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Decomposition Algorithm for Reverse Convex Programs

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Abstract. A decomposition method is proposed for reverse convex programs with a special low rank nonconvex structure. When specialized to linear programs with an additional reverse convex constraint, this method gives an improved version of the polyhedral annexation method earlier developed by the first author for reverse convex programs.

1. Introduction

We are concerned with the problem

$$\min\{\langle c,x\rangle \mid x \in D,\ \varphi(g(x)) \leq 1\} \tag{P}$$

where D is a compact convex set in R^n, $g : X \to R^m$ a continuous mapping on an open convex set $X \supset D$, and $\varphi : Y \to R$ a **quasiconcave** function defined on a convex set Y in R^m containing $g(D)$, such that

(A1) The recession cone of Y contains a polyhedral cone $K = \{y \in R^m_+ \mid y_i = 0,$ $i = p+1, \dots, m\}$ satisfying

$$\varphi(y) \geq \varphi(y') \quad \text{whenever} \quad y, y' \in Y,\ y - y' \in K.$$

(A2) $g(x)$ is K-convex, i.e. for any $x, x' \in X$ and $\alpha \in [0,1]$:

$$g(\alpha x + (1-\alpha)x') \leq_K \alpha g(x) + (1-\alpha)g(x'),$$

where we write $y' \leq_K y$ to mean that $y - y' \in K$, i.e. $y'_i \leq y_i$ $(i = 1, \dots, p)$, $y'_i = y_i$ $(i = p+1, \dots, m)$.

(A3) There is a point $a \in D$ such that

$$\varphi(g(a)) > 1; \quad \langle c,a\rangle < \min\{\langle c,x\rangle \mid x \in D, \varphi(g(x)) \leq 1\}.$$

For any set $E \subset R^m$ containing 0 denote $E^* = -E^\circ = \{t \in R^m | \langle t, y\rangle \geq -1 \; \forall y \in E\}$. If E is a cone then $E^* = \{t \in R^m | \langle t, y\rangle \geq 0 \; \forall y \in E\}$, so $K^* = \{t \in R^m | \; t_i \geq 0, \; i = 1, \ldots, p\}$. Assumption (**A2**) ensures that for any $t \in K^*$ the function $x \mapsto \langle t, g(x)\rangle$ is convex. Clearly (**A2**) holds if $g(x) = (g_1(x), \ldots, g_m(x))$ with $g_1(x), \ldots, g_p(x)$ convex and $g_{p+1}(x), \ldots, g_m(x))$ affine. Assumption (**A3**) is innocuous, because it is satisfied whenever the constraint $\varphi(g(x)) \leq 1$ is essential, i.e.,

$$\min\{\langle c, x\rangle | \; x \in D\} < \min\{\langle c, x\rangle | \; x \in D, \varphi(g(x)) \leq 1\}.$$

Important special cases of problem (P) that have been previously studied include:

(1) Convex program with an additional convex multiplicative constraint:

$$\min\{\langle c, x\rangle | \; x \in D, \; \prod_{i=1}^{m} g_i(x) \leq 1\} \tag{1.1}$$

where $g_i(x)$ are convex positive-valued on $X \supset D$ [4-7] (see also [3]). In this example, $\varphi(y) = \prod_{i=1}^m y_i$, $K = R^m_+$.

(2) Linear program with an additional low-dimensional reverse convex constraint:

$$\min\{\langle c, x\rangle | \; x \in D, \; \psi(x_1, \ldots, x_{m-1}) + \langle d, x\rangle \leq 1\}, \tag{1.2}$$

where $D \subset R^n$ is a polytope, $\psi(x_1, \ldots, x_{m-1})$ is a concave function of $m-1$ variables ($m < n$) (see [3,12]). In this example $Y = R^m, g_i(x) = x_i$ ($i = 1, \ldots, m-1$), $g_m(x) = \langle d, x\rangle$, $\varphi(y) = \psi(y_1, \ldots, y_{m-1}) + y_m$, $K = \{y \in R^m_+ | \; y_i = 0, \; i = 1, \ldots, m-1\}$.

(3) Linear program with an additional monotonic reverse convex constraint:

$$\min\{\langle c, x\rangle | \; x \in D, \; f(x) \leq 1\}, \tag{1.3}$$

where $D \subset R^n$ is convex, $f(x)$ is a quasiconcave function for which there exists an $m \times n$ matrix Q of rank m such that $f(x') \geq f(x)$ whenever $Q_i x' \geq Q_i x$ ($i = 1, \ldots, p$), $Q_i x' = Q_i x$ ($i = p+1, \ldots, m$) (see [9], and also [3, 12]). In this case, $f(x) = \varphi(Qx)$ where $\varphi(.)$ is a quasiconcave function defined by $\varphi(y) = f(x)$ for all x satisfying $Qx = y$.

Note that problems (1.1) are also special cases of problems (1.3). By reduction to concave minimization over convex sets, variants of outer approximation method for solving (1.1) were developed by Thach et al. [7] and Kuno et al. [4, 5]. This method cannot, however, be extended easily to the general problem (P). Later, in Konno et al. [3] and Tuy [12] proposed a decomposition method based on polyhedral annexation for solving problems (1.2) and (1.3). The aim of the present paper is to extend the latter method to problem (P) and to improve it by a more transparent presentation and a more thorough discussion on implementation issues. In addition, results of computational experiments will be reported to show the efficiency of this decomposition approach.

The paper is organized into six sections. After the Introduction, in Sec. 2, we will show how the basic subproblem of transcending the incumbent can be reduced to the minimization of a quasiconcave monotonic function over a convex set. Based on this reduction, a global optimality criterion will be formulated which suggests a decomposition strategy for solving (P). In Sec. 3, we will discuss some crucial issues regarding the implementation of this decomposition strategy. In Sec. 4, the detailed algorithm will be described and its convergence established. In Sec. 5, we will examine the case when D is a polytope and $g(x)$ is affine. It turns out that, when specialized to this case, the algorithm gives an improved version of the decomposition method earlier presented in [3] and [13] for linear programs with an additional monotonic reverse convex contraint. Finally, in Sec. 5, we will report computational experiments to show the performances of the proposed algorithm on randomly generated problems with n up to 140 and m up to 6.

2. Basic Subproblem and Decomposition Strategy

Let $\gamma_0 \in (-\infty, +\infty]$ be an upper bound of the optimal value, i.e., $\gamma_0 \geq \min\{\langle c, x\rangle |\ x \in D,\ \varphi(g(x)) \leq 1\}$. The key subproblem towards solving (P) is the following:

(SP$_0$) *Find a feasible solution with an objective function value less than γ_0 if there is one (or else prove that no such solution exists, i.e. γ_0 is the global optimal value).* (It is agreed that a global optimal value equal to $+\infty$ means that the problem (P) is infeasible.)

An answer to this question can be obtained by solving the following subproblem:

$$\min\{\varphi(g(x))|\ y = g(x),\ x \in D,\ \langle c, x\rangle \leq \gamma_0\}. \qquad (\mathrm{Q}_0)$$

To see this, assume that the problem is *regular*, i.e.

(A4) Any feasible solution of (P) is the limit of a sequence of interior feasible solutions (by interior feasible solution we mean a vector $x \in \mathrm{int}D$ satisfying $\varphi(g(x)) < 1$).

Theorem 1. *If there is a feasible solution x^1 to* (Q$_0$) *with $\varphi(g(x^1)) < 1$, then the point $\hat{x}^1 \in [a, x^1]$ such that $\varphi(g(\hat{x}^1)) = 1$ is a feasible solution to (P) satisfying $\langle c, \hat{x}^1\rangle < \gamma_0$.*

Conversely, under the regularity assumption **(A4)**, *if the optimal value of* (Q$_0$) *is no less than 1, then γ_0 is the global optimal value.*

Proof. If $\varphi(g(x^1)) < 1$, then, since $\varphi(g(a)) > 1$, the line segment $[a, x^1] \subset D$ contains a point $\hat{x}^1$ such that $\varphi(g(\hat{x}^1)) = 1$. Since $\langle c, a\rangle < \gamma_0$, we must have $\langle c, \hat{x}^1\rangle < \gamma_0$, so $\hat{x}^1$ is a feasible solution with $\langle c, \hat{x}^1\rangle < \gamma_0$.

Conversely, if there is $x^1 \in D$ satisfying $\varphi(g(x^1)) \leq 1, \langle c, x^1\rangle < \gamma_0$, then, since by **(A4)** $x^1 = \lim x^\nu$ where $x^\nu \in \mathrm{int}D, \varphi(g(x^\nu)) < 1$, we must have $\langle c, x^\nu\rangle < \gamma_0$ for some ν. Hence, the optimal value of (Q$_0$) must be inferior to 1. ■

Now define

$$E_0 = \{y \in Y | \ y = g(x), \ x \in D, \langle c, x\rangle \le \gamma_0\} \subset Y \subset R^m,$$
$$C = \{y \in Y | \ \varphi(y) \ge 1\}$$

From the assumptions, it is easily seen that the set C is convex and closed and that

$$g(a) \in E_0 \cap \operatorname{int} C. \tag{2.1}$$

Furthermore, for every $u \in K$, we have $\varphi(g(a) + u) \ge \varphi(g(a)) > 1$. Hence,

$$g(a) + K \subset C. \tag{2.2}$$

We can now formulate the following criterion.

Theorem 2. *We have*

$$1 = \min\{\varphi(g(x)) | \ x \in D, \langle c, x\rangle \le \gamma_0\}$$
$$\Leftrightarrow E_0 \subset C \Leftrightarrow [C - g(a)]^* \subset [E_0 - g(a)]^*. \tag{2.3}$$

Proof. Clearly 1 is the optimal value of (Q_0) if and only if there is no $x \in D$ such that $\langle c, x\rangle \le \gamma_0$ and $\varphi(g(x)) < 1$, i.e. if and only if $E_0 \subset C$. Furthermore, from convex analysis, $E_0 \subset C$ if and only if $[C - g(a)]^* \subset [E_0 - g(a)]^*$, proving Theorem 2. ■

The above criterion suggests the following outer approximation scheme for solving (SP_0). (Recall that γ_0 is an upper bound for the objective function value. Also note that $[C - g(a)]^* \subset K^*$ since $K \subset C - g(a)$.)

Step 0. Construct an initial simplex M_1 in R^m such that $[C - g(a)]^* \subset M_1 \subset K^*$ (see Proposition 1).

Step 1. Check whether $M_1 \subset [E_0 - g(a)]^*$ (see Proposition 2). If yes, γ_0 is the optimal value.

Step 2. Otherwise, a $t^1 \in M_1$ is found such that $t^1 \notin [E_0 - g(a)]^*$. Then there is $y^1 \in E_0$ satisfying $\langle t^1, y^1 - g(a)\rangle < -1$, i.e., $y^1 = g(x^1)$ with $x^1 \in D, \langle c, x^1\rangle \le \gamma_0$, $\langle t^1, g(x^1) - g(a)\rangle < -1$. If $\varphi(g(x^1)) < 1$, then by Theorem 1, we can derive a feasible solution $\hat{x}^1$ with objective function value $\langle c, \hat{x}^1\rangle < \gamma_0$, and (SP_0) is solved.

Step 3. If $\varphi(g(x^1)) \ge 1$, i.e. $g(x^1) \in C$, then $t^1 \notin [C - g(a)]^*$ (because $\langle t^1, g(x^1) - g(a)\rangle < -1$), and since $[C - g(a)]^*$ is a convex closed set, a linear inequality can be constructed to cut off t^1 from $[C - g(a)]^*$ (see Proposition 3) and form a polytope M_2 smaller than M_1 but still containing $[C - g(a)]^*$.

Step 4. Return to Step 1, with $M_1 \leftarrow M_2$.

Clearly, since $M_1^* \subset C - g(a)$, i.e., $g(a) + M_1^* \subset C$, the above procedure amounts to constructing a sequence of expanding polytopes $g(a) + M_1^* \subset g(a) +$

$M_2^* \subset \cdots \subset C$ until a polytope $g(a) + M_k^* \supset E_0$ is obtained (which then implies $E_0 \subset C$). Thus the outer approximation procedure for verifying the inclusion $[C - g(a)]^* \subset [E_0 - g(a)]^*$ is equivalent to an inner approximation (polyhedral annexation) for verifying the inclusion $E_0 \subset C$. To make this conceptual method implementable the following operations have to be specified:
- Step 0: constructing the initial simplex M_1;
- Step 1: checking that $M_1 \subset [E_0 - g(a)]^*$;
- Step 3: determining the cut that separates t^1 from $[C - g(a)]^*$.

These issues will be taken up in the next section.

3. Implementation Issues

Before discussing these issues, note that the above method assumes regularity of problem (P) (assumption (**A4**)). For the general case when this assumption may fail to hold, it is useful to introduce the following concept of approximate solution.

A feasible solution x_ε to the problem

$$\min\{\langle c, x\rangle |\ x \in D,\ \varphi(g(x)) \leq 1 + \varepsilon\} \qquad (\mathrm{P}_\varepsilon)$$

such that

$$\langle c, x_\varepsilon\rangle \leq \min\{\langle c, x\rangle |\ x \in D,\ \varphi(g(x)) \leq 1\} \qquad (3.1)$$

is called an *ε-approximate optimal solution* of (P). Setting

$$C_\varepsilon = \{y \in Y |\ \varphi(y) \geq 1 + \varepsilon\},$$

it is easily seen that in any case (with or without (**A4**)) a feasible solution x_ε of (P_ε) with $\langle c, x_\varepsilon\rangle = \gamma_0$ will be an ε-approximate optimal solution of (P) if

$$\{y |\ y = g(x),\ x \in D, \langle c, x\rangle \leq \gamma_0\} \subset C_\varepsilon,$$

or equivalenty, if

$$[C_\varepsilon - g(a)]^* \subset [E_0 - g(a)]^*.$$

In the sequel, ε denotes a small positive number such that (**A3**) still holds for (P_ε), i.e.,

$$\varphi(g(a)) > 1 + \varepsilon, \quad \langle c, a\rangle < \min\{\langle c, x\rangle |\ x \in D, \varphi(g(x)) \leq 1 + \varepsilon\}.$$

Now let $e^i, i = 1, \ldots, m$, be the ith unit vector of R^m and $e_0 = -(e^1 + \cdots + e^m)$. For every $i = 0, p+1, \ldots, m$, compute $\alpha_i = \sup\{\alpha |\ g(a) + \alpha e^i \in C_\varepsilon\}$ (if this supremum is infinite, let α_i be an arbitrary positive number, as large as convenient). Denote

$$S_1 = [\alpha_0 e^0, \alpha_{p+1} e^{p+1}, \ldots, \alpha_m e^m] + \mathrm{cone}\{e^1, \ldots, e^p\}.$$

Proposition 1.

(i) *The set* $M_1 := S_1^*$ *is an m-simplex defined by the inequalities:*

$$t_i \geq 0 \ (i = 1, ..., p), \ t_i \geq -1/\alpha_i \ (i = p+1, ..., m), \ \textstyle\sum_{i=1}^m t_i \leq 1/\alpha_0, \tag{3.2}$$

(ii) M_1 *satisfies* $[C_\varepsilon - g(a)]^* \subset M_1 \subset K^*$ *and its vertex set* $V(M_1)$ *consists of the points*

$$-(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_m}), \ -(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_{i-1}}, \beta_i, \frac{1}{\alpha_{i+1}}, \frac{1}{\alpha_m}), \ i = 1, ..., m, \tag{3.3}$$

where $1/\alpha_i = 0$ *for* $i = 1, ..., p$ *and* $\beta_i = -\sum_{j=0, j\neq i}^m (1/\alpha_j)$.

Proof. Clearly $0 \in \text{int} S_1$, hence M_1 is compact. Since $S_1^* = -S_1^\circ$, (i) follows from well-known properties of polars of polyhedrons. To prove (ii), observe from (3.2) that $M_1 \subset K^*$; furthermore, since $S_1 \subset C_\varepsilon - g(a)$, it follows that $[C_\varepsilon - g(a)]^* \subset S_1^* = M_1$. The last part concerning $V(M_1)$ can be checked by direct computation. ∎

Proposition 2. *For any polytope* M *such that* $[C_\varepsilon - g(a)]^* \subset M \subset K^*$, *we have* $M \subset [E_0 - g(a)]^*$ *if and only if*

$$\min\{\langle t, g(x) - g(a)\rangle | \ x \in D, \langle c, x\rangle \leq \gamma_0\} \geq -1 \quad \forall t \in V(M), \tag{3.4}$$

where $V(M)$ *denotes the vertex set of* M.

Proof. Clearly $M \subset [E_0 - g(a)]^*$ if and only if $\min\{\langle t, y\rangle | \ y \in E_0 - g(a)\} \geq -1 \ \forall t \in M$, i.e., if and only if $\min\{\langle t, g(x) - g(a)\rangle | \ x \in D, \langle c, x\rangle \leq \gamma_0\} \geq -1 \ \forall t \in M$. But, since the function $t \mapsto \min\{\langle t, g(x) - g(a)\rangle | \ x \in D\}$ is concave, it follows that $\min\{\langle t, g(x) - g(a)\rangle | \ x \in D\} \geq -1 \quad \forall t \in M$, if and only if $\min\{\langle t, g(x) - g(a)\rangle | \ x \in D\} \geq -1 \ \forall t \in V(M)$. ∎

Note that by virtue of assumption (**A2**) for each $t \in V(M)$, $M \subset K^*$, the function $x \mapsto \langle t, g(x) - g(a)\rangle$ is convex, so (3.4) is a convex problem. Now in step 3, let $\min\{\langle t^1, g(x) - g(a)\rangle | \ x \in D, \langle c, x\rangle \leq \gamma_0\} = \langle t^1, g(x^1) - g(a)\rangle < -1$ with $x^1 \in D, \langle c, x^1\rangle \leq \gamma_0$ and $\varphi(g(x^1)) \geq 1$.

Proposition 3. *Let* $\theta_1 \geq 1$ *be any number such that* $g(a) + \theta_1(g(x^1) - g(a)) \in C_\varepsilon$. *Then*

$$\langle t^1, g(x^1) - g(a)\rangle < -\frac{1}{\theta_1}, \ \ \langle t, g(x^1) - g(a)\rangle \geq -\frac{1}{\theta_1} \ \forall t \in [C - g(a)]^* \tag{3.5}$$

Therefore, the cut

$$\langle t, g(x^1) - g(a)\rangle \geq -\frac{1}{\theta_1} \tag{3.6}$$

will cut off t^1 *while not excluding any point of* $[C_\varepsilon - g(a)]^*$.

Proof. Note that since $g(x^1) \in C_\varepsilon$ while $g(a) \in \text{int} C_\varepsilon$, a number $\theta_1 \geq 1$ as described always exists. The first inequality of (3.5) follows from $\langle t^1, g(x^1) -$

$g(a)\rangle < -1$ and $\theta_1 \geq 1$. If $t \in [C_\varepsilon - g(a)]^*$ then by definition $\langle t, y\rangle \geq -1 \quad \forall y \in C_\varepsilon - g(a)$, which yields the second inequality because $\theta_1(g(x^1) - g(a)) \in C_\varepsilon$. ■

4. Algorithm and Convergence

The above development leads to the following algorithm for finding an ε-approximate optimal solution of (P), without the regularity assumption.

Algorithm

Step 0. Let $\bar{x}^1$ be the best feasible point available, $\gamma_1 = \langle c, \bar{x}^1\rangle$ (set $\bar{x}^1 = \emptyset, \gamma_1 = +\infty$ if no feasible solution is available). Construct M_1 as indicated in Proposition 1. Let $V_1 = V(M_1)$. Set $k = 1$.

Step 1. For every $t \in V_k \setminus \{0\}$ solve the subproblem

$$\min\{\langle t, g(x) - g(a)\rangle |\ x \in D, \langle c, x\rangle \leq \gamma_k\} \tag{4.1}$$

to obtain the optimal value $\mu(t)$ and an optimal solution $x(t)$ of it. (If this step is entered from step 4, it suffices to solve (4.1) only for every new $t \in V_k$). Let $t^k \in \operatorname{argmin}\{\mu(t)|\ t \in V_k\}$. If $\mu(t^k) \geq -1$, then terminate: $\bar{x}^k$ is ε-approximate optimal if $\gamma_k < +\infty$, or (P) is infeasible otherwise.

Step 2. If $\mu(t^k) < -1$ and $x^k := x(t^k)$ satisfies $\varphi(g(x^k)) < 1$, then from x^k, derive a solution $\bar{x}^{k+1}$ such that $\varphi(g(\bar{x}^{k+1})) = 1 + \varepsilon$ and $\langle c, \bar{x}^{k+1}\rangle < \gamma_k$ (see Remark 2 below). Let $\gamma_{k+1} = \langle c, \bar{x}^{k+1}\rangle, M_{k+1} = M_k, V_{k+1} = V_k$. Return to step 1 with $k \leftarrow k+1$.

Step 3. If $\varphi(g(x^k)) \geq 1$, compute $\theta_k = \sup\{\theta|\ g(a) + \theta(g(x^k) - g(a)) \in C_\varepsilon\}$ and define

$$M_{k+1} = M_k \cap \left\{t |\ \langle t, g(x^k) - g(a)\rangle \geq -\frac{1}{\theta_k}\right\}.$$

Step 4. Compute the vertex set V_{k+1} of M_{k+1}. Set $k \leftarrow k+1$ and return to step 1.

Remark 1. In Step 4, since M_{k+1} differs from M_k by just one additional linear constraint, the vertex set V_{k+1} of M_{k+1} can be derived from V_k by using any subroutine for on-line vertex enumeration, e.g. the one by Chen - Hansen - Jaumard [12, Sec. 6.2].

Remark 2. In Step 2, the solution $\bar{x}^{k+1}$ can be taken to be the point $\hat{x}^k$ where the line segment $[a, x^k]$ meets the surface $\varphi(g(x)) = 1 + \varepsilon$ (see Theorem 1), or any better feasible solution of (P_ε) obtained for example by performing a local search from $\hat{x}^k$.

The convergence of the above Algorithm can be established on the basis of the following simplified version of the Basic Outer Approximation Theorem:

Theorem 3. *Let G be an arbitrary closed subset of R^q with a non-empty interior, let $\{u^k\}$ be an infinite sequence in R^q, and for each k, let $l_k(u)$ be an affine* *function satisfying*

$$l_k(u^k) > 0, \quad l_k(u) \leq 0 \ \forall u \in G.$$

If the sequence $\{u^k\}$ *is bounded, if* $w \in \mathrm{int}G$ *and for every* k *there exist* $v^k \in [w, u^k] \setminus \mathrm{int}G$, *such that* $l_k(v^k) \geq 0$, *then*

$$u^k - v^k \to 0 \quad (k \to +\infty).$$

Proof. See [10] (also [13]). ■

Theorem 4. *If the Algorithm is infinite it generates an infinite sequence* $\{\bar{x}^k\}$ *of which every accumulation point is an* ε*-approximate optimal solution.*

Proof. Denote $\bar{\gamma} = \lim \gamma_k$ $(k \to +\infty)$, $E = \{y \in Y \mid y = g(x),\ x \in D,\ \langle c, x\rangle \leq \bar{\gamma}\}$ and let $\bar{x} = \lim \bar{x}^{k_\nu}(\nu \to +\infty)$. By passing to subsequences if necessary, we can assume that $t^\nu \to \bar{t}$ (where ν stands for k_ν to simplify the notation). We first show that step 2 can occur only in finitely many iterations ν. Indeed, suppose the contrary, that $\varphi(g(x^\nu)) < 1$ for all ν. Let $\hat{x}^\nu$ be the point of $[a, x^\nu]$ where $\varphi(g(\hat{x}^\nu)) = 1 + \varepsilon$. Applying the Basic Outer Approximation Theorem to the set $G := \{x \mid x \in D, \varphi(g(x)) \geq 1 + \varepsilon, \langle c, x\rangle \leq \bar{\gamma}\}$ with $a \in \mathrm{int}G$, the sequences $u^\nu = x^\nu, v^\nu = \hat{x}^\nu$, and the cuts $l_\nu(u) = \langle c, u - \hat{x}^\nu\rangle$, we see that $x^\nu - \hat{x}^\nu \to 0$, hence $\varphi(g(\bar{x})) = 1 + \varepsilon$, conflicting with the fact $\varphi(g(x^\nu)) < 1\ \forall\nu$. Thus, step 2 occurs only in finitely many iterations ν and without loss of generality, we can assume that step 3 occurs at every iteration ν. This implies $\gamma_\nu = \bar{\gamma}\ \forall\nu$. Define $l_\nu(t) = \langle t, -g(x^\nu) + g(a)\rangle - 1$. Then

$$l_\nu(t^\nu) > 0, \quad l_\nu(t) \leq 0\ \forall t \in [E - g(a)]^*. \tag{4.2}$$

Indeed, the left inequality follows from $\mu(t^\nu) = \langle t^\nu, g(x^\nu) - g(a)\rangle < -1$, while for every $t \in [E - g(a)]^*$, we have $\langle t, y\rangle \geq -1\ \forall y \in E - g(a)$, hence $\langle t, g(x^\nu) - g(a)\rangle \geq -1$, proving the right inequality in (4.2). Furthermore, since E is compact, $[E - g(a)]^*$ has an interior point w. Then clearly w satisfies $l_\nu(w) < 0$, while $l_\nu(t^\nu) > 0$, so there exists a point $s^\nu \in [w, x^\nu]$ such that $l_\nu(s^\nu) = 0$. By virtue of the Basic Outer Approximation Theorem applied to $G = [E - g(a)]^*, w \in \mathrm{int}G$, the cuts $l_\nu(t)$, and the sequences $\{t^\nu, s^\nu\}$ we then obtain $t^\nu - s^\nu \to 0$, i.e., $t^\nu \to \bar{t}$ such that $\langle \bar{t}, g(\bar{x}) - g(a)\rangle = -1$. But for every ν and every $t \in M_\nu$,

$$\min\{\langle t, g(x) - g(a)\rangle \mid x \in D, \langle c, x\rangle \leq \bar{\gamma}\} \geq \langle t^\nu, g(x^\nu) - g(a)\rangle.$$

Hence, for every $t \in [C_\varepsilon - g(a)]^* \subset \cap_{\nu=1}^{+\infty} M_\nu$,

$$\min\{\langle t, g(x) - g(a)\rangle \mid x \in D, \langle c, x\rangle \leq \bar{\gamma}\} \geq \langle \bar{t}, g(\bar{x}) - g(a)\rangle = -1.$$

That is,

$$[C_\varepsilon - g(a)]^* \subset [E - g(a)]^*, \tag{4.3}$$

proving the ε-approximate optimality of $\bar{x}$. ■

Remark 3. As $\varepsilon \to 0$ is an ε-approximate optimal solution tends to an exact optimal solution.

5. The Case of Linear Programs with a Low Rank Reverse Convex Constraint

An important special case for the applications is when D is a polytope and $g(x)$ is affine: $g_i(x) = \langle c^i, x\rangle$, $i = 1, \dots, m$. The algorithm can then be made finite by the following arrangements and improvements:

(1) In Step 1, each problem (4.1) is a linear program:

$$\min\{\sum_{i=1}^{m} t_i\langle c^i, x - a\rangle |\ x \in D, \langle c, x\rangle \le \gamma_k\},$$

so an optimal solution of it can be taken to be a vertex of the polytope $D_k = \{x \in D|\ \langle c, x\rangle \le \gamma_k\}$.

(2) In step 2, $x^k = x(t^k)$ is a vertex of D_k which is not optimal for the linear program $\min\{\langle c, x\rangle|\ x \in D_k\}$ (because the intersection of the segment $[a, x^k]$ with the surface $\varphi(g(x)) = 1 + \varepsilon$ is a better solution). Therefore, starting from x^k, one can find (e.g., by using the simplex procedure applied to the above linear program) a better solution lying on the intersection of the surface $\varphi(g(x)) = 1 + \varepsilon$ with an edge of D. This better solution should be taken to be $\bar{x}^{k+1}$.

(3) In step 3, $\theta_k = \sup\{\theta|\ \varphi(g(a + \theta(x^k - a)) \ge 1 + \varepsilon\}$.

Under these conditions the incumbent solution $\bar{x}^k$ at iteration k is always an intersection point of an edge of D with the surface $\varphi(g(x)) = 1 + \varepsilon$. The finiteness of the algorithm then follows from the finiteness of the set of edges of D.

Remark 4. Let $D = \{x \in R^n|\ Ax \le b,\ x \ge 0\}$ and $g(x) = Gx$, where G is the $m \times n$ matrix with rows $c^1, \dots, c^m$. Assuming $\text{rank}G = m$ we can write $G = [G_B, G_N]$, $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$, where G_B is an $m \times m$ non-singular matrix. Then setting

$$Z = \begin{bmatrix} G_B^{-1} \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} -G_B^{-1}G_N x_N \\ x_N \end{bmatrix},$$

the equation $Gx = y$ yields

$$x = Zy + u \quad \text{with } Gu = -G_N x_N + G_N x_N = 0.$$

Thus, under the specified assumptions (D polyhedral and $g(x)$ affine) the problem (P) can always be rewritten as

$$\min\{\langle c, Zy + u\rangle|\ AZy + Au \le b, Zy + u \ge 0,\ \varphi(y) \le 1\}.$$

Note that this is a problem of type (2) as described in the Introduction.

6. Computational Experience

The proposed algorithm has been tested on problems of the form

$$\min\{\langle c, x\rangle |\ x \in D, h(x_1, x_2, ..., x_r) + \langle d, x\rangle \leq 0\},$$

where $h(x_1, x_2, ..., x_r)$ is a concave function of $r < n$ variables and $D = \{x \in R^n : Ax \leq b, x \geq 0\}$ with $A \in R^{p\times n}$ and $b \in R^p$. This choice of the test problems is in part justified by Remark 4.

The function $h(x)$ used in the tested problems is a concave function of the following types:

(0) $-\sum_{i=1}^{r} q_i x_i^2$

(1) $-\sum_{i=1}^{r}(x_i^2)/3 - (2/3) * \max\{x_i : i = 1, 2, ..., n\}$

(2) $\sum_{i=1}^{r}\{i^2(x_i + i/n)^{1/i} + i * x_i\}$

(3) $-\sum_{i=1}^{r} x_i^2/i - [\sum_{i=1}^{r}(x_i + i)]^{3/2}$

(4) $\ln(\sum_{i=1}^{r}(i * x_i) + 1) - \sum_{i=1}^{r} x_i^2/(n - i + 1)$

(5) $-\exp[\sum_{i=1}^{r}(x_i^2/i^2 - i * x_i)/10^5] - \sum_{i=1}^{r} x_i^2/(n - i + 1)$

(6) $\sqrt{(\sum_{i=1}^{r} i * x_i) * (\sum_{i=1}^{r} x_i + n - i)} - \sum_{i=1}^{r} x_i^2/i.$

The algorithm coded in PASCAL and ran on an IBM PC Intel Pentium MMX 166. We solved more than 100 randomly generated problems. The computational results on a number of these problems are reported in Table 1, where the following notations are used:

n	number of variables
p	number of linear constraints
$r + 1$	rank of reverse convex constraint
It	number of iterations
NOpt	index of iteration in which the optimal solution is found
NCut	number of cuts
NIm	number of iterations in which the incumbent has improved
LP	number of solved linear subproblems
IX	maximal number of vertices at one iteration
IV	total number of generated vertices
Time	computational time (in seconds)
Type	type of function $h(x)$

Table 1

n	p	r	It	NOpt	NCut	NImp	LP	IX	IV	Time	Type
20	10	7	30	27	17	12	2848	704	5342	95.24	0
30	18	6	28	27	14	13	1040	251	1739	136.38	5
30	12	7	16	15	9	6	508	242	992	24.44	0
30	10	7	20	19	7	12	273	120	428	15.00	0
40	17	6	8	4	3	4	84	31	78	14.22	0
50	15	5	18	16	16	1	343	118	1126	56.41	1
50	20	6	31	30	12	18	1710	158	1036	461.64	0
50	18	7	20	19	10	9	2489	364	1674	565.29	0
50	10	7	38	36	19	18	2897	642	5998	208.94	1
60	25	5	11	9	4	6	88	24	76	40.04	0
60	20	6	9	6	3	5	71	32	77	31.70	0
60	15	4	5	4	0	4	26	5	5	5.82	0
70	25	4	6	4	1	4	35	8	13	25.70	0
70	10	5	30	29	11	18	568	74	462	66.41	0
70	10	6	8	7	0	7	54	7	7	6.86	0
80	15	5	10	9	5	4	91	34	112	28.01	0
80	10	6	56	53	13	42	2121	215	1454	231.78	1
80	5	7	5	3	1	3	36	14	22	1.97	0
80	25	4	11	7	3	7	61	12	26	58.38	0
90	25	4	12	9	5	6	68	21	78	59.10	0
90	5	6	7	3	3	3	80	28	73	3.19	0
90	8	7	4	3	0	3	36	8	8	4.01	0
90	20	4	8	7	6	1	72	30	118	42.08	1
90	10	6	12	10	4	7	303	58	146	43.94	1
100	9	6	7	4	2	4	65	21	44	8.90	0
110	25	4	13	12	4	8	86	17	55	119.57	4
120	5	5	20	19	8	11	354	76	358	20.04	5
120	35	2	12	11	2	9	39	4	11	96.23	2
120	35	3	6	5	0	5	28	4	4	86.84	3
120	10	6	9	7	2	6	74	21	40	12.90	6
130	35	2	9	8	1	7	31	4	7	136.92	2
130	35	3	6	5	0	5	25	4	4	86.62	3
130	4	6	16	14	4	11	496	58	153	22.90	1
130	5	6	26	14	20	5	1520	437	3975	104.19	1
130	8	7	15	14	6	8	265	86	274	38.39	0
140	15	4	21	20	4	16	205	21	63	108.20	4
140	10	5	6	4	1	4	36	12	18	6.97	5
140	20	5	11	10	5	5	98	30	102	70.58	5
140	5	5	12	11	5	6	110	36	122	8.19	0
140	8	6	7	5	2	4	62	21	44	7.69	0
140	3	7	7	5	1	5	51	18	26	2.63	0

Example 1. $n = 10,\ p = 6,\ r = 3,\ \varepsilon = 0.000010$

$$A = \begin{pmatrix} -3.0 & 2.3 & -4.2 & 4.3 & -0.2 & -1.4 & 4.4 & 1.0 & 0.2 & 0.1 \\ -1.3 & 3.0 & -0.3 & 0.0 & 1.7 & -2.6 & -1.1 & -2.0 & 1.9 & 1.0 \\ 0.0 & -0.4 & -0.4 & -0.6 & -4.4 & -0.6 & 1.5 & 1.6 & -3.9 & -1.9 \\ 0.8 & 2.0 & -1.5 & 3.3 & 2.2 & 2.7 & -3.5 & 4.2 & -1.4 & -1.0 \\ -4.7 & 3.6 & 1.0 & -3.1 & 0.5 & 3.6 & 2.4 & -0.8 & 0.9 & -1.9 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{pmatrix}$$

$b = (4.290, 0.560, 0.800, 2.850, 3.190, 500.000),$

$c = (72, -50, 270, 90, 16, 129, -83, 67, 159, 78),$

$h(x) = -8x_1^2 - 10x_2^2 - 2x_3^2 - 16.436x_1 + 83.821x_2 + 51.448x_3 + 15.000 \le 0.$

Computational Result

Optimal solution:

$x^* = (0.685, 0.000, 0.000, 0.000, 0.315, 0.000, 1.456, 0.000, 0.000, 0.000)$

Number of iterations: 7

Computational time: 0.44 sec.

Optimal value: -66.530648

Value of reverse function $h(x^*) = 0.000057$

Optimal solution obtained at iteration: 4

Number of LP : 27

Number of improvements : 4

Number of cuts : 2

Maximal number of vertices : 6

Total number of generated vertices : 16

Example 2. $n = 10,\ p = 8,\ r = 4,\ \varepsilon = 0.00001$

$$A = \begin{pmatrix} 3.5 & 3.2 & -0.7 & 2.5 & 0.9 & -4.6 & -0.6 & -2.2 & 2.1 & -3.1 \\ 4.7 & -4.6 & 2.5 & 3.2 & -2.8 & -2.7 & 0.6 & -1.7 & -4.0 & -2.7 \\ -1.2 & -2.8 & -1.1 & 2.5 & -3.0 & -2.1 & 2.3 & -0.8 & 2.6 & -2.0 \\ 1.1 & 0.3 & -2.3 & -1.9 & -3.7 & -2.3 & 0.1 & 3.8 & 4.6 & 1.1 \\ -3.0 & -3.2 & -4.6 & 3.8 & -2.4 & -0.4 & 2.1 & 0.8 & 3.1 & -1.9 \\ 1.5 & 0.5 & -4.9 & -2.7 & 0.0 & 1.0 & -1.5 & -2.8 & -2.3 & -0.4 \\ -2.7 & 0.2 & 2.6 & -3.7 & -4.7 & 3.8 & -4.6 & 4.1 & -2.3 & 0.8 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{pmatrix}$$

$b = (1.890, -4.810, 0.440, 2.030, 2.640, -5.590, -4.850, 500.000),$

$c = (-47.0, 184.0, 82.0, 74.0, 105.0, -3.0, -123.0, -105.0, 56.0, 104.0),$

$h(x) = -8.0x_1^2 - 7.0x_2^2 - 7.0x_3^2 - 3.0x_4^2 + 63.190x_1 + 40.478x_2.$
$+65.653x_3 + 31.306x_4 + 90000.000$

Computational Result

Optimal solution:

$x^* = (95.237, 60.851, 0.0, 0.0, 0.0, 99.218, 222.329, 22.364, 0.0, 0.000)$

Number of iterations: 17

Computational time: 2.04 sec.
Optimal value: -23271.932023
Value of reverse function $h(x^*) = -0.000011$
Optimal solution obtained in iteration: 16
Number of LP: 107
Number of improvements: 10
Number of cuts: 6
Maximal number of vertices: 24
Total number of generated vertices: 100

References

1. R. Horst and H. Tuy, *Global Optimization* (Deterministic Approaches), third edition, Springer-Verlag, Berlin, New York, 1996.
2. H. Konno and T. Kuno, Multiplicative programming, in: *Handbook on Global Optimization*, R. Horst and P. M. Pardalos (eds.), Kluwer Academic Publishers, Dordrecht - Boston - London, 1995, pp. 369-403.
3. H. Konno, P. T. Thach, and H. Tuy, *Optimization on Low Rank Nonconvex Structures*, Kluwer Academic Publishers, Dordrecht - Boston - London, 1996.
4. T. Kuno, H. Konno and Y. Yamamoto, A parametric successive underestimation method for convex programming problems with an additional convex multiplicative constraint, *Journal of the Operations Research Society of Japan* **35** (1992) 290-299.
5. T. Kuno, Y. Yajima, Y. Yamomoto, and H. Konno, Convex programs with an additional constraint on the product of several convex functions, *European Journal of Operations Research* **77** (1991) 314-324.
6. U. Pferschy and H. Tuy, Linear programs with an additional rank two reverse convex constraint, *Journal of Global Optimization* **4** (1994) 347-366.
7. P. T. Thach, R. E. Burkard, and W. Oettli, Mathematical programs with a two-dimensional reverse convex constraint, *Journal of Global Optimization* **1** (1991) 145-154.
8. H. Tuy, Polyhedral Annexation, Dualization and dimension reduction technique in global optimization, *Journal of Global Optimization* **1** (1992) 229-244.
9. H. Tuy, The Complementary convex structure in global optimization, *Journal of Global Optimization* **2** (1992) 21-40.
10. H. Tuy, D.C. Optimization: Theory, methods and algorithms, in: *Handbook on Global Optimization*, R. Horst and P. M. Pardalos (eds.), Kluwer Academic Publishers, Dordrecht - Boston - London, 1995, pp. 149-216.
11. H. Tuy, Canonical D.C. Programming: Outer approximation methods revisited, *Operations Research Letters* **18** (1995) 99-106.
12. H. Tuy, *Convex Analysis and Global Optimization*, Kluwer Academic Publishers, Dordrecht - Boston - Lóndon, 1998.
13. H. Tuy, Parametric methods in global optimization, in: *Proceedings of the 5th Conference on Parametric Optimization*, Tokyo, 1998.