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Endomorphism Rings of Harada Modules*

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Abstract. In this paper we prove that the endomorphism ring of a Harada left R-module, modulo its Jacobson radical, is a direct product of left full linear rings.

F. Kasch, in his unpublished lecture [4], has proved the following:

Theorem. *If M is a Harada module and $S = \text{End}(M)$, then $S/J(S)$ is isomorphic to a direct product of full linear rings.*

It is well known that, discrete modules are Harada modules [5, Theorem 4.15 and Corollary 5.5]. However, the converse is not true. For example, any module which has a finite LE decomposition is a Harada module [1, Corollary 12.7], but it may not be a discrete module, e.g., $(Z_p^\infty)^{(n)}$, for any natural number n, is a Harada module which is not discrete. Recently, Ali and Zelmanowitz [1] proved that if M is a discrete module and $S = \text{End}(M)$, then $S/J(S)$ is a left continuous ring. Later Zelmanowitz [6], improving this result, proved Kasch's Theorem for the particular case of discrete modules using results from Mohamed–Muller [5] and Kasch [4].

In this paper we give a simple proof of Kasch's Theorem using a well known property of Harada modules.

Throughout, rings will be with identity, modules unital left modules and endomorphisms of modules will act on the right. For notations and terminology, we refer the reader to [2] and [5].

Definition 1. *A module M is said to be an* LE *module if it has local ring of*

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endomorphisms. A decomposition $M = \oplus_A M_a$ is said to be an LE *decomposition if each M_a is an* LE *module.*

Definition 2. *A module M is called a Harada module if it has an* LE *decomposition that complements direct summands.*

It follows from Corollary 12.5 of [2] that any LE decomposition of a Harada module complements direct summands. Also from Lemma 12.3 of [2], it follows that a direct summand of Harada module is a Harada module.

Definition 3. *Let $M = \oplus_A M_a$ be a decomposition of an arbitrary module. For fixed $a_1 \in A$, we set $A_1 = \{a \in A : M_a \cong M_{a_1}\}$. Then $M_1 = \oplus_{A_1} M_a$ will be called a homogeneous component of M with respect to the decomposition $M = \oplus_A M_a$.*

Notations. We now fix the notations for all results to follow.

Let $M = \oplus_A M_a$ be an LE decomposition of a Harada module. Then $M = \oplus_I M_i$, where for each $i \in I$, $M_i = \oplus_{A_i} M_a$, for some $A_i \subseteq A$, is a homogeneous component of M with respect to the decomposition $M = \oplus_A M_a$. We set $S = \text{End}(M)$ and $S_i = \text{End}(M_i)$, for each $i \in I$. For any $B \subseteq A$, we denote by $l_B : \oplus_B M_b \to \oplus_A M_a$ and $\pi_B : \oplus_A M_a \to \oplus_B M_b$ the natural injection and projection, respectively. We will identify, whenever convenient, S with the ring of all row-summable matrices $(f_{ij})_{i,j\in I}$, where $f_{ij} : M_i \to M_j$ is a homomorphism under the map $f \mapsto (l_i f \pi_j)_{i,j\in I}$.

To prove the Theorem we show that:

(a) $S/J(S) \cong \prod_i S_i/J(S_i)$ and

(b) for each $i \in I$, $S_i/J(S_i)$ is isomorphic to a full linear ring.

Lemma 1. *With the notations above, let*

$$T = \{f \in S : l_a f \pi_a \text{ is a non-isomorphism for each } a, b \text{ in } A\}.$$

Then $J(S) = T$.

Proof. From Theorem 2.25 in [5], it follows that $\{M_\alpha : \alpha \in A\}$ is a locally semi-T-nilpotent family and then by Kanbara [3, Corollary 1], $J(S) = T$.

Lemma 2. *With the notations above, $\theta_i : S \to S_i/J(S_i)$ defined as*

$$\theta_j(f) = l_A f \pi_{A_i} + J(S_i)$$

is a ring epimorphism with $J(S) \subseteq \text{Ker}(\theta_i)$.

Proof. Obviously θ_i is additive and onto. We now show that, for $f, g \in S$, $\theta_i(fg) = \theta_i(f)\theta_i(g)$. Observe that

$$\begin{aligned} l_{A_i} f g \pi_{A_i} &= l_{A_i} f g \pi_{A_i} + l_{A_i} f \pi_{A_i} l_{A_i} g \pi_{A_i} - l_{A_i} f \pi_{A_i} l_{A_i} g \pi_{A_i} \\ &= l_{A_i} f \pi_{A_i} l_{A_i} g \pi_{A_i} + l_{A_i} f (1 - \pi_{A_i} l_{A_i}) g \pi_{A_i}. \end{aligned}$$

So it suffices to show that $l_{A_i}f(1-\pi_{A_i}l_{A_i})g\pi_{A_i} \in J(S_i)$ and for which, in view of Lemma 1, we show that, for a, $b \in A$,

$$l_a l_{A_i}(1-\pi_{A_i}l_{A_i})g\pi_{A_i}\pi_b = l_a f(1-\pi_{A_i}l_{A_i})g\pi_b$$

is a non-isomorphism. Suppose, on the contrary, there exists $\alpha : M_b \to M_a$ such that $l_a f(1-\pi_{A_i}l_{A_i})g\pi_b\alpha = I_{M_a}$. Thus, $l_a f(1-\pi_{A_i}l_{A_i}) : M_a \to \oplus_{A\setminus A_i}M_a$ is a split monomorphism. Hence, M_a is isomorphic to summand of $\oplus_{A\setminus A_i}M_a$. Thus, for some $c \in A \setminus A_i$, $M_a \cong M_c$. This is a contradiction.

Let $\alpha \in J(C)$. By Lemma 1, we have that, for all $a, b \in A$, $l_a\alpha\pi_b$ is a non-isomorphism. In particular, for all $a_i, b_i \in A_i$, $l_{a_i}l_{A_i}\alpha\pi_{A_i}\pi_{b_i} = l_{a_i}\alpha\pi_{b_i}$ is a non-isomorphism. Hence, by Lemma 1, $l_{A_i}\alpha\pi_{A_i} \in J(S_i)$ and thus $J(S) \subseteq \mathrm{Ker}(\theta_i)$.

Proof of the Theorem. We prove the assertions (a) and (b) above.

(a) For $i \in I$, let $\theta_i : S \to S_i/J(S_i)$ be as in Lemma 2, and $\theta = \prod_I \theta_i : S \to \prod_I S_i/J(S_i)$. In view of Lemma 2, θ is a ring homomorphism and $J(S) \subseteq \mathrm{Ker}(\theta)$. By the matrix identification of S (see notations above), it is clear that θ is onto. Suppose that $f = (f_{ab})_{a,b\in A} \in \mathrm{Ker}(\theta)$. In order to show that $j \in J(S)$,we show that each f_{ab} is a non-isomorphism (see Lemma 1). Suppose, on the contrary, $f_{ab} : M_a \to M_b$ is an isomorphism, for some $a, b \in A$. Thus, for some $i \in I$, $a, b \in A_i$. So $f_{ab} = l_a f\pi_b = l_a l_{A_i} f\pi_{A_i}\pi_b$ is an isomorphism. This, in view of Lemma 1, violates the fact that $l_{A_i}f\pi_{A_i} \in J(S_i)$.

(b) We need to show that, if $M = N^{(A)}$, for some set A, is locally semi-T-nilpotent LE decomposition of M with $S = \mathrm{End}(M)$, then $S/J(S)$ is isomorphic to a full linear ring. Let $L = \mathrm{End}(N)$ and $D = L/J(L)$. As L is local, so D is a division ring.

Let Y be the ring of all $A \times A$ row finite matrices with entries from D. We define $\phi : S \to Y$ as $\phi(f_{ab})_{a,b\in A} = (\overline{f}_{ab})_{a,b\in A}$. Then ϕ is a ring epimorphism. By Lemma 1, $J(S) = T$. Obviously, $\mathrm{Ker}(\phi) = T = J(S)$. Hence, $S/J(S) \cong Y$. This completes the proof. ∎

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