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Bézout Identities with Inequality Constraints

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We dedicate this paper to Ambikeshwar Sharma with esteem, admiration and gratitude for lighting our way.

Abstract. This paper examines the set $\mathcal{B}(P) = \{Q : P \cdot Q = 1,\ Q \in \mathcal{R}^m\}$, where $P \in \mathcal{R}^m$ is unimodular and $\mathcal{R}$ is either the algebra $\mathcal{P}_R$ of algebraic polynomials which are real-valued on the cube $\mathbb{I}^d$ or the algebra $\mathcal{L}_R$ of Laurent polynomials which are real-valued on the torus $\mathbb{T}^d$. We sharpen previous results for the case $m = 2$, $d = 1$ by showing that if P is non-negative, then there exists a positive $Q \in \mathcal{B}(P)$ whose length is bounded by a function of the length of P and the separation between the zeros of P. In the general case we employ the Quillen–Suslin theorem, the Swan theorem, the Weierstrass approximation theorem, and the Michael selection theorem to prove a result about the existence of solutions to the Bézout identity with inequality constraints.

1. Introduction

In this paper we present several improvements of the results in our recent paper [30] which was motivated by our interests in orthonormal wavelet construction [33, 34] and conjugate quadrature filter design [22, 31]. As explained in [30], this leads one to the problem of solving certain algebraic identities with inequality constraints over Laurent polynomial rings. These inequality constraints state that the univariate Laurent polynomials have non-negative values on the unit circle and this implies they have spectral factors. The Bézout equation implies that these spectral factors are conjugate quadrature filters associated with orthonormal wavelet bases. In this paper, a special case, provided by Corollary 4.1, of inequality constraints for multivariate Laurent polynomials state that they have non-negative values on the torus. Although they generally do not have spectral factors, the Bézout equation implies that they are associated with interpolatory positive definite functions. These functions are potentially useful for interpolating autocovariance functions from their values on a lattice.

The general form of the identity we study is the equation

$$f_1 g_1 + \cdots + f_m g_m = 1, \tag{1.1}$$

where, for each $k = 1, \ldots, m$, the products $f_k g_k$ belong to a ring $\mathcal{R}$ with identity 1. For us, $f_1, \ldots, f_m$ are specified and a solution $g_1, \ldots, g_m$ satisfying certain auxiliary conditions is sought.

The earliest known equation of this form was studied by the ancient Hindu mathematician Aryabhata [49]. He characterized the set of integer solutions x, y of the equation

$$ax + by = 1, \tag{1.2}$$

where a and b are integers. The modern notion involving polynomials originated from the algebraic investigations of Bézout [9, 10, 23], after whom the identities are often named. In deference to common practice we will refer to these equations as Bézout identities.

Bézout identities play a central role in mathematics and its applications to science and engineering. A description of some of their multitude of applications can be found in several of the references. For example, Bézout identities involving polynomial rings have applications in transcendental number theory [2], spectral theory [16, 17], infinite dimension realization theory [18], and robust stabilization [21]. Bézout identities involving rings of entire functions with growth conditions arise in the problem of finding all distributional solutions to a system of linear partial differential equations, and in image and signal processing wavelet construction [38] applications requiring deconvolution [1–7]. Bézout identities involving the ring H^∞ of bounded analytic functions on the disc are the focus of the corona theorem [12, 20]. Bézout identities involving matrices also occur in signal processing and control theory. In particular, if $f_1, \ldots, f_m$ are $p \times q$ matrices with coefficients in $\mathbb{F}[x]$, the ring of polynomials in the indeterminate x over the field F, then there exist $q \times p$ matrices $g_1, \ldots, g_m$ with coefficients in $\mathbb{F}[x]$ that satisfy the Bézout identity (1.1) if and only if $f_1, \ldots, f_m$ are right–coprime [47, p. 693]; 19, p. 239-241]. Multivariable extensions are discussed in [11, p. 54-72]. Bézout identities also play an important role in the graphical display of curves and surfaces [13, 32, 43]. The construction of locally supported multivariate interpolation schemes [37, 43] as well as the locally finite decomposition of transtlation invariant spaces [35, 36] also leads to the study of Bézout identities.

Of the many important theorems which address the existence of solutions of the Bézout identity, the most fundamental is the Nullstellensatz which states that when $f_1, \ldots, f_m$ are in the ring of algebraic polynomials

$$\mathcal{R} = \mathbb{F}[x_1, \ldots, x_d],$$

where $\mathbb{F}$ is a subfield of the field $\mathbb{C}$ of complex numbers, there exists a solution $g_1, \ldots, g_m \in \mathcal{R}$ of Eq. (1.1) if and only if the set of common roots of the polynomials $f_1, \ldots, f_m$

$$V := \{z : z \in \mathbb{C}^d,\ f_1(z) = \cdots = f_m(z) = 0\}$$

is empty, that is, $V = \phi$. For the ring of Laurent polynomials

$$\mathcal{R} = \mathbb{F}[x_1, \ldots, x_d, x_1^{-1}, \ldots, x_d^{-1}],$$

the existence of a solution to Eq. (1.1) is guaranteed if and only if

$$V \subset \{z : z = (z_1, \ldots, z_d) \in \mathbb{C}^d,\ z_1 \cdots z_d = 0\}$$

(see [15, 28, 46, 48]).

The material in this paper falls naturally into two categories. In the first part, we review the proof of our result presented in [30]. This allows us to contrast it with a new and simpler proof presented here. Moreover, some additional observations are presented in the univariate case concerning the degrees of the solution of a given Bézout identity which were inaccessible by the methods of [30]. In the second part, we fully exploit this alternate derivation to obtain a general multivariate version of our main result from [30]. To accomplish this objective, we first parameterize the set of all solutions of a given Bézout identity using the Quillen–Suslin and Swan theorems, [27, 28, 41, 44, 46], then we use the Michael selection theorem, [39], to show the existence of a continuous solution to this identity that satisfies general inequality constraints and finally, by invoking the Stone–Weierstrass approximation theorem [14], we obtain a desired polynomial solution.

To begin this program, we start by reviewing our notation and terminology from [30]. Let us use $\mathbb{Z}_+$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{T}$ to denote the non-negative integers, integers, reals, complex numbers, unit disc, and unit circle, respectively. We shall also use $\mathcal{L}$ to denote the algebra of complex-valued Laurent polynomial functions on $\mathbb{C} \setminus \{0\}$. Recall that a Laurent polynomial

$$P : \mathbb{C} \setminus \{0\} \to \mathbb{C}$$

is in $\mathcal{L}$ provided there exists a finitely supported complex-valued coefficient sequence $\{p_n : n \in \mathbb{Z}\}$ such that

$$P(z) = \sum_{n \in \mathbb{Z}} p_n z^n, \quad z \in \mathbb{C} \setminus \{0\}. \tag{1.3}$$

For any Laurent polynomial P, we define

$$n_\ell(P) := \min\{n : n \in \mathbb{Z}_+, p_n \neq 0\}, \quad n_g(P) := \max\{n : n \in \mathbb{Z}_+, p_n \neq 0\},$$

and the *length* of P by

$$\ell(P) := n_g(P) - n_\ell(P). \tag{1.4}$$

We also use $\mathcal{L}_R$, $\mathcal{L}_N$, $\mathcal{L}_P$ to denote the subsets of Laurent polynomials whose restriction to the unit circle $\mathbb{T}$ is real-valued, non-negative, and positive, respectively. It is obvious that any Laurent polynomial P is in $\mathcal{L}_R$ if and only if its coefficient sequence is hermitian-symmetric, that is, for $n \in \mathbb{Z}$, $p_{-n} = \overline{p_n}$. In this case, we have that $n_\ell(P) = -n_g(P)$ and $\ell(P)$ is an even integer.

2. Univariate Laurent Polynomials: Existence

In our recent paper [30] we showed that if P_1 and P_2 are in $\mathcal{L}_N$ and have no common zeros in $\mathbb{C} \setminus \{0\}$, then there exist Q_1 and Q_2 in $\mathcal{L}_N$ such that

$$P_1 Q_1 + P_2 Q_2 = 1. \tag{2.1}$$

Let us now describe a construction of these Laurent polynomials which differs slightly from that given in [30]. In what follows we always assume neither P_1 nor P_2 is identically constant as this case is apparent, in all respects.

First, we choose any pair of Laurent polynomials $B_\ell \in \mathcal{L}_R$, $\ell = 1, 2$, that satisfies the Bézout identity

$$P_1 B_1 + P_2 B_2 = 1. \tag{2.2}$$

Using these Laurent polynomials, we introduce two rational functions

$$R_1 := -\frac{B_2}{P_1}, \quad R_2 := \frac{B_1}{P_2} \tag{2.3}$$

and define a positive constant

$$\widetilde{\epsilon} := \frac{1}{2} (\max\{P_1(z) P_2(z) : z \in \mathbb{T}\})^{-1}. \tag{2.4}$$

Equation (2.2) implies that R_1 is *bounded above* on $\mathbb{T}$, R_2 is *bounded below* on $\mathbb{T}$, and

$$\widetilde{\epsilon} \leq \frac{R_2(w) - R_1(w)}{2}, \quad w \in \mathbb{T}. \tag{2.5}$$

Second, we define the function $\widetilde{f}$ by the equation

$$\widetilde{f} := \frac{\widetilde{R}_1 + \widetilde{R}_2}{2}, \tag{2.6}$$

where $\widetilde{R}_1$ is defined by the equation

$$\widetilde{R}_1(w) := \max\{R_1(w), \ \min\{R_2(z) : z \in \mathbb{T}\} - 2\widetilde{\epsilon}\}, \quad w \in \mathbb{T}, \tag{2.7}$$

and $\widetilde{R}_2$ is defined by the equation

$$\widetilde{R}_2(w) := \min\{R_2(w), \ \max\{R_1(z) : z \in \mathbb{T}\} + 2\widetilde{\epsilon}\}, \quad w \in \mathbb{T}. \tag{2.8}$$

Our choice of these functions guarantees that the inequality

$$\widetilde{R}_1(w) + \widetilde{\epsilon} \leq \widetilde{f}(w) \leq \widetilde{R}_2(w) - \widetilde{\epsilon}, \quad w \in \mathbb{T} \tag{2.9}$$

holds and ensures that the function $\widetilde{f}$ is *continuous* on $\mathbb{T}$.

Third, we construct a Laurent polynomial F that is real-valued on $\mathbb{T}$ and approximates $\widetilde{f}$ uniformly on $\mathbb{T}$ to within $\widetilde{\epsilon}$, that is

$$||\widetilde{f} - F||_{\mathbb{T}} < \widetilde{\epsilon}, \tag{2.10}$$

where we use $||g||_{\mathbb{T}}$ to denote the maximum norm of a function g on $\mathbb{T}$. The existence of the Laurent polynomial F is assured by the Weierstrass approximation theorem.

Fourth, we define Laurent polynomials

$$Q_1 := B_1 - FP_2, \quad Q_2 := B_2 + FP_1. \tag{2.11}$$

Certainly Q_1 and Q_2 satisfy the Bézout identity (2.1). Furthermore, since inequalities (2.9) and (2.10) imply that

$$R_1(w) \leq F(w) \leq R_2(w), \; w \in \mathbb{T} \tag{2.12}$$

we conclude that Q_1 and Q_2 are in $\mathcal{L}_P$, fulfilling our expectations. This proves the assertion. It should be noted that, although the function $\widetilde{f}$ is continuous, it is only piecewise analytic.

We will describe an alternative four-step procedure to construct non-negative Laurent polynomials Q_1 and Q_2 that satisfy the Bézout identity (2.1). Our next proof uses a *matrix extension* method which will be fully exploited for the multidimensional case.

As in the proof above, we use Lemma 2.2 of [30] to identify $B_1, B_2 \in \mathcal{L}_R$ which satisfy the Bézout identity (2.2). Let us consider the matrix

$$M := \begin{bmatrix} P_1 & P_2 \\ -B_2 & B_1 \end{bmatrix} \tag{2.13}$$

and its inverse

$$M^{-1} = \begin{bmatrix} B_1 & -P_2 \\ B_2 & P_1 \end{bmatrix}. \tag{2.14}$$

In addition to the rational functions in Eq. (2.3) we require the rational function

$$g := \frac{1}{P_1 + P_2} \tag{2.15}$$

and the positive constant

$$\epsilon := \min\{-R_1(z), R_2(z) \, : \, z \in \mathbb{T}\}. \tag{2.16}$$

Note that g is analytic in a neighborhood of $\mathbb{T}$ as well as positive on $\mathbb{T}$. Likewise, the function

$$f := g(B_1 - B_2) \tag{2.17}$$

is analytic in a neighborhood of $\mathbb{T}$, real-valued on $\mathbb{T}$, and satisfies the equation

$$\begin{bmatrix} g \\ g \end{bmatrix} := M^{-1} \begin{bmatrix} 1 \\ f \end{bmatrix}. \tag{2.18}$$

We now choose a Laurent polynomial F which is real-valued on $\mathbb{T}$ and approximates f uniformly on $\mathbb{T}$ to within ϵ, that is,

$$\|f - F\|_{\mathbb{T}} < \epsilon. \tag{2.19}$$

Again, we observe that the Laurent polynomials Q_1 and Q_2 in Eqs. (2.11) satisfy Bézout identity (2.1). Furthermore, we combine Eqs. (2.11) and (2.14) to obtain

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = M^{-1} \begin{bmatrix} 1 \\ F \end{bmatrix}. \tag{2.20}$$

Therefore, Eqs. (2.18) and (2.20) imply that

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} g \\ g \end{bmatrix} - M^{-1} \begin{bmatrix} 0 \\ f - F \end{bmatrix} \tag{2.21}$$

and consequently, using Eq. (2.14) and inequalities (2.19), we conclude that $Q_1, Q_2 \in \mathcal{L}_P$. In contrast to our first proof, the rational function f is analytic on an open neighborhood of $\mathbb{T}$. This fact will be exploited in the next section.

We end this section by observing that the above result extends to a finite collection of univariate Laurent polynomials $P_1, \dots, P_m \in \mathcal{L}_N$ having no common zeros in $\mathbb{C} \setminus \{0\}$.

Lemma 2.1. *Let $P_1, \dots, P_m \in \mathcal{L}_N$ have no common zeros in $\mathbb{C} \setminus \{0\}$. Then there exist Laurent polynomials $Q_1, \dots, Q_m \in \mathcal{L}_P$ such that*

$$\sum_{i=1}^{m} P_i Q_i = 1.$$

We will give a proof of this fact using induction on m and the following fact.

Lemma 2.2. *Given $P_1, P_2 \in \mathcal{L}_N$, there exist $p_1, p_2 \in \mathcal{L}_N$ having no common zeros and $P \in \mathcal{L}_N$ such that $P_1 = p_1 P$ and $P_2 = p_2 P$.*

Let us explain the reasoning we used to establish this lemma. A well-known result of Riesz-Fejer states that a Laurent polynomial P is in $\mathcal{L}_N$ if and only if there exists an algebraic polynomial S such that

$$P(z) = |S(z)|^2, \quad z \in \mathbb{T}. \tag{2.22}$$

Alternatively, every $P \in \mathcal{L}_N$ admits the *canonical factorization*

$$P(z) = c(P) \prod_{n=1}^{\ell(P)/2} (z - \lambda_n)(z^{-1} - \overline{\lambda_n}), \quad z \in \mathbb{C} \setminus \{0\}, \tag{2.23}$$

where λ_n, $n = 1, \ldots, \ell(P)/2$, are the roots of S and $c(P)$ is a positive constant. In other words,

$$S(z) = \sqrt{c(P)} \prod_{n=1}^{\ell(P)/2} (z - \lambda_n), \quad z \in \mathbb{C} \setminus \{0\}.$$

Furthermore, we may choose S so that its roots lie in the closed unit disc. In this case, Jensen's theorem [25] implies that $c(P)$ equals the *geometric mean* of P, that is

$$c(P) = \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \log P(e^{i\theta}) d\theta\right]. \tag{2.24}$$

In fact, $c(P)$ is the geometric mean of P *if and only if* S has all its zeros in the closed unit disc. With this information at hand we now prove Lemma 2.2.

Proof of Lemma 2.2. For $\ell = 1, 2$ and $z \in \mathbb{C} \setminus \{0\}$, we express $P_\ell(z)$ in the form

$$P_\ell(z) := S_\ell(z) \overline{S_\ell(1/\overline{z})},$$

where each S_ℓ, $\ell = 1, 2$ is an algebraic polynomial that has all its zeros in the closed unit disc.

Let U be the algebraic polynomial obtained as the greatest degree common divisor of S_1 and S_2. Therefore

$$S_\ell = U D_\ell, \ \ell = 1, 2,$$

where D_1 and D_2 are algebraic polynomials having no common zeros in $\mathbb{C}$. Hence

$$P_\ell = P p_\ell, \ \ell = 1, 2, \tag{2.25}$$

where

$$P(z) := U(z) \overline{U(1/\overline{z})}, \quad z \in \mathbb{C} \setminus \{0\},$$

and

$$p_\ell(z) := D_\ell(z) \overline{D_\ell(1/\overline{z})}, \quad z \in \mathbb{C} \setminus \{0\}.$$

The proof is concluded by observing that p_1 and p_2 have no common zeros. ∎

We now return to the proof of Lemma 2.1 which uses induction on m.

Proof of Lemma 2.1. Having established the result for $m = 2$, we now assume $m \geq 3$ and that the result has been established for any collection of $m - 1$ Laurent polynomials having no common zeros in $\mathbb{C} \setminus \{0\}$ which are non-negative on $\mathbb{T}$. We now suppose that $P_1, \ldots, P_m$ are Laurent polynomials in $\mathcal{L}_N$ with no common zeros in $\mathbb{C} \setminus \{0\}$. We apply Lemma 2.2 to the Laurent polynomials

P_1 and P_2 to obtain Laurent polynomials p_1, p_2, and P that satisfy Eq. (2.25). Since the two collections of Laurent polynomials $\{p_1, p_2\}$ and $\{P, P_3, \ldots, P_m\}$ have no common zeros in $\mathbb{C}\setminus\{0\}$, the induction hypothesis implies that there exist Laurent polynomials $q_1, q_2 \in \mathcal{L}_P$ such that

$$p_1 q_1 + p_2 q_2 = 1$$

and there exist Laurent polynomials $Q, Q_3, \ldots, Q_m \in \mathcal{L}_P$ such that

$$PQ + P_3 Q_3 + \cdots + P_m Q_m = 1.$$

We define, for $\ell = 1, 2$, Laurent polynomials in $\mathcal{L}_N$ by the equation

$$Q_\ell := q_\ell Q.$$

Then $Q_1, \ldots, Q_m$ satisfy the Bézout identity

$$P_1 Q_1 + \cdots + P_m Q_m = 1.$$

This advances the induction hypothesis and proves the result. ■

It is important to realize that this method of proof does *not* extend to *multivariate* Laurent polynomials. The reason for this is two-fold. First, we relied on the fact that univariate Laurent polynomials form a *principle ideal domain*, a property not available in the multivariate case. Furthermore, by *Bézout's intersection theorem* [9, 10, 24, p. 670], for algebraic varieties, any collection of m polynomials in at least m dimensions generally has common zeros. We shall overcome these difficulties and provide substantial generalizations of Lemma 2.1 in Sec. 4.

3. Univariate Laurent Polynomials: Length Bounds

In this section we estimate the lengths of the Laurent polynomials Q_1 and Q_2 in $\mathcal{L}_P$ whose existence was established in the previous section. To this end, corresponding to a pair P_1 and P_2 of Laurent polynomials in $\mathcal{L}_N$ having no common zeros, we define positive integers L, B, and D by the equations

$$L := \max\{\ell(P_1),\, \ell(P_2)\}\,, \tag{3.1}$$

$$B := \min\{\max\{\ell(Q_1), \ell(Q_2)\} \,:\, Q_1, Q_2 \in \mathcal{L}_P,\, P_1 Q_1 + P_2 Q_2 = 1\}, \tag{3.2}$$

and

$$D := \min\{\ell(F) : F \in \mathcal{L}_N,\, \|f - F\|_{\mathbb{T}} < \epsilon\}, \tag{3.3}$$

where ϵ is the positive constant and f is the rational function described at the end of Sec. 2. We also introduce a measure ρ of the separation of zeros of P_1 and P_2 defined by

$$\rho := \min\{|\mu - \lambda| : \mu, \lambda \in \mathbb{C},\, P_1(\mu) = P_2(\lambda) = 0\}\,. \tag{3.4}$$

Since P_1 and P_2 are in $\mathcal{L}_N$, it follows that

$$\rho = \min\{|\mu - \lambda| : \mu, \lambda \in \mathbb{D},\ P_1(\mu) = P_2(\lambda) = 0\},$$

and so it follows that $\rho \in [0,2]$. Moreover, the solutions $B_1, B_2 \in \mathcal{L}_R$ of the Bézout identity (2.2) were constructed above to satisfy the inequalities

$$\ell(B_1) \leq \ell(P_2), \quad \ell(B_2) \leq \ell(P_1). \tag{3.5}$$

Therefore, Eq. (2.11) implies that

$$B \leq D + L \tag{3.6}$$

and if $\ell(F) > 1$, we obtain the equation

$$B = D + L. \tag{3.7}$$

Our main result in this section provides an upper bound for B.

Theorem 3.1. *There exists a function*

$$G : (0, \infty) \times \mathbb{Z}_+ \to \mathbb{Z}_+$$

such that for any pair of Laurent polynomials P_1 and P_2 in $\mathcal{L}_R$ having no common zeros in $\mathbb{C} \setminus \{0\}$,

$$B \leq G(\rho, L).$$

Furthermore, for a fixed ρ, the function G may be chosen to satisfy, the asymptotic bound

$$G(\rho, L) = O\left(L^5\, 5^{L/2} \left(\frac{256}{\rho}\right)^{L^2/4}\right), \quad L \to \infty,$$

and, for a fixed L, the asymptotic bound

$$G(\rho, L) = O(\rho^{-L^2/4} \log \rho^{-1}), \quad \rho \to 0.$$

We have computational evidence that indicates these bounds are pessimistic. However, without the alternate proof of the existence of positive solutions of the Bézout identity (2.1) presented here, even these bounds would be inaccesible.

To prove Theorem 3.1 we derive a series of upper bounds for B in terms of various quantities and then relate these quantities to ρ and L. Although these quantities require knowledge of P_1 and P_2, they provide convenient methods to obtain upper bounds for B without having to compute positive solutions of the Bézout identity. We approach the problem by using the notion of *resultant.* Before starting the proof we show by example that B may tend to infinity as ρ tends to zero, even if L is *fixed.*

Proposition 3.1. *For any positive integer n, there exist Laurent polynomials P_1 and P_2 in $\mathcal{L}_N$ such that $\ell(P_1) = \ell(P_2) = 4n$, and*

$$B \geq \frac{4}{\pi}\rho^{-1} - 4n\,. \tag{3.8}$$

Proof. We choose distinct numbers λ_1 and λ_2 in $\mathbb{T}$ such that $\lambda_1 \neq -\lambda_2$, and represent these numbers as

$$\lambda_\ell = e^{i\theta_\ell}\,,\ \ell = 1, 2,$$

where

$$\theta_2 \in (\theta_1, \theta_1 + \pi).$$

Using these numbers and any positive integer n, we define P_1 and P_2 in $\mathcal{L}_N$ of length $4n$ by the equation

$$P_1(z) := \prod_{\ell=1}^{2}(z - \lambda_\ell)^n(z^{-1} - \overline{\lambda_\ell})^n, \quad P_2(z) := P_1(-z),\ z \in \mathbb{C}\setminus\{0\}.$$

Since P_1 and P_2 have no common zeros, there exist Laurent polynomials Q_1 and Q_2 in $\mathcal{L}_N$ that satisfy the Bézout identity (2.1). We introduce $W \in \mathcal{L}_N$ by the formula

$$W(z) := \frac{1}{2}\left(Q_1(z) + Q_2(-z)\right),\ z \in \mathbb{C}\setminus\{0\},$$

so that

$$P_1(z)W(z) + P_1(-z)W(-z) = 1,\ z \in \mathbb{C}\setminus\{0\}.$$

Next, we define $V \in \mathcal{L}_N$ by setting

$$V := P_1 W$$

and the trigonometric polynomial T by the equation

$$T(\theta) := V(e^{i\theta}),\ \theta \in \mathbb{R}.$$

Clearly, T is non-negative and satisfies the equation

$$T(\theta) + T(\theta + \pi) = 1,\ \theta \in \mathbb{R}.$$

Hence, we conclude that

$$T(\theta) \leq 1,\ \theta \in \mathbb{R}.$$

Moreover, T has an n-th order zero at θ_2 and $1 - T(\cdot + \pi)$ has an n-th order zero at θ_1. Therefore, there exists

$$\phi \in (\theta_2, \theta_1 + \pi)$$

such that

$$T^{(2n-1)}(\phi) = \frac{(-1)^{n-1}}{(\theta_1 + \pi - \theta_2)^{2n-1}}.$$

By Bernstein's inequality [8, Vol. I, p. 26],

$$|T^{2n-1}(\phi)| \le \left[\frac{\ell(V)}{2}\right]^{2n-1}$$

from which we conclude that

$$\ell(V) \ge \frac{2}{(\theta_1 + \pi - \theta_2)} > \frac{4}{\pi|\lambda_1 + \lambda_2|}.$$

We also observe that

$$\ell(V) = \ell(P_1 W) = \ell(P_1) + \ell(W) = 4n + \ell(W)$$

and

$$\ell(W) \le \max\{\ell(Q_1), \ell(Q_2)\}.$$

Since Q_1 and Q_2 satisfy the Bézout identity, we obtain that $\ell(Q_1) = \ell(Q_2)$. Therefore, for $\ell = 1, 2$, we conclude that

$$\ell(Q_\ell) \ge \frac{4}{\pi}\rho^{-1} - 4n\,.$$

■

We now turn to the problem of deriving upper bounds on B. Since $\ell(Q_1)$ and $\ell(Q_2)$ are unchanged by multiplying P_1 and P_2 by positive real numbers, we may assume without loss of generality that the constants $c(P_1)$ and $c(P_2)$ in the canonical factorization of P_1 and P_2 described by Eq. (2.23) are both equal to one. With this condition we obtain the estimate

$$\|P_\ell\|_{\mathbb{T}} \le 2^\ell, \quad \ell = 1, 2. \tag{3.9}$$

We let B_1 and B_2 be the Laurent polynomials, ϵ the positive constant, and g and f be the rational functions described at the end of Sec. 2. In our preliminary estimates we also make use of the following constant

$$\beta := \max\{\|B_\ell\|_{\mathbb{T}} \;:\; \ell = 1, 2\}. \tag{3.10}$$

Later, we shall estimate β by some function of ρ and L.

We define the Laurent polynomial P in $\mathcal{L}_P$ by the equation

$$P := P_1 + P_2,$$

and observe that

$$\min_{z \in \mathbb{T}} P(z) \ge \beta^{-1}. \tag{3.11}$$

Moreover, since $c(P_1) = c(P_2) = 1$, Jensen's theorem and the arithmetic–geometric inequality imply that

$$c(P) = c(P_1 + P_2) \geq 2\sqrt{c(P_1)c(P_2)} \geq 2. \tag{3.12}$$

To proceed further we need a measure of the separation between the zero set of P and the unit circle $\mathbb{T}$. To this end, we introduce the positive constant

$$\kappa := \min\left\{\frac{1}{2}, \min\left\{1 - |\lambda| \,:\, P(\lambda) = 0,\, \lambda \in \mathbb{D}\right\}\right\}. \tag{3.13}$$

We note the quantities ϵ defined by Eq. (2.16), β defined by Eq. (3.10), κ defined by Eq. (3.13), and ρ defined by Eq. (3.4), are functions of P_1 and P_2 (albeit, complex functions of P_1 and P_2). We derive an upper bound on the lengths of Q_ℓ, $\ell = 1, 2$ by deriving bounds on ϵ, β, and κ in terms of ρ and L.

To this end, we also make use of the error of best uniform approximation on $\mathbb{T}$, to the rational function f, defined in Eq. (2.17), by trigonometric polynomials which is given by

$$E_n(f) := \min\{\|f - G\|_{\mathbb{T}} \,:\, G \in \mathcal{L}_R, \ell(G) \leq 2n\}. \tag{3.14}$$

Our first lemma provides a bound for this quantity as a function of β, κ, and L.

Lemma 3.1. *Let f be the rational function defined in Eq.* (2.17). *For any integer $n \geq L$ and for any r in the open interval $(1-\kappa, 1)$, there holds the inequality*

$$E_n(f) \leq 2\beta \frac{r^{n+1-L}}{(1-r)(r-1+\kappa)^L}. \tag{3.15}$$

Proof. For $n \geq L$, we have that

$$E_n(f) \leq 2\beta\, E_N(g), \tag{3.16}$$

where $N := n - L$. Furthermore, the canonical factorization of P and the fact $c(P) \geq 2$ imply that

$$E_n(f) \leq \beta E_N(h), \tag{3.17}$$

where

$$h(z) := \frac{1}{\prod_{\ell=1}^{\ell(P)/2}(z - \lambda_\ell)(z^{-1} - \overline{\lambda}_\ell)}, \quad z \in \mathbb{C} \setminus \{0\},$$

and λ_ℓ, $\ell = 1, \ldots, \ell(P)/2$ are roots of P inside the unit disc. Note that, for any r in the interval $(1-\kappa, 1)$, h is analytic in the annulus

$$\mathbb{A}_r := \{z \,:\, r \leq |z| \leq r^{-1}\}$$

and uniformly bounded on it by the constant $(r - 1 + \kappa)^{-\ell(P)}$. A standard result ensures that

$$E_N(h) \leq \frac{2\max\{|h(z)| \,:\, z \in \mathbb{A}_r\}\, r^{N+1}}{1-r}.$$

Combining the above inequalities and the fact that $\ell(P) \leq L$ concludes the proof. ∎

The bound given by the right side of inequality (3.15) is minimized by choosing r to be equal to the unique root of the equation

$$\frac{n+1-L}{x} + \frac{1}{1-x} - \frac{L}{x-1+\kappa} = 0$$

in the open interval $(1-\kappa, 1)$. A simple bound, that describes the asymptotic decay of $E_n(f)$, is obtained by choosing

$$r = (1-\kappa)(1 + \frac{L}{n+1-2L}), \quad n \geq L\left(1 + \frac{1}{\kappa}\right). \tag{3.18}$$

With this bound we are led to the following corollary.

Corollary 3.1. *Let f be the rational function defined in Eq. (2.17). Then*

$$E_n(f) \leq \frac{2\beta}{\kappa}(1-\kappa)^n \left(\frac{4ne}{L}\right)^L, \quad n \geq \frac{2L}{\kappa}. \tag{3.19}$$

Proof. We substitute the value of r in Eq. (3.18) into inequality (3.15) to obtain

$$E_n(f) \leq \frac{2\beta(1-\kappa)^{n+1-2L}(1+\frac{L}{n+1-2L})^{n+1-2L}(n+1-L)^L}{(\kappa-(1-\kappa)\frac{L}{n+1-2L})L^L}.$$

The result follows by a direct computation using the fact that $(1+x/m)^m < e^x$ for all $m \geq 0$, $x > 0$. ∎

Our next result bounds B in terms of L, κ and β.

Proposition 3.2. *The minimal length B defined by Eq. (3.2) satisfies the inequality*

$$B \leq L + \max\left\{40a, \frac{5}{4}(b + a\log a)\right\}, \tag{3.20}$$

where a and b are defined by the equations

$$a := \frac{L}{\kappa} \tag{3.21}$$

and

$$b := \frac{1}{\kappa}\log\left[\frac{4\beta}{\kappa}\left(\frac{16e}{L}\right)^L\right]. \tag{3.22}$$

Proof. Using the bound on the norm on P_ℓ, $\ell = 1, 2$, given in Eq. (3.9), we obtain the inequality

$$\epsilon \geq 2^{-2L-1}. \tag{3.23}$$

Therefore, by Corollary (3.1), if $n \geq 2L/\kappa$ and

$$\frac{2\beta}{\kappa}(1-\kappa)^n \left(\frac{4ne}{L}\right)^L \leq 2^{-2L-1}, \tag{3.24}$$

then $B \leq L + n$. The inequality in (3.24) is equivalent to demanding that

$$n - a \log n \geq b\,, \tag{3.25}$$

where a and b are defined in Eqs. (3.21) and (3.22). To unravel this inequality we introduce the function

$$\psi(x) := x - \log x\,, \quad x \in [1, \infty)\,. \tag{3.26}$$

This function is a strictly increasing bijection of $(1, \infty)$ and satisfies the inequality

$$\frac{4}{5}x \leq \psi(x), \quad x \in [40, \infty)\,. \tag{3.27}$$

Using this function we can guarantee inequality (3.25) provided that

$$\psi(\frac{n}{a}) \geq \frac{1}{a}(b + a \log a)\,.$$

From this observation and Eq. (3.27) the result follows. ■

We recall our objective to bound B above by a function of ρ and L. Proposition 3.2 implies that it suffices to derive upper bounds for κ and β in terms of ρ and L. The next result provides an upper bound for κ in terms of β. To this end, we define the constant $\beta_0 := \max\{1, \beta\}$.

Lemma 3.2. *The quantity κ defined by Eq.* (3.13) *satisfies the estimate*

$$2 \leq \kappa^{-1} \leq \beta_0 L 5^{L/2}. \tag{3.28}$$

Proof. Equation (3.13) implies that $\kappa \leq 1/2$, and therefore, the left-hand side of the above inequality is valid. For the upper bound we observe that

$$1 - |\lambda| \geq \min\left\{\frac{1}{2}, \frac{\min\{|P(z)| \,:\, z \in \mathbb{T}\}}{\max\{|P'(z)| \,:\, z \in \mathbb{A}\}}\right\}, \tag{3.29}$$

where $\mathbb{A} := \{z : 1/2 \leq |z| \leq 1\}$. Recall inequality (3.11) which states that

$$\min\{|P(z)| \,:\, z \in \mathbb{T}\} \geq \beta^{-1}\,.$$

Moreover, the canonical factorizations of P_1 and P_2 imply that

$$\max\{|P'(z)| \,:\, z \in \mathbb{A}\} \leq L 5^{L/2}\,.$$

Combining these last two inequalities with inequality (3.29) proves the result. ■

Proposition 3.2 and Lemma 3.2 show that B is bounded above by a function of β, ρ, and L. We now conclude the proof of Theorem 3.1 by deriving an upper bound for β in terms of ρ and L. The method we use relies on the notion of a *resultant* of two polynomials.

We begin with a lemma about two algebraic polynomials P_1 and P_2 having no common zeros in $\mathbb{C}$. We will use this lemma to get the desired bound on β.

Lemma 3.3. *Let*

$$P_1(z) := \gamma \prod_{j=1}^{m} (z - \lambda_j)\,, \quad z \in \mathbb{C}$$

and

$$P_2(z) := \delta \prod_{j=1}^{n} (z - \mu_j)\,, \quad z \in \mathbb{C}$$

be two polynomials with no common zeros in $\mathbb{C}$. Suppose k is an integer such that $0 \le k \le m + n - 1$. Let Q_1 and Q_2 be the unique polynomials of degree $n-1$, $m-1$, respectively, which satisfy the Bézout identity

$$P_1(z)Q_1(z) + P_2(z)Q_2(z) = z^k\,, \quad z \in \mathbb{C}. \tag{3.30}$$

Then there hold the inequalities

$$\|Q_1\|_{\mathbb{T}} \le \frac{n\,\|P_1\|_{\mathbb{T}}^{n-1}\,\|P_2\|_{\mathbb{T}}^{m}}{|\gamma\delta| \prod_{i=1}^{n}\prod_{j=1}^{m} |\lambda_j - \mu_i|}\,, \tag{3.31}$$

and

$$\|Q_2\|_{\mathbb{T}} \le \frac{m\,\|P_1\|_{\mathbb{T}}^{n}\,\|P_2\|_{\mathbb{T}}^{m-1}}{|\gamma\delta| \prod_{i=1}^{n}\prod_{j=1}^{m} |\lambda_j - \mu_i|}\,. \tag{3.32}$$

Proof. We write

$$Q_1(z) = r_0 + r_1 z + \cdots + r_{n-1} z^{n-1}, \quad z \in \mathbb{C},$$
$$Q_2(z) = s_0 + s_1 z + \cdots + s_{m-1} z^{m-1}, \quad z \in \mathbb{C}.$$

Define two vectors in $\mathbb{R}^{m+n}$ by setting

$$x = [r_0, r_1, \ldots, r_{n-1}, s_0, s_1, \ldots, s_{m-1}]^T$$

and set

$$e = [0, \ldots, 0, 1, 0, \ldots, 0]^T\,,$$

where one occurs in the $k + 1$ coordinate of the vector e. Let W be the $(m+n)\times(m+n)$ matrix which represents the resultant of P_1 and P_2 (see [50, p. 29-30]). Hence, Eq. (3.30) is equivalent to the linear system $Wx = e$ and it follows that

$$x_j = \frac{\det W_j}{\det W}\,, \; j = 0, \ldots, m + n - 1,$$

where W_j is the $(m+n)\times(m+n)$ matrix formed by replacing the $(j+1)$th column of W by the vector e. Note that all the columns of W_j have the sum of squared moduli equal to either $|P_1|^2_{\mathbb{T}}$, $|P_2|^2_{\mathbb{T}}$, or one (for the $j+1$st column) where $|\cdot|_{\mathbb{T}}$ denotes the H^2-norm on $\mathbb{T}$. Moreover, since $\lambda_1, \lambda_2, \ldots, \lambda_m$, $\mu_1, \mu_2, \ldots, \mu_s$, are the zeros of P_1 and P_2, respectively, we know that

$$\det W = \gamma\,\delta \prod_{i=1}^{n}\prod_{j=1}^{m}(\lambda_j - \mu_i),$$

(see [50, p. 30]). Hence, the Hadamard inequality [26, p. 472] implies that

$$|r_i| \le \frac{|P_1|^{n-1}_{\mathbb{T}}\ |P_2|^{m}_{\mathbb{T}}}{\prod_{i=1}^{n}\prod_{j=1}^{m}|\lambda_j - \mu_i|}, \quad i = 0, \ldots, n-1.$$

$$|s_i| \le \frac{|P_1|^{n}_{\mathbb{T}}\ |P_2|^{m-1}_{\mathbb{T}}}{\prod_{i=1}^{n}\prod_{j=1}^{m}|\lambda_j - \mu_i|}, \quad i = 0, \ldots, m-1.$$

This proves the lemma. ∎

The Laurent polynomial version of the above result is stated next.

Lemma 3.4. *For $i = 1, 2$, let*

$$P_i(z) = \sum_{k=-N_i}^{N_i} p_{ki} z^k, \quad z \in \mathbb{C}\setminus\{0\},$$

be Laurent polynomials with zeros λ_{ji}, $j = 1, \ldots, N_i$. Suppose

$$A_i(z) = \sum_{k=-N_{3-i}}^{N_{3-i}-1} a_{ki} z^k, \quad z \in \mathbb{C}\setminus\{0\}, \quad i = 1, 2,$$

are the unique Laurent polynomials of their respective degrees that satisfy the Bézout identity

$$P_1(z)A_1(z) + P_2(z)A_2(z) = 1, \quad z \in \mathbb{C}\setminus\{0\}.$$

Then there hold the inequalities

$$\|A_1\|_{\mathbb{T}} \le \frac{\ell(P_2)\,\|P_1\|_{\mathbb{T}}^{\ell(P_2)-1}\,\|P_2\|_{\mathbb{T}}^{\ell(P_1)}}{p_{N_1,1}p_{N_2,2}\prod_{i=1}^{N_1}\prod_{j=1}^{N_2}|\lambda_{i1} - \lambda_{j2}|} \tag{3.33}$$

and

$$\|A_2\|_{\mathbb{T}} \le \frac{\ell(P_1)\,\|P_1\|_{\mathbb{T}}^{\ell(P_2)}\,\|P_2\|_{\mathbb{T}}^{\ell(P_1)-1}}{p_{N_1,1}p_{N_2,2}\prod_{i=1}^{N_1}\prod_{j=1}^{N_2}|\lambda_{i1} - \lambda_{j2}|}. \tag{3.34}$$

In the next lemma we bound β in terms of ρ and L when P_1 and P_2 are Laurent polynomials in $\mathcal{L}_R$ both having geometric mean one.

Lemma 3.5. *Suppose P_1 and P_2 are Laurent polynomials in $\mathcal{L}_R$ with geometric means equal to one and having no common zeros in $\mathbb{C}\setminus\{0\}$. We have the estimate*

$$\beta_0 \le L\left(\frac{4}{\rho^{1/4}}\right)^{L^2}. \tag{3.35}$$

Proof. Recall that

$$B_\ell = \Re A_\ell, \quad \ell = 1,2,$$

where A_1 and A_2 are the Laurent polynomials identified in Lemma 3.4. Hence, we conclude that

$$\beta \le \max\{\|A_1\|_{\mathbb{T}},\, \|A_2\|_{\mathbb{T}}\}\,.$$

This means that the desired bound for β will follow from Lemma 3.4. For this purpose, we recall the bounds

$$\|P_\ell\|_{\mathbb{T}} \le 2^{\ell(P_\ell)}, \quad \ell = 1,2,$$

from Eq. (3.9) for the maximum norm of P_ℓ, $\ell = 1,2$, on the unit circle. To finish the proof it suffices to note that the canonical factorization of P_i implies that $|p_{N_i,i}| \le 1$, $i = 1,2$. ∎

We have now derived the inequalitites necessary to prove Theorem 3.1.

Proof of Theorem 3.1. To complete the proof we combine inequality (3.20), which bounds L_Q above as a function of L, β, and κ^{-1}, with inequality (3.2), which bounds κ^{-1} as a function of L and β, and inequality (3.35), which bounds β as a function of L and ρ. These inequalities establish both claims in the theorem. ∎

An upper bound for β in terms of ρ and L can be obtained by using a contour integral representation for the solution of the Bézout identity. We do not pursue this method here but do record the following fact.

Lemma 3.6. *Let P_1 and P_2 be Laurent polynomials with no common zeros in $\mathbb{C}\setminus\{0\}$. Suppose Γ_ℓ, $\ell = 1,2$, are simple closed curves whose interior Ω_ℓ, $\ell = 1,2$, contains the roots of $P_{3-\ell}$ and whose exterior contains the roots of P_ℓ, $\ell = 1,2$. We define analytic functions C_ℓ, $\ell = 1,2$, on the open region $\mathbb{C}\setminus\overline{\Omega}$, where*

$$\Omega := \Omega_1 \cup \Omega_2\,,$$

by the equation

$$C_\ell(z) := \frac{P_{3-k}(z)}{2\pi i}\int_{\Gamma_k}\frac{d\zeta}{(z-\zeta)P_1(\zeta)P_2(\zeta)}, \quad z \in \mathbb{C}\setminus\overline{\Omega}. \tag{3.36}$$

Then the functions C_ℓ, $\ell = 1,2$, are the restrictions to the complement of $\overline{\Omega}$ of Laurent polynomials A_ℓ, $\ell = 1,2$, which satisfy the Bézout identity

$$P_1A_1 + P_2A_2 = 1. \tag{3.37}$$

Proof. The fact that each C_ℓ, $\ell = 1,2$, equals the restriction to the complement of $\overline{\Omega}$ of a Laurent polynomial follows from Cauchy's integral formula. Moreover, it can be seen, by noting the asymptotic growth of the integrand as $z \to \infty$, that $n_\ell(A_\ell) \geq n_\ell(P_{3-\ell})$ and $n_g(A_\ell) \leq n_g(P_{3-\ell}) - 1$. To confirm the Bézout identity we introduce Laurent polynomials $T := P_1P_2$ and $S := P_1A_1 + P_2A_2$. It suffices to show that $S(z) = 1$ for $z \notin \overline{\Omega}$. First note that

$$S(z) = \frac{T(z)}{2\pi i}\int_\Gamma \frac{d\zeta}{(\zeta - z)T(\zeta)},$$

where Γ is the boundary of Ω. Next, observe that since the integrand is a rational function which is analytic in the extended complex plane, except at the zeros of T and the point z, the sum of its residues is zero. Since Ω contains the zeros of T and $z \notin \overline{\Omega}$, the integral equals the sum of the residues of the integrand at the zeros of T. The result follows from the fact that the residue of the integrand at z is $-1/T(z)$. ■

4. Multivariate Laurent Polynomials

Our objective in this section is to extend the results from Sec. 2. For this purpose, fix an integer $d \geq 1$. We let $\mathcal{L}$ denote the algebra of Laurent polynomials with coefficients in $\mathbb{C}$ defined on $(\mathbb{C} \setminus \{0\})^d$ and let $\mathcal{P}$ denote its subalgebra consisting of algebraic polynomials. Let $\mathbb{X}$ denote either the d-torus $\mathbb{T}^d$ or the d-cube $\mathbb{I}^d := [-1,1]^d$ and $\mathcal{R}$ either the subalgebra $\mathcal{L}_R$ of $\mathcal{L}$, consisting of functions that are real-valued on $\mathbb{T}^d$, or the subalgebra $\mathcal{P}_R$ of $\mathcal{P}$, consisting of functions that are real-valued on $\mathbb{I}^d$. Let $C(\mathbb{X})$ denote the Banach algebra of all continuous real-valued functions (under pointwise multiplication), equipped with the maximum norm (denoted by $\|\cdot\|_\mathbb{X}$), on $\mathbb{X}$. We use the notation $C(\mathbb{X})^m$ for the m-fold product of $C(\mathbb{X})$ with the maximum component norm. For $R \in C(\mathbb{X})^m$, we define the set

$$\mathcal{B}_c(P) := \{R : R \in C(\mathbb{X})^m, P \cdot R = 1\},$$

where

$$F \cdot G := F_1G_1 + \cdots + F_mG_m$$

for

$$F = [f_1, \ldots, f_m],\ G = [g_1, \ldots, g_m] \in C(\mathbb{X})^m.$$

Similarly, for $P \in \mathcal{R}^m$ we introduce the set

$$\mathcal{B}_r(P) := \{Q : Q \in \mathcal{R}^m, P \cdot Q = 1\}.$$

We let $C(\mathbb{X})^m_+$ denote the cone consisting of elements in $C(\mathbb{X})^m$ whose components are non-negative and use $\operatorname{int} C(\mathbb{X})^m_+$ for its interior. The results in this section include the fact that for any $P \in C(\mathbb{X})^m_+ \cap \mathcal{R}^m$, the set $\operatorname{int} C(\mathbb{X})^m_+ \cap \mathcal{B}_r(P)$ is non-empty. Our method of proof proceeds in three steps. First, we parameterize the set $\mathcal{B}_r(P)$ by using the Quillen–Suslin and Swan theorems. Next, we apply the Stone–Weierstrass theorem on $C(\mathbb{X})^m$ to show that $\mathcal{B}_r(P)$ is dense in

$\mathcal{B}_c(P)$. Finally, we employ the Michael selection theorem to construct an element in $\mathcal{B}_r(P)$ that satisfies the desired inequalities.

Our first concern is to parameterize the Bézout set $\mathcal{B}_r(P)$ of a vector P in $\mathcal{R}^m$. For this purpose, we say an element $P \in \mathcal{R}^m$ is *unimodular* provided that $\mathcal{B}_r(P)$ is non-empty. For the algebra $\mathcal{P}$, the Nullstellensatz implies that P is unimodular if and only if its components have no common zeros in $\mathbb{C}^d$, while for the algebra $\mathcal{L}$, the Nullstellensatz implies P is unimodular if and only if the components of P have no common zeros in $(\mathbb{C} \setminus \{0\})^d$. Moreover, for any unimodular $P \in \mathcal{P}_R^m$, the Quillen–Suslin theorem implies that there exists an $m \times m$ matrix M over $\mathcal{P}_R$ such that M has an inverse with entries in $\mathcal{P}_R$ and has the first row that coincides with P [41, 44, 45].

For any unimodular $P \in \mathcal{P}_R^m$, the following result parameterizes the Bézout set $\mathcal{B}_r(P)$ in terms of the entries of M^{-1}. To state this result we find the following notation convenient. For every $x \in \mathbb{C}^{m-1}$, we let x_* denote the vector in $\mathbb{C}^m$ whose first component is one and the remaining components are the components of x. For every $x \in \mathbb{C}^m$, we let $x_\uparrow$ denote the vector in $\mathbb{C}^{2m}$ whose first m components are the components of x and the remaining m components are zero. In addition, for any $y \in \mathbb{C}^{2m}$, we let $y_\downarrow$ denote the vector whose components are the first m components of y.

Proposition 4.1. *Let $P \in \mathcal{P}_R^m$ be unimodular. Then*

$$\mathcal{B}_r(P) = \{F : F = G_* M^{-T}, G \in \mathcal{P}_R^{m-1}\}. \tag{4.1}$$

Proof. For $G \in \mathcal{P}_R^{m-1}$, we define the vector $F = G_* M^{-T}$, where M is the matrix defined above by the Quillen–Suslin theorem. Since the first row of M is P, we observe that

$$PM^{-1} = 0_* .$$

We compute

$$P \cdot F = PM^{-1} G_*^T = 1,$$

in other words, $F \in \mathcal{B}_r(P)$. Conversely, for any $F \in \mathcal{B}_r(P)$, we define $G \in \mathcal{P}_R^{m-1}$ by the equation

$$MF^T = G_*^T$$

which concludes the proof. ∎

We now turn our attention to a description of $\mathcal{B}_r(P)$ for $\mathcal{R} = \mathcal{L}_R$ and a unimodular vector $P \in \mathcal{R}^m$. This case presents some difficulty. Recall that, for any unimodular $P \in \mathcal{L}^m$, the theorem of Swan implies that there exists an $m \times m$ matrix M over $\mathcal{L}$ such that M has an inverse having entries in $\mathcal{L}$ and the first row of M coincides with P [28, 46]. However, it is unknown if the entries of the matrix M can be chosen to be in $\mathcal{L}_R$ when $P \in \mathcal{L}_R^m$ (see [29]). To overcome this problem, we observe that p is in $\mathcal{L}$ if and only if there exist p_R and p_I in $\mathcal{L}_R$ such that $p = p_R + ip_I$. We use this fact in the following way. For any unimodular $P \in \mathcal{L}_R^m$, let M be its matrix completion (over $\mathcal{L}$) given by the Swan theorem and express it in the form

$$M = M_R + iM_I,$$

where M_R and M_I are $m\times m$ matrices over $\mathcal{L}_R$. We construct the $2m\times 2m$ matrix $\widetilde{M}$ in block form by the equation

$$\widetilde{M} := \begin{bmatrix} M_R & M_I \\ -M_I & M_R \end{bmatrix}. \tag{4.2}$$

Note that the inverse of $\widetilde{M}$ is given by

$$\widetilde{M}^{-1} := \begin{bmatrix} A_R & A_I \\ -A_I & A_R \end{bmatrix}, \tag{4.3}$$

where A_R and A_I are $m\times m$ matrices over $\mathcal{L}_R$ defined by the equation

$$A_R + iA_I := M^{-1}.$$

Since $P \in \mathcal{L}_R^m$, the first row of $\widetilde{M}$ equals the $2m$-vector $P_\uparrow$. Using these facts, we now parameterize $\mathcal{B}_r(P)$ in terms of the entries of $\widetilde{M}^{-1}$.

Proposition 4.2. *Let $P \in \mathcal{L}_R^m$ be unimodular. Then*

$$\mathcal{B}_r(P) = \{F_\downarrow : F = G_*\widetilde{M}^{-T},\ G \in \mathcal{L}_R^{2m-1}\}. \tag{4.4}$$

Proof. The proof of this result is similar to the proof of Proposition 4.1. Specifically, for $G \in \mathcal{L}_R^{2m-1}$, we define the vector

$$F := G_*\widetilde{M}^{-T}.$$

Since the first row of $\widetilde{M}$ equals $P_\uparrow$, we conclude that

$$P_\uparrow \widetilde{M}^{-1} = 0_*.$$

Therefore, we have that

$$P \cdot F_\downarrow = P_\uparrow \cdot F = P_\uparrow \widetilde{M}^{-1} \cdot G_* = 0_* \cdot G_* = 1,$$

in other words, $F_\downarrow$ is in the Bézout set $\mathcal{B}_r(P)$. Conversely, for any $F \in \mathcal{B}_r(P)$, we define $G \in \mathcal{L}_R^{2m-1}$ by the equation

$$G_*^T := \widetilde{M} F_\uparrow^T.$$

Consequently, we obtain that

$$F_\uparrow = G_* \widetilde{M}^{-T}$$

which concludes the proof. ∎

Next, we demonstrate that, for any unimodular vector P in $\mathcal{R}^m$, the Bézout set $\mathcal{B}_r(P)$ is dense in $\mathcal{B}_c(P)$ relative to the maximum norm on $\mathbb{X}$.

Proposition 4.3. *Let $P \in \mathcal{R}^m$ be unimodular. Then $\mathcal{B}_r(P)$ is dense in $\mathcal{B}_c(P)$.*

Proof. We divide the proof into two cases. First we deal with the case where $\mathcal{R} = \mathcal{P}_R$. Given $H \in \mathcal{B}_c(P)$, we define the vector-valued function $F \in C(\mathbb{X})^{m-1}$ by the equation

$$F_* := HM^T, \tag{4.5}$$

where M is the $m \times m$ matrix over $\mathcal{P}_R$ given by the Quillen–Suslin theorem, as appearing in Proposition 4.1. By the Weierstrass approximation theorem, for every $\epsilon > 0$, there exists $G \in \mathcal{P}_R^{m-1}$ such that

$$\|F - G\|_{\mathbb{X}} < \epsilon.$$

We define the vector J in the set $\mathcal{B}_r(P)$ by the equation $J := G_* M^{-T}$, and note that

$$\|J - H\|_{\mathbb{X}} \leq \|M^{-T}\| \, \|F - G\|_{\mathbb{X}} \leq \epsilon \, \|M^{-T}\|,$$

where

$$\|M^{-T}\| := \max \left\{ \sum_{j=1}^{m} \| \left(M^{-T}\right)_{i,j} \|_{\mathbb{X}} : i = 1, \ldots, m \right\}$$

which concludes the proof of the first case.

The case $\mathcal{R} = \mathcal{L}_R$ is argued similarly. Given $H \in \mathcal{B}_c(P)$ we define a vector-valued function $F \in C(\mathbb{X})^{2m-1}$ by the equation

$$F_* := H_{\uparrow} \widetilde{M}^T, \tag{4.6}$$

where $\widetilde{M}$ is the $2m \times 2m$ matrix over $\mathcal{L}_R$ given by the Swan theorem, as appearing in Proposition 4.2. By the Weierstrass approximation theorem, for every $\epsilon > 0$, there exists $G \in \mathcal{L}_R^{2m-1}$ such that

$$\|F - G\|_{\mathbb{X}} < \epsilon.$$

We define the vector $J \in \mathcal{L}_R^{2m}$ by the equation

$$J := G_* \widetilde{M}^{-T}$$

and note that $J_{\downarrow}$ is in $\mathcal{B}_c(P)$. Therefore, we obtain that

$$\|J_{\downarrow} - H\|_{\mathbb{T}} \leq \|\widetilde{M}^{-T}\| \;\; \|F - G\|_{\mathbb{X}} < \epsilon \|\widetilde{M}^{-T}\|$$

which concludes the proof of the second case and the proposition. ∎

We now address the existence of solutions to the Bézout equation with inequality constraints. To this end, we require some facts about cones. We let $\mathbb{B}^m$ denote the closed unit ball and let $\mathbb{S}^{m-1}$ denote the unit sphere in $\mathbb{R}^m$. For any non-empty closed subset $\mathbb{H}$ of $\mathcal{H}$ and vector $b \in \mathbb{B}^m$, we define

$$d(b, \mathbb{H}) := \inf \{ |b - h| : h \in \mathbb{H} \} .$$

Furthermore, we let $\mathcal{H}$ denote the set of all non-empty closed subsets of $\mathbb{B}^m$ equipped with the *Hausdorff topology* defined by the metric

$$d(\mathbb{H}_1, \mathbb{H}_2) := \sup\{\, d(h, \mathbb{H}_2) \,:\, h \in \mathbb{H}_1 \,\} + \sup\{\, d(h, \mathbb{H}_1) \,:\, h \in \mathbb{H}_2 \,\}.$$

For any subset $\mathbb{H}$ of $\mathbb{R}^m$, we let $[\mathbb{H}]$ denote the convex hull of $\mathbb{H}$ and let $\mathrm{cl}\,\mathbb{H}$ be its closure.

A *cone* K is any subset of $\mathbb{R}^m$ containing all non-negative multiples of its elements and is *non-trivial* if it contains a non-zero element. We say it is *admissible* if it is closed, convex, and does not contain a line, that is, a one-dimensional linear subspace. The *dual cone* of any cone $\mathbb{K}$ is given by the equation

$$\mathbb{K}^\perp := \{v : v \in \mathbb{R}^m,\ u \cdot v \geq 0,\ u \in \mathbb{K}\}. \tag{4.7}$$

Recall the fact that any closed convex cone $\mathbb{K}$ satisfies $\mathbb{K}^{\perp\perp} = \mathbb{K}$.

We use the continuous function

$$\theta : (\mathbb{R}^m \backslash \{0\}) \times (\mathbb{R}^m \backslash \{0\}) \to [0, \pi]$$

defined by the formula

$$\theta(u, v) := \cos^{-1} \frac{u \cdot v}{|u|\,|v|},\quad u, v \in (\mathbb{R}^m \backslash \{0\}), \tag{4.8}$$

where $|\cdot|$ denotes euclidean length of a vector. The function $\theta(u, v)$ is the *angle* between the vectors u and v. Also, it equals the geodesic distance between the vectors $u/|u|$ and $v/|v|$ in $\mathbb{S}^{m-1}$.

For any closed subset $\mathbb{K}$ of $\mathbb{R}^m$ containing non-zero elements and a vector $u \in \mathbb{R}^m \setminus \{0\}$, we set

$$\theta(u, \mathbb{K}) := \inf\{\, \theta(u, v) : v \in \mathbb{K} \setminus \{0\}\}.$$

Let $\mathcal{K}$ denote the set of all admissible cones in $\mathbb{R}^m$ equipped with the topology induced by the metric

$$\begin{aligned} \theta(\mathbb{K}_1, \mathbb{K}_2) := &\max\{\, \theta(x, \mathbb{K}_2) : x \in \mathbb{K}_1 \setminus \{0\}\} \\ &+ \max\{\, \theta(x, \mathbb{K}_1) : x \in \mathbb{K}_2 \setminus \{0\}\},\qquad \mathbb{K}_1, \mathbb{K}_2 \in \mathcal{K}. \end{aligned} \tag{4.9}$$

Our main result in this section is the following theorem.

Theorem 4.1. *Let* s *be any continuous (set-valued) function*

$$\mathrm{s} : \mathbb{X} \to \mathcal{K}.$$

Suppose $P \in \mathcal{R}^m$ *is unimodular and satisfies*

$$P(x) \in \mathrm{s}(x),\ x \in \mathbb{X}.$$

Then there exists $Q \in \mathcal{B}_r(P)$ *such that*

$$Q(x) \in \operatorname{int}\, \mathrm{s}(x)^{\perp}\,,\ x \in \mathbb{X}.$$

Our objective in the remainder of this section is to prove this result and to note some interesting special cases. To this end we first relate the topological spaces $\mathcal{K}$ and $\mathcal{H}$.

Lemma 4.1. *The following functions are continuous*
$\phi : \mathcal{K} \to \mathcal{H}$ *defined by* $\phi(\mathbb{K}) := \mathbb{K} \cap \mathbb{S}^{m-1}\,,\quad \mathbb{K} \in \mathcal{K};$
$\psi : \mathcal{K} \to \mathcal{H}$ *defined by* $\psi(\mathbb{K}) := \left[\mathbb{K} \cap \mathbb{S}^{m-1}\right]\,,\quad \mathbb{K} \in \mathcal{K};$
$\Gamma : \mathcal{K} \to \mathbb{R}$ *defined by* $\Gamma(\mathbb{K}) := d(0, \left[\mathbb{K} \cap \mathbb{S}^{m-1}\right]),\quad \mathbb{K} \in \mathcal{K}.$
Furthermore, the last function is positive-valued.

Proof. First, observe that, for any $\mathbb{K}_1, \mathbb{K}_2 \in \mathcal{K}$,

$$\theta(\mathbb{K}_1, \mathbb{K}_2) = \theta(\mathbb{K}_1 \cap \mathbb{S}^{m-1}, \mathbb{K}_2 \cap \mathbb{S}^{m-1}),$$

and for every pair of closed subsets $\mathbb{D}_1, \mathbb{D}_2$ of $\mathbb{S}^{m-1}$,

$$\frac{2}{\pi}\theta(\mathbb{D}_1, \mathbb{D}_2) \le d(\mathbb{D}_1, \mathbb{D}_2) \le \theta(\mathbb{D}_1, \mathbb{D}_2)\,. \tag{4.10}$$

Therefore, we conclude that

$$d(\phi(\mathbb{K}_1), \phi(\mathbb{K}_2)) \le \theta(\mathbb{K}_1, \mathbb{K}_2)\,.$$

The function ψ satisfies the same inequality since, for every pair of closed subsets $\mathbb{H}_1, \mathbb{H}_2$ of $\mathbb{R}^m$, we have that

$$d\left([\mathbb{H}_1], [\mathbb{H}_2]\right) \le d(\mathbb{H}_1, \mathbb{H}_2)\,.$$

The final claim is proved by using the triangle inequality to confirm that the function Γ satisfies the same inequality as ϕ and ψ. ■

We also need facts about admissible cones. To this end, for any non-trivial cone $\mathbb{K}$, we define

$$\Theta(K) := \max\left\{\, \theta(u, v) :\ u, v \in \mathbb{K}\backslash\{0\}\right\} \tag{4.11}$$

and set $\Theta(\{0\}) := 0$.

Lemma 4.2. *Suppose* $\mathbb{K}$ *is a closed convex cone.*
The following properties are equivalent: $\mathbb{K} \in \mathcal{K}$, $\Theta(\mathbb{K}) < \pi$, $\operatorname{int}\, \mathbb{K}^{\perp} \neq \phi$.
If $\mathbb{K}$ *is admissible,* $u \in \mathbb{K}$ *and* $v \in \operatorname{int}\, \mathbb{K}^{\perp}$, *then* $u \cdot v > 0$.
If $\mathbb{J}$ *is a closed convex cone such that* $\mathbb{K} \subset \{0\} \cup \operatorname{int}\, \mathbb{J}$, *then* $\mathbb{J}^{\perp} \subset \{0\} \cup \operatorname{int}\, \mathbb{K}^{\perp}$.
If $\mathbb{K}_1, \mathbb{K}_2 \in \mathcal{K}$, *then*

$$d(K_1^{\perp} \cap \mathbb{B}^m, K_2^{\perp} \cap \mathbb{B}^m) = d(K_1 \cap \mathbb{B}^m, K_2 \cap \mathbb{B}^m).$$

If $\mathbb{K}_1, \mathbb{K}_2 \in \mathcal{K}$, *then*

$$\theta(\mathbb{K}_1^\perp, \mathbb{K}_2^\perp) \le \pi\, \theta(\mathbb{K}_1, \mathbb{K}_2)\,.$$

Proof. We briefly describe the proof of the last three claims since the others are argued in a similar manner to the third. Since $\mathbb{J}^\perp \subset \mathbb{K}^\perp$, it suffices to prove that if u is a non-zero vector in the *boundary* of $\mathbb{K}^\perp$, then $u \notin \mathbb{J}^\perp$. Let $w \in \mathbb{R}^m \backslash \{0\}$ determine a supporting hyperplane for $\mathbb{K}^\perp$ at u. Thus, $u \cdot w = 0$ and $w \cdot v \ge 0$ for all $v \in \mathbb{K}^\perp$. Since $w \in \mathbb{K}^{\perp\perp} = \mathbb{K}$, it follows that $w \in \operatorname{int} \mathbb{J}$. Hence, there exists $y \in \mathbb{J}$ such that $u \cdot y < 0$. This shows that u cannot be in $\mathbb{J}^\perp$ and proves the third claim.

To show the fourth claim, we first observe that the above discussion implies that, for any $\mathbb{K} \in \mathcal{K}$ and for any $x \in \mathbb{R}^m$,

$$d(x, \mathbb{K}^\perp) = \max\{-w \cdot x : w \in \mathbb{K} \cap \mathbb{B}^m\}\,.$$

In fact, if $x \notin \mathbb{K}^\perp$, then the vector u which achieves the maximum of the right-hand side of the above equation is given by

$$u = \frac{y-x}{|y-x|}\,,$$

where y is the closest point in $\mathbb{K}^\perp$ to x. Therefore, if $\mathbb{K}_1$ and $\mathbb{K}_2$ are in $\mathcal{K}$ and $x \in \left(\mathbb{K}_1^\perp \backslash \mathbb{K}_2^\perp\right) \cap \mathbb{B}^m$, then

$$d(x, \mathbb{K}_2^\perp) = -u \cdot x \le \max\{-u \cdot v : v \in \mathbb{K}_1^\perp \cap \mathbb{B}^m\} = d(u, \mathbb{K}_1)\,.$$

From this inequality, it follows that

$$\max\{d(x, \mathbb{K}_2^\perp \cap \mathbb{B}^m) : x \in \mathbb{K}_1^\perp \cap \mathbb{B}^m\} \le \max\{d(v, \mathbb{K}_1 \cap \mathbb{B}^m) : v \in \mathbb{K}_2 \cap \mathbb{B}^m\}.$$

Interchanging the roles of $\mathbb{K}_1$ and $\mathbb{K}_2$ and then replacing $\mathbb{K}_1$ by $\mathbb{K}_1^\perp$ and $\mathbb{K}_2$ by $\mathbb{K}_2^\perp$ concludes the proof of the fourth assertion.

To show the fifth claim we first prove, for any non-trivial cone $\mathbb{K}$ and any vector $x \in \mathbb{S}^{m-1}$, that

$$d(x, \mathbb{K} \cap \mathbb{B}^m) \le d(x, \mathbb{K} \cap \mathbb{S}^{m-1}) \le 2\, d(x, \mathbb{K} \cap \mathbb{B}^m)\,. \tag{4.12}$$

The first inequality follows from the inclusion $\mathbb{S}^{m-1} \subset \mathbb{B}^m$. To show the second inequality, we let $\pi(x)$ denote the closest point in $\mathbb{K}$ to x. If $\pi(x) = 0$, then $d(x, \mathbb{K}) = 1$ and since $d(x, \mathbb{K} \cap \mathbb{S}^{m-1}) \le 2$ the claim holds. Otherwise, we set

$$z = \frac{\pi(x)}{|\pi(x)|}$$

and observe that $z \in \mathbb{K} \cap \mathbb{S}^{m-1}$. Moreover, since $x \cdot \pi(x) = |\pi(x)|^2$, we compute

$$d(x, \mathbb{K})^2 = 1 - |\pi(x)|^2 \ge 1 - |\pi(x)| = \frac{1}{2}|z - x|^2 \ge d(x, \mathbb{K} \cap \mathbb{S}^{m-1})^2$$

which concludes the proof of inequality (4.12).

To finish the proof of the fifth assertion we observe that inequality (4.12) implies that

$$d(\mathbb{K}_1^\perp \cap \mathbb{S}^{m-1}, \mathbb{K}_2^\perp \cap \mathbb{S}^{m-1}) \le 2\, d(\mathbb{K}_1^\perp \cap \mathbb{B}^m, \mathbb{K}_2^\perp \cap \mathbb{B}^m).$$

Appealing to the fourth claim we obtain that

$$d(\mathbb{K}_1^\perp \cap \mathbb{S}^{m-1}, \mathbb{K}_2^\perp \cap \mathbb{S}^{m-1}) \le 2\, d(\mathbb{K}_1 \cap \mathbb{B}^m, \mathbb{K}_2 \cap \mathbb{B}^m).$$

We now combine this inequality with inequalities (4.10) to conclude that

$$\theta(\mathbb{K}_1^\perp, \mathbb{K}_2^\perp) \le \frac{\pi}{2}\, d(\mathbb{K}_1^\perp \cap \mathbb{S}^{m-1}, \mathbb{K}_2^\perp \cap \mathbb{S}^{m-1}) \le \pi\, d(\mathbb{K}_1 \cap \mathbb{B}^m, \mathbb{K}_2 \cap \mathbb{B}^m).$$

Using the fact that, for any admissible cone $\mathbb{K}$, the function

$$x \mapsto d(x, \mathbb{K} \cap \mathbb{B}^m)$$

is a convex function on $\mathbb{R}^m$, we have that

$$\theta(\mathbb{K}_1^\perp, \mathbb{K}_2^\perp) \le \pi\, d(\mathbb{K}_1 \cap \mathbb{S}^{m-1}, \mathbb{K}_2 \cap \mathbb{S}^{m-1}) \le \pi\, \theta(\mathbb{K}_1, \mathbb{K}_2)$$

which concludes the proof of the fifth and final assertion. ■

We will also require the following specialization of the Michael selection theorem from Theorem 3.1 of [39].

Michael Selection Theorem. *If $\mathbb{X}$ is a compact metric space and if $\mathbf{t}$ is a continuous function from $\mathbb{X}$ into $\mathcal{H}$, then $\mathbf{t}$ admits a continuous selection, that is, a continuous function*

$$g : \mathbb{X} \to \mathbb{B}^m$$

such that, for all $x \in \mathbb{X}$, $g(x) \in \mathbf{t}(x)$.

We are now prepared to prove our main result.

Proof of Theorem 4.1. Let

$$\mathbf{s} : \mathbb{X} \to \mathcal{K}$$

satisfy the hypothesis of Theorem 4.1. Choose any positive ϵ such that

$$\epsilon < \frac{\pi - \max\{\, \Theta(\mathbf{s}(x)) \,:\, x \in \mathbb{X}\,\}}{2},$$

and define the function $\mathbf{s}_\epsilon$ by the equation

$$\mathbf{s}_\epsilon(x) := \mathrm{cl}\,\{u \,:\, u \in \mathbb{R}^m \setminus \{0\}, \theta(u, \mathbf{s}(x)) \le \epsilon\,\}, \quad x \in \mathbb{X}.$$

We first prove that this function maps $\mathbb{X}$ into $\mathcal{K}$ continuously and satisfies the inclusion

$$\mathbf{s}(x) \subset \{0\} \cup \mathrm{int}\ \mathbf{s}_\epsilon(x),\ x \in \mathbb{X}. \tag{4.13}$$

The fact that $\mathbf{s}_\epsilon$ maps into $\mathcal{K}$ is a consequence of our choice of ϵ and also the first part of Lemma 4.2. The inclusion (4.13) uses the fact that, for $x \in \mathbb{X}$,

$$\operatorname{int} \mathbf{s}_\epsilon(x) := \{u \,:\, u \in \mathbb{R}^m \setminus \{0\},\ \theta(u, \mathbf{s}(x)) < \epsilon\}.$$

We shall now show that $\mathbf{s}_\epsilon$ is, indeed, uniformly continuous. Choose $\mu > 0$. It suffices to show that there exists $\delta > 0$ such that, for any $x, y \in \mathbb{X}$, $|x - y| < \delta$ implies

$$\theta(\mathbf{s}_\epsilon(x), \mathbf{s}_\epsilon(y)) \le \mu\,.$$

Since $\mathbf{s}$ is continuous and $\mathbb{X}$ is compact, there exists $\delta > 0$ such that, for any $x, y \in \mathbb{X}$, $|x - y| < \delta$ implies that

$$\theta(\mathbf{s}(x), \mathbf{s}(y)) < \frac{\mu}{2}\,.$$

For any $x, y \in \mathbb{X}$ with $|x - y| < \delta$ and $u \in \mathbf{s}_\epsilon(x) \backslash \mathbf{s}_\epsilon(y)$, it suffices to show that

$$\theta(u, \mathbf{s}_\epsilon(y)) < \frac{\mu}{2}\,,$$

since the roles of x and y can be interchanged. To this end, we observe that there exist $v \in \mathbf{s}(x)$ and $w \in \mathbf{s}(y)$ such that $\theta(u, v) \le \epsilon$ and $\theta(v, w) < \mu/2$. Therefore, by the triangle inequality, it follows that

$$\theta(u, w) < \epsilon + \frac{\mu}{2}\,. \tag{4.14}$$

By our choice of u, we are assured that

$$L := [\{u, w\}] \cap \operatorname{cl} \mathbf{s}_\epsilon(y)$$

is a proper closed convex subset of the line segment $[\{u, w\}]$ that joins u and w. Therefore, L is a line segment having the form $[\{z, w\}]$ where z is in $[\{u, w\}]$ and on the boundary of $\operatorname{cl} \mathbf{s}_\epsilon(y)$. Consequently, z also has the property that

$$\theta(z, w) = \epsilon\,. \tag{4.15}$$

Our choice of z ensures that $\theta(u, w) = \theta(u, z) + \theta(z, w)$. The reason for this is that the unit vectors

$$\widehat{z} := z/|z|\,, \quad \widehat{u} := u/|u|\,, \quad \widehat{w} := w/|w|$$

have the property that $\widehat{z}$ is on the geodesic connecting $\widehat{u}$ and $\widehat{w}$ in $\mathbb{S}^{m-1}$. Therefore, it follows that

$$\theta(u, w) = \theta(\widehat{u}, \widehat{w}) = \theta(\widehat{u}, \widehat{z}) + \theta(\widehat{z}, \widehat{w}) = \theta(u, z) + \theta(z, w). \tag{4.16}$$

Consequently, combining Eqs. (4.14)–(4.16) yields

$$\theta(u, \mathbf{s}_\epsilon(y)) \le \theta(u, z) < \frac{\mu}{2}\,,$$

and establishes the assertions concerning the function $\mathbf{s}_\epsilon$.

Next, we introduce the function $\mathbf{t} : \mathbb{X} \to \mathcal{H}$ by the equation

$$\mathbf{t}(x) := \left[\mathbf{s}_\epsilon(x)^\perp \cap \mathbb{S}^{m-1}\right], \ \ x \in \mathbb{X},$$

and observe that Lemmas 4.1 and 4.2 imply that $\mathbf{t}$ is continuous. Therefore, the Michael selection theorem implies that there exists a continuous function

$$g : \mathbb{X} \to \mathbb{R}^m,$$

such that

$$g(x) \in \mathbf{t}(x) \subset \mathbf{s}_\epsilon(x)^\perp \backslash \{0\} \subset \operatorname{int} \mathbf{s}(x)^\perp .$$

Part two of Lemma 4.1 implies that the function $P \cdot g$ is positive on $\mathbb{X}$ which ensures that the function R defined by the equation

$$R(x) := \frac{g(x)}{P(x) \cdot g(x)}, \ \ x \in \mathbb{X}, \tag{4.17}$$

is in $\mathcal{B}_c(P)$. Next, we construct the subset $\mathcal{F}$ of functions in $C(\mathbb{X})^m$ by the equation

$$\mathcal{F} := \{F : F(x) \in \operatorname{int} \mathbf{s}(x)^\perp, \ x \in \mathbb{X}\}.$$

Since $\mathbf{s}$ is continuous and $\mathbb{X}$ is compact, $\mathcal{F}$ is open. Proposition 4.3 asserts that $\mathcal{B}_c(P)$ is dense in $\mathcal{B}_r(P)$ and hence there exists $Q \in \mathcal{B}_r(P)$ such that $Q \in \mathcal{F}$ and the proof is complete. ■

We record two corollaries of Theorem 4.1.

Corollary 4.1. *Let $P \in C(\mathbb{X})^m_+ \cap \mathcal{R}^m$. Then there exists $Q \in \operatorname{int} C(\mathbb{X})^m_+ \cap \mathcal{B}_r(P)$.*

Proof. The cone $\operatorname{int} C(\mathbb{X})^m_+$ is admissible and self-dual. Moreover, the function

$$\mathbf{s} : \mathbb{X} \to \mathcal{K}$$

defined by

$$\mathbf{s}(x) := \mathbb{K}, \ \ x \in \mathbb{X}$$

is continuous. Theorem 4.1 implies that there exists $Q \in \mathcal{B}_r(P)$ such that

$$Q(x) \in \operatorname{int} C(\mathbb{X})^m_+, \ x \in \mathbb{X},$$

which concludes the proof. ■

Corollary 4.2. *Let $P \in \mathcal{R}^m$ be unimodular and let $0 < \epsilon < \pi/2$. Then there exists $Q \in \mathcal{B}_r(P)$ such that*

$$\theta(P(x), Q(x)) < \epsilon, \ x \in \mathbb{X}.$$

Proof. For each $x \in \mathbb{X}$, define the cone

$$\mathrm{s}(x) := \mathrm{cl}\left\{u \,:\, u \in \mathbb{R}^m \backslash \{0\},\, \theta(P(x), u) \leq \frac{\pi}{2} - \epsilon\right\}.$$

Since this cone is admissible and contains $P(x)$ Theorem 4.1 implies that there exists $Q \in \mathcal{B}_r(P)$ such that, for all $x \in \mathbb{X}$,

$$Q(x) \in \mathrm{int}\ \mathrm{s}(x)^{\perp} = \{u \,:\, u \in \mathbb{R}^m \backslash \{0\},\, \theta(P(x), u) < \epsilon\}.$$

This concludes the proof. ∎

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