin\theta|\,(\ln\varepsilon|\sin\theta|)^2}.$$

It follows that

$$\Big|\int_{\partial B_\varepsilon} F_{\phi,\lambda}(x,y) B_2(v,\phi)\,ds\Big| \leq C \int_{\rho=\varepsilon} |F_{\phi,\lambda}(x,y)|.\Big(|\nu_1| + |\nu_2.\,x^{-2}e^{-1/|x|}|\Big)\,ds$$

$$\leq C \int_{\rho=\varepsilon} \frac{|A_{+,\phi}^{-(1+\lambda)/2}||A_{-,\phi}^{-(1-\lambda)/2}|\,(x^{-2}e^{-2/|x|} + x^{-2}e^{-1/|x|}|y|)\,ds}{\varepsilon\big|\ln\varepsilon|\sin\theta|\,big|\,(x^{-4}e^{-4/|x|} + y^2)^{1/2}}$$

$$\leq \frac{C}{|\ln\varepsilon|}\int_{-\pi}^{\pi} |A_{+,\phi}^{-(1+\lambda)/2}||A_{-,\phi}^{-(1-\lambda)/2}|\,(|\sin\theta| + |\cos\theta|)\,d\theta \to 0 \text{ as } \varepsilon \to 0. \qquad (8)$$

Next, we evaluate $B_1(F_{\phi,\lambda},\phi,\lambda)$. We have

$$B_1(F_{\phi,\lambda},\phi,\lambda)=\begin{cases}-\left(e^{i\phi-\frac{1}{|x|}}+iy\right)^{\frac{3+\lambda}{2}}\left(e^{i\phi-\frac{1}{|x|}}-iy\right)^{-\frac{3-\lambda}{2}}\times\\ \times\left(x^{-4}e^{-4/|x|}+y^2\right)^{-\frac{1}{2}}\left(x^{-2}e^{-\frac{2}{|x|}}\left(e^{2i\phi-\frac{2}{|x|}}+y^2\right)+\right.\\ \left.+x^{-3}e^{i\phi-\frac{1}{|x|}}\left(i\lambda y-e^{i\phi-\frac{1}{|x|}}\right)\left(e^{-\frac{2}{|x|}}+y^2\right)\right),\ x\geq 0,\\ (-1)^{\frac{2+\lambda_+-\lambda_-}{2}}\left(e^{i\phi-\frac{1}{|x|}}+iy\right)^{-\frac{3+\lambda}{2}}\left(e^{i\phi-\frac{1}{|x|}}-iy\right)^{\frac{3-\lambda}{2}}\times\\ \times\left(x^{-4}e^{-4/|x|}+y^2\right)^{-\frac{1}{2}}\left(-x^{-2}e^{-\frac{2}{|x|}}\left(e^{2i\phi-\frac{2}{|x|}}+y^2\right)+\right.\\ \left.+x^{-3}e^{i\phi-\frac{1}{|x|}}\left(i\lambda y-e^{i\phi-\frac{1}{|x|}}\right)\left(e^{-\frac{2}{|x|}}+y^2\right)\right),\ x\leq 0.\end{cases}$$

Therefore,

$$-\int_{\rho=\varepsilon}v(x,y)B_1(F_{\phi,\lambda},\phi,\lambda)\,ds=\int_0^{\pi}v(\varepsilon,\theta)A_{+,\phi}^{\frac{1+\lambda_+}{2}}A_{-,\phi}^{-\frac{1-\lambda_+}{2}}\sin\theta\,d\theta$$

$$+(-1)^{\frac{2+\lambda_+-\lambda_-}{2}}\int_{-\pi}^{0}v(\varepsilon,\theta)A_{+,\phi}^{-\frac{1+\lambda_-}{2}}A_{-,\phi}^{-\frac{1-\lambda_-}{2}}|\sin\theta|\,d\theta$$

$$+\int_0^{\pi}v(\varepsilon,\theta)e^{i\phi}\ln(\varepsilon\sin\theta)A_{+,\phi}^{-\frac{3+\lambda_+}{2}}A_{-,\phi}^{-\frac{3-\lambda_+}{2}}\left(e^{i\phi}\sin\theta-i\lambda_+\cos\theta\right)\,d\theta$$

$$+(-1)^{\frac{2+\lambda_+-\lambda_-}{2}}\int_{-\pi}^{0}v(\varepsilon,\theta)e^{i\phi}\ln(\varepsilon|\sin\theta|)A_{+,\phi}^{-\frac{3+\lambda_-}{2}}A_{-,\phi}^{-\frac{3-\lambda_-}{2}}\left(e^{i\phi}|\sin\theta|-i\lambda_-\cos\theta\right)\,d\theta.$$

Using the expansion of $v(x,y)$ in Taylor's series around $(0,0)$

$$v(x,y)=v(0,0)+x\frac{\partial v(0,0)}{\partial x}+y\frac{\partial v(0,0)}{\partial y}+O(x^2+y^2)\quad\text{as}\quad x^2+y^2\to 0,$$

we have

$$v(\varepsilon,\theta)=v(0,0)-\frac{\operatorname{sign}(\sin\theta)}{\ln(\varepsilon|\sin\theta|)}\frac{\partial v(0,0)}{\partial x}+O\left((1/\ln\varepsilon)^2\right)\quad\text{as}\quad\varepsilon\to 0.$$

Hence, we deduce that

$$
\begin{aligned}
&-\int\limits_{\rho=\varepsilon} v(x,y)B_1(F_{\phi,\lambda},\phi,\lambda)\,ds = \Bigg(\int\limits_0^\pi A_{+,\phi}^{-\frac{3+\lambda_+}{2}} A_{-,\phi}^{-\frac{3-\lambda_+}{2}} \left(e^{i\phi}\sin\theta - i\lambda_+\cos\theta\right)\,d\theta \\
&+(-1)^{\frac{2+\lambda_+-\lambda_-}{2}}\int\limits_{-\pi}^{0} A_{+,\phi}^{-\frac{3+\lambda_-}{2}} A_{-,\phi}^{-\frac{3-\lambda_-}{2}} \left(e^{i\phi}|\sin\theta| - i\lambda_-\cos\theta\right)\,d\theta\Bigg) e^{i\phi}v(0,0)\ln\varepsilon \\
&+\Bigg(\int\limits_0^\pi A_{+,\phi}^{-\frac{1+\lambda_+}{2}} A_{-,\phi}^{-\frac{1-\lambda_+}{2}}\sin\theta\,d\theta + (-1)^{\frac{2+\lambda_+-\lambda_-}{2}}\int\limits_{-\pi}^{0} A_{+,\phi}^{-\frac{1+\lambda_-}{2}} A_{-,\phi}^{-\frac{1-\lambda_-}{2}}|\sin\theta|\,d\theta \\
&+e^{i\phi}\int\limits_0^\pi \ln(\sin\theta) A_{+,\phi}^{-\frac{3+\lambda_+}{2}} A_{-,\phi}^{-\frac{3-\lambda_+}{2}} \left(e^{i\phi}\sin\theta - i\lambda_+\cos\theta\right)\,d\theta \\
&+(-1)^{\frac{2+\lambda_+-\lambda_-}{2}} e^{i\phi}\int\limits_{-\pi}^{0} \ln|\sin\theta| A_{+,\phi}^{-\frac{3+\lambda_-}{2}} A_{-,\phi}^{-\frac{3-\lambda_-}{2}} \left(e^{i\phi}|\sin\theta| - i\lambda_-\cos\theta\right)\,d\theta\Bigg) v(0,0) \\
&+\Bigg(-e^{i\phi}\int\limits_0^\pi A_{+,\phi}^{-\frac{3+\lambda_+}{2}} A_{-,\phi}^{-\frac{3-\lambda_+}{2}} \left(e^{i\phi}\sin\theta - i\lambda_+\cos\theta\right)\,d\theta + (-1)^{\frac{2+\lambda_+-\lambda_-}{2}} e^{i\phi}\times \\
&\quad\times\int\limits_{-\pi}^{0} A_{+,\phi}^{-\frac{3+\lambda_-}{2}} A_{-,\phi}^{-\frac{3-\lambda_-}{2}} \left(e^{i\phi}|\sin\theta| - i\lambda_-\cos\theta\right)\,d\theta\Bigg)\frac{\partial v(0,0)}{\partial x} \\
&+O\left(1/\ln\varepsilon\right) \text{ as } \varepsilon\to 0. \qquad (9)
\end{aligned}
$$

By a theorem of Schwartz, without computation we can deduce that the coefficient of $\ln\varepsilon$ on the right side of (9) is equal to zero. Alternatively we can see it by using Corollary 2 and Corollary 3. Thus

$$
\begin{aligned}
&\int\limits_0^\pi A_{+,\phi}^{-\frac{3+\lambda_+}{2}} A_{-,\phi}^{-\frac{3-\lambda_+}{2}} \left(e^{i\phi}\sin\theta - i\lambda_+\cos\theta\right)\,d\theta \\
&\quad+(-1)^{\frac{2+\lambda_+-\lambda_-}{2}}\int\limits_{-\pi}^{0} A_{+,\phi}^{-\frac{3+\lambda_-}{2}} A_{-,\phi}^{-\frac{3-\lambda_-}{2}} \left(e^{i\phi}|\sin\theta| - i\lambda_-\cos\theta\right)\,d\theta \\
&\quad= 2e^{-i\phi}\left(\cos\frac{\pi\lambda_+}{2} + (-1)^{\frac{2+\lambda_+-\lambda_-}{2}}\cos\frac{\pi\lambda_-}{2}\right) = 0.
\end{aligned}
$$

Therefore, we obtain

$$
-\int\limits_{\rho=\varepsilon} v(x,y)B_1(F_{\phi,\lambda},\phi,\lambda)\,ds \to T_{\phi,\lambda}v(0,0) - 4\cos\frac{\pi\lambda_+}{2}\frac{\partial v(0,0)}{\partial x} \quad \text{as } \varepsilon\to 0. \qquad (10)
$$

Now from (7), (8), and (10), we have

$$\begin{aligned}(G_{\phi,\lambda}F_{\phi,\lambda}(x,y), v(x,y)) &= (F_{\phi,\lambda}(x,y), G_{\phi,-\lambda}v(x,y)) \\ &= \lim_{\varepsilon\to 0}\int\limits_{\rho\geq\varepsilon} F_{\phi,\lambda}(x,y)G_{\phi,-\lambda}v(x,y)dxdy \\ &= T_{\phi,\lambda}v(0,0) - 4\cos\frac{\pi\lambda_+}{2}\frac{\partial v(0,0)}{\partial x}.\end{aligned}$$

Hence, $G_{\phi,\lambda}F_{\phi,\lambda}(x,y) = T_{\phi,\lambda}\delta(x,y) + 4\cos(\pi\lambda_+/2)\,\partial\delta(x,y)/\partial x$. ∎

Now we would like to find the values λ_+, λ_- such that the right side of (5) vanishes. It is obvious that the first necessary condition is $\cos(\pi\lambda_+/2) = 0$, or in other words, $\lambda_+ = 2k+1, \lambda_- = 2l+1$, where $k, l \in \mathbb{Z}$. We have the following:

Theorem 2. *Assume that* $-\pi/2 < \phi < \pi/2$. *Then* $T_{\phi,\lambda} = 0$ *when* $\lambda_+ = 2k+1, \lambda_- = 2l+1$, *or* $\lambda_+ = -2k-1, \lambda_- = -2l-1$, *where* k *and* l *are non-negative integers.*

Proof. Let us introduce the following function:

$$\begin{aligned}L(z,\mu) = &\int_0^\pi B_{+,z}^{-(1+\mu)/2}\, B_{-,z}^{-(1-\mu)/2}\sin\theta\, d\theta + \\ &+ \int_0^\pi \ln(\sin\theta) B_{+,z}^{-(3+\mu)/2}\, B_{-,z}^{-(3-\mu)/2}\left(z^2\sin\theta - iz\mu\cos\theta\right) d\theta,\end{aligned}$$

where we have used $B_{+,z} = z\sin\theta + i\cos\theta, B_{-,z} = z\sin\theta - i\cos\theta$. We will need the following

Lemma 4. *Assume that* $\operatorname{Re} z > 0$. *Then, for any integer* $k \geq 0$, *we have*

$$L(z, 2k+1) = (-1)^k\pi; L(z, -2k-1) = (-1)^k\pi.$$

Proof. First using Lemma 1, Corollary 1 and Lemma 2, we obtain

$$L(1,\mu) = \left(\frac{2(1+\mu^2)}{1-\mu^2} + 2\ln 2 + \Gamma'(1) - \psi\left(\frac{3+\mu}{2}\right) - \psi\left(\frac{3-\mu}{2}\right)\right)\cos\frac{\pi\mu}{2}.$$

Therefore, we deduce that

$$L(1, 2k+1) = (-1)^k\pi, \ \text{ if } \ k\geq 0; L(1, -2k-1) = (-1)^k\pi, \ \text{ if } \ k \geq 0.$$

Since for any fixed μ the function $L(z,\mu)$ is analytic with respect to z, in the half-plane $\operatorname{Re} z > 0$, it is sufficient to prove that $\left.\frac{d^n L(z,\mu)}{dz^n}\right|_{z=1} = 0$ for every positive integer n when μ is an odd integer. By the Leibniz formula, for an integer $n \geq 1$ we have

$$\begin{aligned}\frac{(-1)^n d^n L(z,\mu)}{dz^n}\Big|_{z=1} &= \sum_{m=0}^{n} C_n^m \Bigg(\left(\frac{1+\mu}{2}, m\right)\left(\frac{1-\mu}{2}, n-m\right) \\ &\quad \times \int_0^\pi B_{+,1}^{-\mu+n-2m} \sin^{n+1}\theta \, d\theta + \left(\frac{3+\mu}{2}, m\right)\left(\frac{3-\mu}{2}, n-m\right) \\ &\quad \times \int_0^\pi \ln(\sin\theta) B_{+,1}^{-\mu+n-2m} \left(\sin^{n+1}\theta - i\mu \sin^n\theta\cos\theta\right) d\theta \Bigg) \\ &\quad - n\sum_{m=0}^{n-1} C_{n-1}^m \left(\frac{3-\mu}{2}, n-m-1\right)\left(\frac{3+\mu}{2}, m\right) \\ &\quad \times \int_0^\pi \ln(\sin\theta) B_{+,1}^{-\mu+n-2m-1} \left(2\sin^n\theta - i\mu\sin^{n-1}\theta\cos\theta\right) d\theta \\ &\quad + n(n-1)\sum_{m=0}^{n-2} C_{n-2}^m \left(\frac{3+\mu}{2}, m\right)\left(\frac{3-\mu}{2}, n-m-2\right) \\ &\quad \times \int_0^\pi \ln(\sin\theta) B_{+,1}^{-\mu+n-2m-2} \sin^{n-1}\theta \, d\theta.\end{aligned}$$

After some computation, using Lemma 1, Corollary 1 and Lemma 2 we see that

$$\frac{(-1)^n d^n L(z,\mu)}{dz^n}\Big|_{z=1} = S_1^n(\mu) + S_2^n(\mu) + S_3^n(\mu) + S_4^n(\mu) + S_5^n(\mu) + S_6^n(\mu) + S_7^n(\mu),$$

where

$$S_1^n(\mu) = \frac{\Gamma(n+2)\cos(\pi\mu/2)}{2^{n+1}} \sum_{m=0}^{n} \frac{C_n^m}{[(1+\mu)/2+m][(1-\mu)/2+n-m]},$$

$$S_2^n(\mu) = \frac{\Gamma(n)\cos(\pi\mu/2)}{2^{n-1}(n+1)(1-\mu^2)} \sum_{m=0}^{n} n(\mu-n+2m)\mu C_n^m,$$

$$S_3^n(\mu) = -\frac{\Gamma(n)\cos(\pi\mu/2)}{2^{n-2}(1-\mu^2)} \sum_{m=0}^{n-1} (\mu-n+2m+1)\mu C_{n-1}^m,$$

$$\begin{aligned}S_4^n(\mu) = \frac{n\cos(\pi\mu)/2)\,(\Gamma(n)\ln 2 + \Gamma'(n))}{2^{n-1}\,(1-\mu^2)} &\Big(\sum_{m=0}^{n} (n+1+(n-2m-\mu)\mu) C_n^m \\ &- \sum_{m=0}^{n-1} 2(2n+(n-2m-\mu-1)\mu) C_{n-1}^m + (n-1)2^n\Big),\end{aligned}$$

$$S_5^n(\mu) = \frac{\Gamma(n)\cos(\pi\mu/2)}{2^{n-1}(n+1)(1-\mu^2)} \sum_{m=0}^{n} (2n+1)\Big(n+1+(n-2m-\mu)\mu\Big)C_n^m,$$

$$S_6^n(\mu) = -\frac{\Gamma(n)\cos(\pi\mu/2)}{2^{n-1}(1-\mu^2)} \sum_{m=0}^{n-1} 2(2n+(n-2m-\mu-1)\mu)C_{n-1}^m,$$

$$S_7^n(\mu) = -\frac{\Gamma(n+1)\cos(\pi\mu/2)}{2^n(1-\mu^2)} \times$$

$$\times \Bigg(\sum_{m=0}^{n}(n+1+(n-2m-\mu)\mu)\Big(\psi\Big(\frac{3-\mu}{2}+n-m\Big)+\psi\Big(\frac{3+\mu}{2}+m\Big)\Big)C_n^m$$

$$-\sum_{m=0}^{n-1} 2(2n+(n-2m-\mu-1)\mu)\Big(\psi\Big(\frac{1-\mu}{2}+n-m\Big)+\psi\Big(\frac{3+\mu}{2}+m\Big)\Big)C_{n-1}^m$$

$$+\sum_{m=0}^{n-2} 4(n-1)\Big(\psi\Big(\frac{-1-\mu}{2}+n-m\Big)+\psi\Big(\frac{3+\mu}{2}+m\Big)\Big)C_{n-2}^m\Bigg).$$

It is not difficult to show that

$$S_2^n(1)+S_5^n(1) = S_2^n(-1)+S_5^n(-1) = \pi\Gamma(n+1),$$
$$S_3^n(1)+S_6^n(1) = S_3^n(-1)+S_6^n(-1) = -\pi\Gamma(n+1),$$
$$S_1^n(1) = S_1^n(-1) = \frac{\pi\Gamma(n+1)}{2^{n+1}},\ S_7^n(1) = S_7^n(-1) = -\frac{\pi\Gamma(n+1)}{2^{n+1}},$$
$$S_4^n(\mu) = 0 \text{ for every odd integer } \mu,$$
$$S_2^n(\mu) = S_3^n(\mu) = S_5^n(\mu) = S_6^n(\mu) = 0 \text{ for every odd integer } \mu \neq \pm 1,$$
$$S_1^n(2k+1) = S_1^n(-2k-1) = \frac{(-1)^k\pi\Gamma(n+1)C_n^k}{2^{n+1}} \text{ for every integer } k \geq 1,$$
$$S_7^n(2k+1) = S_7^n(-2k-1) = -\frac{(-1)^k\pi\Gamma(n+1)C_n^k}{2^{n+1}} \text{ for every integer } k \geq 1.$$

Therefore, we deduce that $\frac{d^n L(z,\mu)}{dz^n}\big|_{z=1} = 0$ for every positive integer n when μ is an odd integer. ■

(*Continuing the proof of Theorem 2*) Note that

$$T_{\phi,\lambda} = L(e^{i\phi},\lambda_+) + (-1)^{\frac{2+\lambda_+-\lambda_-}{2}} L(e^{i\phi},\lambda_-).$$

Therefore, applying Lemma 4 yields the desired result. ■

Remark 1. Combining Theorems 1 and 2 gives all the values λ_+, λ_-, where $G_{\phi,\lambda}$ is not hypoelliptic as stated in [13]. It seems that our non-smooth solutions can be also obtained by a method in [13] with the cut-off function $\chi_-(\xi) \in C^\infty(\mathbb{R})$ replaced by the Heaviside function.

Remark 2. Since the coefficients of $G_{\phi,\lambda}$ belong to the Gevrey class $G^2(\mathbb{R}^2)$, it makes sense to say about the s-Gevrey hypoellipticity of $G_{\phi,\lambda}$ for $s \geq 2$. From

Theorems 1 and 2, we also see that $G_{\phi,\lambda}$ is not s-Gevrey hypoelliptic ($s \geq 2$) when $\lambda_+ = 2k+1, \lambda_- = 2l+1$, or $\lambda_+ = -2k-1, \lambda_- = -2l-1$, where k and l are non-negative integers.

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