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Survey

Diagonal Subalgebras and Blow-ups of Projective Spaces*

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Abstract. This is a survey on recent results on the relationship between diagonal subalgebras and blow-ups of projective spaces.

1. Introduction

Diagonal subalgebra is a rather new object in commutative algebra. The motivation for the study of diagonal subalgebras comes from algebraic geometry.

In the last ten years many authors have studied rational surfaces obtained by blowing-up $\mathbb{P}^2$ at collections of points. They showed that these surfaces embedded in certain projective spaces often have good algebraic properties [6, 9-12, 15, 17]). For instance, Geramita, Gimigliano and Harbourne [10] considered the blow-up of $\mathbb{P}^2$ at points which are the intersection of two curves f and g meeting transversely. They embedded the obtained rational surface in projective spaces by very ample divisors which correspond to the linear systems of curves of a fixed degree c passing through the points of multiplicity at least e, where e is a given positive number. Their main results can be summarized in the following theorem.

Theorem 1.1. *Let R be the coordinate ring of such an embedded surface. Suppose that $d_1 = \deg f \geq d_2 = \deg g$.*

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(i) *If* $d_1 = d_2 = d$, $c = d+1$, *and* $e = 1$, *then* R *is defined by* 3 *quadrics and* d *forms of degree* d.
(ii) R *is normal.*
(iii) R *is a Cohen–Macaulay ring if and only if* $c \geq d_1 + ed_2 - 2$.

Moreover, they raised the problem of computing the defining equations for R in (i).

Diagonal subalgebra was introduced in order to solve this problem [26] and it turned out that digonal subalgebra is an effective algebraic tool for the study of embedded rational n-folds obtained by blowing-up $\mathbb{P}^{n-1}$ along a subvariety [7]. In this paper we will give a survey of the methods and the results of [26] and [7] and we shall see that Theorem 1.1 can be greatly generalized.

We will consider the more general situation of blowing-up $\mathbb{P}^{n-1}$ at an arbitrary subvariety Z. Let V denote the obtained rational $(n-1)$-fold. Then V can be described algebraically as follows.

Let $I \subset k[X] = k[x_1, \dots, x_n]$ be the defining ideal of Z. One can associate with I the Rees algebra:

$$R(I) := \oplus_{v\geq 1} I^v t^v$$

which is the subalgebra of $k[X,t]$ generated over $k[X]$ by the elements ft, $f \in I$. Since I is a homogeneous ideal, $R(I)$ has a natural bigraded structure by setting

$$R(I)_{(u,v)} = (I^v)_u t^v$$

for all $(u,v) \in \mathbb{N}^2$, where $(I^v)_u$ denotes the k-vector space of forms of degree u of I^v. Therefore, we can associate with $R(I)$ the projective scheme

$$\begin{aligned}&\operatorname{Biproj} R(I) = \\ &\{P \in \operatorname{Spec} R(I) |\ P \text{ is bihomogeneous, } P \not\supseteq R(I)_{(1)} \text{ and } P \not\supseteq R(I)_{(2)}\},\end{aligned}$$

where $R(I)_{(1)}$ and $R(I)_{(2)}$ denote the ideals of $R(I)$ generated by the homogeneous elements of degree (u,v) with $u \geq 1$ and $v \geq 1$, respectively. The following lemma is more or less a standard fact:

Lemma 1.2. $V \cong \operatorname{Biproj} R(I)$.

Let c and e be two fixed positive integers. We shall see when V can be embedded into a projective space by the linear system $(I^e)_c$ of forms of degree c in the ideal I^e. Let

$$R := k[(I^e)_c]$$

be the subalgebra of $k[X]$ generated by the elements of $(I^e)_c$. Since R is generated by forms of the same degree, it is a graded algebra and we can define the projective scheme

$$\operatorname{Proj} R := \{\wp \in \operatorname{Spec} R |\ \wp \text{ is homogeneous and } \wp \not\supseteq R_+\},$$

where R_+ denotes the ideal of R generated by the homogeneous elements of positive degree.

Let $I = (f_1, \dots, f_r)$, where f_j is a homogeneous polynomial with $d_j = \deg f_j$, $j = 1, \dots, r$. Put

$$d := \max\{d_1, \dots, d_r\}.$$

One can easily prove the following relationship between $\operatorname{Biproj} R(I)$ and $\operatorname{Proj} R$:

Lemma 1.3. $\operatorname{Biproj} R(I) \cong \operatorname{Proj} R$ *if* $c \geq de + 1$.

So we obtain $V \cong \operatorname{Proj} R$ if $c \geq de + 1$. In that case, the linear system $(I^e)_c$ corresponds to a very ample divisor of V and R is the coordinate ring of the embedding of V into a projective space by this very ample divisor.

Now we shall see that, given two positive integers c, e, one can associate with every bigraded algebra a subalgebra and that R is just this subalgebra of $R(I)$ if $c \geq de + 1$.

Let $\Delta = \{(ci, ei) | i \in \mathbb{Z}^2\}$. Let $S = \oplus_{(u,v)\in \mathbb{N}^2} S_{(u,v)}$ be an arbitrary bigraded ring.

Definition. *Given any bigraded S-module M we define*

$$M_\Delta := \oplus_{i\in \mathbb{Z}} M_{(ci,ei)}$$

and we call M_Δ the Δ-diagonal or the (c,e)-diagonal of M.

It is clear that S_Δ is a graded ring and M_Δ a graded S_Δ-module. We may consider the Δ-diagonal as a functor from the category of bigraded algebras to the category of (simply) graded algebras.

We can intepret the algebra $R = k[(I^e)_c]$ as the Δ-diagonal of the Rees algebra $R(I)$ for certain c and e:

Lemma 1.4. $R = R(I)_\Delta$ *if* $c \geq de$.

This interpretation of the algebra R provides us with an algebraic tool for the study of the blow-ups of projective spaces.

We shall see that many properties of $R(I)_\Delta$ can be read off from the Rees algebra $R(I)$. We would like to mention that the above observations also hold if we consider the blow-up of any projective variety W at any subvariety Z. In this case I is a homogeneous ideal of the coordinate ring of W.

This paper consists of six sections. The next five sections will deal with the presentation, normality, local cohomology, Cohen–Macaulayness, and Koszul property of $R(I)_\Delta$, respectively.

We will keep the notations of this introduction unless otherwise specified.

2. Presentation of $R(I)_\Delta$

Let $k[X,Y] = k[x_1,\dots,x_n,y_1,\dots,y_r]$ be a polynomial ring over k. We consider $k[X,Y]$ as a bigraded algebra by setting

$$\deg x_i = (1,0),\ i = 1,\dots,n,$$
$$\deg y_j = (d_j,1),\ j = 1,\dots,r.$$

Mapping y_j to f_jt we get a bigraded isomorphism

$$R(I) \cong k[X,Y]/J,$$

where J is a bihomogeneous ideal of $k[X,Y]$. It is clear that

$$R(I)_\Delta = k[X,Y]_\Delta/J_\Delta.$$

Thus, to find a presentation for $R(I)_\Delta$ we only need to find a presentation for $k[X,Y]_\Delta$ and the preimage of J_Δ in the polynomial ring of the presentation of $k[X,Y]_\Delta$.

The algebra $k[X,Y]_\Delta$ is generated by the monomials $x_1^{a_1}\cdots x_n^{a_n}y_1^{b_1}\cdots y_r^{b_r}$ whose exponents satisfy a system of equations

$$a_1 + \cdots + a_n + b_1d_1 + \cdots + b_rd_r = ci,$$
$$b_1 + \cdots + b_r = ei,$$

where i can be any non-negative integer. Such an algebra is defined by binomials, hence one can easily find a presentation for $k[X,Y]_\Delta$.

Let us consider the following case which generalizes Theorem 1.1(i).

Case: $d_1 = \ldots = d_r = d,\ c = d+1,\ e = 1.$

In this case I is generated by forms of the same degree. Therefore, we can replace the natural bigrading of $R(I)$ by the simpler bigrading:

$$R(I)_{(u,v)} = (I^v)_{u+dv}t^v,$$

which corresponds to the standard bigrading $\deg x_i = (1,0)$, $\deg y_j = (0,1)$ of $k[X,Y]$. Since the new bigrading is only a linear transformation of the natural bigrading, the ideal J remains bihomogeneous.

Now, let $\Delta = \{(i,i) |\ i \in \mathbb{Z}^2\}$. Then $R(I)_\Delta = k[I_{d+1}]$. We have

$$k[X,Y]_\Delta = k[x_iy_j |\ i = 1,\dots,n,\ j = 1,\dots,r]$$
$$\cong k[T]/I_2(T),$$

where $T = (t_{ij})$ is an $n\times r$ generic matrix and $I_2(T)$ denotes the ideal of $k[T]$ generated by the 2×2 minors of T.

Proposition 2.1. *Let C be an ideal of $k[T]$ generated by the preimages of the elements of a minimal set of generators of J_Δ. Then*

$$R(I)_\Delta \cong k[T]/I_2(T) + C.$$

To compute C we only need to find a set of generators of J_Δ and then compute their preimages in $k[T]$. But such a set of generators for J_Δ can be easily determined.

Lemma 2.2. *Let $J = (H_1, \dots, H_s)$, where H_i is a bihomogeneous polynomial of degree (a_i, b_i), $i = 1, \dots, s$. Put $c_i = \max\{a_i, b_i\}$. Then J_Δ is generated by elements of the forms H_iM, where $M \in k[X, Y]$ is a monomial with $\deg M = (c_i - a_i, c_i - b_i)$, $i = 1, \dots, s$.*

Now we will show how to compute a presentation of $R(I)_\Delta$ in the situation of Theorem 1.1(i).

Example. Let $I = (f, g) \subset k[x_1, x_2, x_3]$, where f, g are two homogeneous polynomials of the same degree d which form a regular sequence (the ideal I need not to be the defining ideal of a set of points). We have

$$R(I) \cong k[x_1, x_2, x_3, y_1, y_2]/(H),$$

where

$$H := gy_1 - fy_2.$$

In this case,

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \\ t_{31} & t_{32} \end{pmatrix}$$

is a 3×2 generic matrix. Hence,

$$I_2(T) = (t_{11}t_{22} - t_{12}t_{21}, t_{11}t_{32} - t_{12}t_{31}, t_{21}t_{32} - t_{22}t_{31})$$

is generated by 3 quadrics. Since $\deg H = (d, 1)$, $(H)_\Delta$ is generated by elements of the form $Hy_1^a y_2^b$, $a + b = d - 1$. Since there are d monomials $y_1^a y_2^b$ with $a + b = d - 1$ and since the preimages of $Hy_1^a y_2^b$ in $k[T]$ is a form of degree d, the preimage C of $(H)_\Delta$ in $k[T]$ is generated by d forms of degree d. Therefore, the defining ideal $I_2(T) + C$ of $R(I)_\Delta$ is generated by 3 quadrics and d forms of degree d. This gives an easy proof for Theorem 1.1(i). Moreover, we can also solve the problem of computing the defining equations explicitly. For instance, if $f = x_1^d$ and $g = x_2^d$, then $H = x_2^d y_1 - x_1^d y_2$ and $t_{21}^{a+1} t_{22}^b - t_{11}^a t_{12}^{b+1}$, $a + b = d - 1$, is a preimage of $Hy_1^a y_2^b$ in $k[T]$. Therefore, we may put

$$C = (t_{12}^d - t_{21}t_{22}^{d-1}, t_{11}t_{12}^{d-1} - t_{21}^2 t_{22}^{d-2}, \dots, t_{11}^{d-1}t_{12} - t_{21}^d).$$

Hence,

$$R(I)_\Delta \cong k[T]/(t_{11}t_{22} - t_{12}t_{21}, t_{11}t_{32} - t_{12}t_{31}, t_{21}t_{32} - t_{22}t_{31},$$
$$t_{12}^d - t_{21}t_{22}^{d-1}, t_{11}t_{12}^{d-1} - t_{21}^2 t_{22}^{d-2}, \ldots, t_{11}^{d-1}t_{12} - t_{21}^d).$$

We would like to raise a more general problem as follows.

Problem. *Compute the Betti numbers of $R(I)_\Delta$ in terms of I.*

3. Normality of $R(I)_\Delta$

Let S be an arbitrary bigraded domain. Let

$$Q(S) = \left\{ \frac{f}{g} \mid f, g \text{ are homogeneous, } g \neq 0 \right\}$$

be the field of homogeneous fractions of S. Then $Q(S)$ is a $\mathbb{Z}^2$-bigraded ring. Let $\overline{S}$ denote the integral closure $\overline{S}$ of S in $Q(S)$. Then $\overline{S}$ also has a bigraded structure. Therefore we can define the Δ-diagonal $\overline{S}_\Delta$.

Let

$$Q(S_\Delta) = \{ \frac{f}{g} \mid f, g \in S_\Delta \text{ are homogeneous, } g \neq 0\}$$

be the field of homogeneous fractions of S_Δ. Let $\overline{(S_\Delta)}$ denote the integral closure of S_Δ in $Q(S_\Delta)$. By degree reasoning we can easily show that

$$\overline{(S_\Delta)} = \overline{S}_\Delta \cap Q(S_\Delta).$$

As a consequence, the normality is preserved by passing to the diagonal subalgebra.

Proposition 3.1. *If S is a normal domain, then so is S_Δ.*

This result can also be proved using the notion of Reynolds operator which is of independent interest. A Reynolds operator of a ring extension $R \subset S$ is an R-module surjection $\phi : S \to R$ such that the composition $R \subset S \to R$ is the identity map. In this case, normality carries from S to R by a result of Hochster and Roberts [19]. A Reynolds operator exists if and only if R is a direct summand of S. Therefore, the ring extension $S_\Delta \subset S$ has a Reynolds operator.

Now we want to apply the above observation to the diagonal $R(I)_\Delta$ of the Rees algebra $R(I)$.

First we will show how to compute the integral closure $\overline{(R(I)_\Delta)}$ of $R(I)_\Delta$ in terms of I. It is not hard to see that

$$Q(R(I)_\Delta) = k\left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}, x_1 f\right)$$

which is the subfield of $k(X)$ generated by the elements $x_2/x_1, \dots, x_n/x_1, x_1 f$ for any element $f \neq 0$ of $(I^e)_{c-1}$.

It is well-known that $\overline{R(I)} = \oplus_{v \in \mathbb{N}} \overline{I^v} t^v$, where $\overline{I^v}$ denotes the integral closure of I^v. Therefore,

$$\overline{R(I)}_\Delta = k[(\overline{I^{ei}})_{ci} |\ i \in \mathbb{N}]$$

which is the subalgebra of $k[X]$ generated by the elements of $(\overline{I^{ei}})_{ci}$, $i \in \mathbb{N}$. So we obtain the following formula for the computation of the integral closure $\overline{(R(I)_\Delta)}$ of $R(I)_\Delta$.

Proposition 3.2. [26] *Let $f \neq 0$ be an arbitrary element in $(I^e)_{c-1}$. Then*

$$\overline{(R(I)_\Delta)} = k\left(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}, x_1 f\right) \cap k[(\overline{I^{ei}})_{ci} |\ i \in \mathbb{N}].$$

Moreover, we can easily generalize Theorem 1.1(ii) as follows.

Let I be the defining prime ideal of a collection of points. Then

$$I = \cap_j \wp_j,$$

where $\wp_j$ are the defining ideals of the points. Assume that I is generated by a regular sequence. Then I^v is an unmixed ideal for all $v \geq 1$. From this it follows that

$$I^v = \cap_j (\wp_j)^v.$$

Hence,

$$R(I) = \oplus_{v \geq 0} I^v t^v = \cap_j \big(\oplus_{v \geq 0} (\wp_j)^v \big).$$

Note that $\oplus_{v \geq 0} (\wp_j)^v$ is the Rees algebra $R(\wp_j)$ of $\wp_j$. This Rees algebra is normal because $\wp_j$ is generated by linear forms. Therefore, as an intersection of normal rings, $R(I)$ must be normal, too. By Proposition 3.1, this implies that $R(I)_\Delta$ is also normal. So we obtain

Theorem 3.3. [26] *Let I be the defining prime ideal of a collection of points in $\mathbb{P}^{n-1}$. Assume that I is generated by a regular sequence. Then $R(I)$ and $R(I)_\Delta$ are normal.*

4. Local Cohomology of $R(I)_\Delta$

Let $S = k[X, Y]$ with the bigrading $\deg x_i = (1, 0)$ and $\deg y_j = (d_j, 1)$. Since $R(I)$ can be represented as a bigraded quotient ring of S, we will consider, instead of $R(I)$, an arbitrary finitely generated bigraded S-module L.

Let E be the additive monoid generated by the tuples $(a_1, \dots, a_n, b_1, \dots, b_r) \in \mathbb{N}^{n+r}$ which are solutions of a system of equations

$$a_1 + \cdots + a_n + b_1 d_1 + \cdots + b_r d_r = ci,$$
$$b_1 + \cdots + b_r = ei,$$

where i can be any non-negative integer.

Let $k[E]$ denote the semigroup ring of E over k. Then

$$S_\Delta \cong k[E].$$

It is easily seen that E is normal, that is, if $m(a_1, \ldots, a_n, b_1, \ldots, b_r) \in E$ for some integer $m > 0$, then $(a_1, \ldots, a_n, b_1, \ldots, b_r) \in E$. Hochster [18] showed that in this case, $k[E]$ is a Cohen–Macaulay ring. Moreover, we can compute the canonical module $\omega_{k[E]}$ of $k[E]$ in terms of E [28]. It follows that S_Δ is a Cohen–,Macaulay ring with

$$\omega_{S_\Delta} = (\omega_S)_\Delta.$$

Hence, there is a canonical homomorphism from $\mathrm{Hom}_S(L, \omega_S)_\Delta$ to $\mathrm{Hom}_{S_\Delta}(L_\Delta, \omega_{S_\Delta})$.

By local duality [14] this canonical homomorphism induces the following link between the local cohomology module $H^{q+1}_{\mathfrak{m}_S}(L)$ of L (with respect to the maximal graded ideal $\mathfrak{m}_S$) and the local cohomology module $H^q_{\mathfrak{m}_{S_\Delta}}(L_\Delta)$ of L_Δ (with respect to the maximal graded ideal $\mathfrak{m}_{S_\Delta}$ of S_Δ).

Proposition 4.1. [7] *There is a canonical graded homomorphism*

$$\phi^q_L : H^q_{\mathfrak{m}_{S_\Delta}}(L_\Delta) \to H^{q+1}_{\mathfrak{m}_S}(L)_\Delta.$$

To obtain more information on the local cohomology modules of L_Δ, we have to consider a minimal free resolution of L:

$$0 \to D_l \to \cdots \to D_1 \to D_0 \to L.$$

Taking the Δ-diagonals we obtain a graded resolution of L_Δ:

$$0 \to (D_l)_\Delta \to \cdots \to (D_1)_\Delta \to (D_0)_\Delta \to L_\Delta.$$

Therefore, we may get information on the local cohomology modules of L_Δ from that of $(D_i)_\Delta$, $i = 1, \ldots, l$. Each module D_i is of the form $\oplus S(a, b)$, where $S(a, b)$ denotes the twisted module S with the degree shifted by (a, b). Since $(D_i)_\Delta = \oplus S(a, b)_\Delta$, we need to know the local cohomology modules of $S(a, b)_\Delta$. It turns out that the local cohomology modules of $S(a, b)_\Delta$ can be computed by means of the notion of Segre products of bigraded modules.

Let A and B be two bigraded algebras over a field k. Let M be a bigraded A-module and N a bigraded B-module. If we set

$$(M \otimes_k N)_{(u,v)} := \oplus_{(u_1+v_1, u_2+v_2)=(u,v)} M_{(u_1,v_1)} \otimes_k N_{(u_2,v_2)}$$

for all $(u, v) \in \mathbb{Z}^2$, then $M \otimes_k N$ is a bigraded module over the bigraded algebra $A \otimes_k B$.

Definition. *The Segre product $M \otimes_\Delta N$ of M and N is defined by*

$$M \otimes_\Delta N := (M \otimes_k N)_\Delta.$$

It is clear that $A \otimes_\Delta B$ is a graded algebra and $M \otimes_\Delta N$ is a graded $A \otimes_\Delta B$-module.

This Segre product of bigraded modules is a natural extension of the usual Segre product of (simply) graded modules.

Example. If A and B are two graded algebras, we may view A and B as bigraded algebras by setting

$$A_{(u,0)} = A_u,\ A_{(u,v)} = 0 \text{ if } v > 0,$$
$$B_{(0,v)} = B_v,\ B_{(u,v)} = 0 \text{ if } u > 0.$$

In this case, if we take Δ to be the $(1,1)$-digagonal of $\mathbb{Z}^2$, then $A \otimes_\Delta B$ is the usual Segre product $M \underline{\otimes} N$ of A and B which is defined by

$$A \underline{\otimes} B := \oplus_{i \in \mathbb{N}} A_i \otimes_k B_i.$$

Goto and Watanabe [13], Stückrad and Vogel [27] already computed the local cohomology modules of the Segre products of modules over graded algebras. Following their approaches we can describe the local cohomology modules of $M \otimes_\Delta N$ by means of those of M and N. For this we will compute the transforms of $M \otimes_\Delta N$ with respect to the maximal graded ideal of $A \otimes_\Delta B$ which are defined as follows (see, e.g., [3]).

Given an ideal $\mathfrak{m}$ of a ring T and a T-module L, we call the module

$$D^q_{\mathfrak{m}}(L) := \lim \operatorname{Ext}^q_T(\mathfrak{m}^i, L)^q$$

the qth $\mathfrak{m}$-transform of L, $q \geq 0$. The $\mathfrak{m}$-transforms of L are related to the local cohomology modules $H^q_{\mathfrak{m}}(L)$ of L with respect to $\mathfrak{m}$ by the exact sequence

$$0 \to H^0_{\mathfrak{m}}(L) \to L \to D^0_{\mathfrak{m}}(L) \to H^1_{\mathfrak{m}}(L) \to 0$$

and the isomorphisms

$$D^q_{\mathfrak{m}}(L) \cong H^{q+1}_{\mathfrak{m}}(L),\ q \geq 1.$$

We have the following formula for the transforms of the Segre product of bigraded modules.

Theorem 4.2. [7] *Let $\mathfrak{m}_{A\otimes_\Delta B}$, $\mathfrak{m}_A$, and $\mathfrak{m}_B$ denote the maximal graded ideals of $A \otimes_\Delta B$, A, and B, respectively. Then*

$$D^q_{\mathfrak{m}_{A\otimes_\Delta B}}(M \otimes_\Delta N) = \bigoplus_{i+j=q} D^i_{\mathfrak{m}_A}(M) \otimes_\Delta D^j_{\mathfrak{m}_B}(N),\ q \geq 0.$$

Now let $A = k[X]$ and $B = k[Y]$ with the bigrading $\deg x_i = (0,1)$ and $\deg y_j = (d_j, 1)$. Then $S = A \otimes_k B$. Hence,

$$S(a,b)_\Delta = A(a,b) \otimes_\Delta B.$$

Since A and B are polynomial rings, we know the local cohomology modules of A and B (which are all zero except $H^n_{\mathfrak{m}_A}(A)$ and $H^r_{\mathfrak{m}_B}(B)$). Using the above theorem we can describe the local cohomology modules of $S(a,b)$ in terms a,b,c,e. From this description we can deduce the following results on the local cohomology modules of L.

Theorem 4.3. [7] *Assume that* $c \geq de+1$. *Then*
(i) ϕ^q_L *is an isomorphism for* $q < n$.
(ii) *There is a positive integer* s_0 *such that* ϕ^q_L *induces an isomorphism*

$$[H^q_{\mathfrak{m}_{S_\Delta}}(L_\Delta)]_s \cong [H^{q+1}_{\mathfrak{m}_S}(L)_\Delta]_s$$

for $|s| \geq s_0$ *and* $q \geq n$.

Theorem 4.4. [7] *Assume that* $\dim L_\Delta = \dim L - 1$ *for* $c \gg e \gg 0$. *Then the following conditions are equivalent:*
(i) L_Δ *is a Buchsbaum module with* $H^q_{\mathfrak{m}_{S_\Delta}}(L_\Delta)_s = 0$ *for* $s \neq 0$, $q < \dim L - 1$ *if* $c \gg e \gg 0$.
(ii) $H^q_{\mathfrak{m}_S}(L)_{(-u,-v)} = 0$ *for* $u \gg v \gg 0$, $q < \dim L$.

5. Cohen–Macaulayness of $R(I)_\Delta$

It is usually hard to determine when the diagonal of a bigraded algebra is Cohen–Macaulay, even in the case of Segre products of graded algebras [13, 27].

If we happen to know a free resolution of the given bigraded algebra, then we can use the knowledge on the local cohomology modules of the twisted modules of the resolution to find out which diagonal of the bigraded algebra is Cohen–Macaulay.

Let us consider the case $I = (f_1, \ldots, f_r)$, where $f_1, \ldots, f_r$ is a regular sequence of homogeneous forms in $k[X]$. In this case, we have the following presentation:

$$R(I) \cong S/I_2(\mathcal{M}),$$

where $S = k[X,Y]$ and $\mathcal{M}$ is the matrix

$$\begin{pmatrix} f_1 & \cdots & f_r \\ y_1 & \cdots & y_r \end{pmatrix}$$

Therefore, the Eagon–Northcott complex gives a minimal free resolution for $R(I)$:

$$0 \to D_{r-1} \to \cdots \to D_1 \to D_0 = S \to R(I),$$

where

$$D_p = \bigoplus_{m=1}^{p} \bigoplus_{1 \leq j_1 \leq \cdots \leq j_{p+1} \leq r} S(-(d_{j_1} + \cdots + d_{j_{p+1}}), -m),\ p = 1, \ldots, r-1.$$

Using the information on the shifts of the twisted modules of D_p, we can compute the local cohomology modules of $(D_p)_\Delta$. This led to the following result which generalizes Theorem 1.1(iii):

Theorem 5.1. [7] *Let $I = (f_1, \dots, f_r) \subset k[x_1, \dots, x_n]$, where $f_1, \dots, f_r$ is a regular sequence of homogeneous elements of degree $d_1, \dots, d_r$. Let $d = \max_j d_j$ and assume that $c \geq de + 1$. Then $R(I)_\Delta$ is a Cohen–Macaulay ring if and only if*

$$c > \sum_{j=1}^{r} d_j + (e-1)d - n.$$

Example. Let I be as above with $n = 3$, $r = 2$. Assume that $d_1 \leq d_2$. By the above theorem, $R(I)_\Delta$ is Cohen-Macaulay if and only if $c > d_1 + ed_2 - 3$. That is exactly Theorem 1.1(iii).

Another approach to the Cohen–Macaulayness of the diagonal subalgebras is the technique of filtration.

Let S be an arbitrary bigraded algebra. Let $\mathcal{F}$ be a filtration of bigraded ideals of S. Then $\mathrm{gr}_\mathcal{F}(S)$ also has a bigraded structure and therefore the diagonal subalgebra $\mathrm{gr}_\mathcal{F}(S)_\Delta$. Let $\mathcal{F}_\Delta$ be the filtration of S_Δ which consists of the Δ-diagonals of the ideals of $\mathcal{F}$. Then we can show that

$$\mathrm{gr}_{\mathcal{F}_\Delta}(S_\Delta) \cong \mathrm{gr}_\mathcal{F}(S)_\Delta.$$

As a consequence, if $\mathrm{gr}_\mathcal{F}(S)_\Delta$ is Cohen–Macaulay, then so is S_Δ.

This technique can be applied to study the Cohen–Macaulayness of $R(I)_\Delta$ when I is generated by a d-sequence or when I is a straightening closed ideal. For more information on these notions, we refer to [4, 5, 20].

Let $I = (f_1, \dots, f_r) \subset k[X]$ be generated by forms of the same degree, say d. Then $R(I)$ can be bigraded by $R(I)_{(u,v)} = (I^v)_{vd+u}t^v$. Consider a presentation

$$R(I) = k[X, Y]/J.$$

Then the lexicographic term order on $k[X, Y]$ induces a filtration $\mathcal{F}$ of bigraded ideals of $R(I)$ with

$$\mathrm{gr}_\mathcal{F}(R(I)) \cong k[X, Y]/J^*,$$

where J^* denotes the ideal generated by the initial forms of the elements of J. Hence

$$\mathrm{gr}_\mathcal{F}(R(I))_\Delta = k[X, Y]_\Delta/(J^*)_\Delta.$$

If $f_1, \dots, f_r$ is a d-sequence or if $k[X]$ is a polynomial ring with straightening law on a finite poset Π and $\Omega = \{f_1, \dots, f_r\} \subset \Pi$ is a straightening closed poset ideal, where $f_1, \dots, f_r$ is a linearization of Ω, then we can compute J^* [16, 25]. It turns out that J^* is the intersection of ideals Q_j such that each quotient ring $k[X, Y]/Q_j$ is the tensor product of two well-determined algebras. Therefore,

$(J^*)_\Delta$ is the intersection of the ideals $(Q_j)_\Delta$ and $k[X,Y]_\Delta/(Q_j)_\Delta$ is the Segre product of these algebras. Under some mild conditions, the Segre products of these algebras are Cohen–Macaulay so that we can deduce that $k[X,Y]_\Delta/(J^*)_\Delta$ and therefore, $R(I)_\Delta$ are Cohen–Macaulay rings. For instance, we can prove the following result:

Theorem 5.2. [26] *Let $k[X]$ be a polynomial ring with straightening law on an upper semimodular semilattice Π. Let $I = \Omega k[X]$, where Ω is a straightening closed ideal in Π of homogeneous elements of the same degree such that* $\operatorname{rank} \Pi \setminus \Omega \geq 2$. *Then the $(1,1)$-diagonal algebra of $R(I)$ is Cohen–Macaulay.*

This theorem is interesting because it contains the case when I is the ideal generated by the maximal minors of a generic matrix.

The Cohen–Macaulayness of $R(I)_\Delta$ has been further studied by other authors. Hyry [21] gave a sufficient condition for the Cohen–Macaulayness of the $(1,1)$-diagonal of a Cohen–Macaulay standard bigraded algebra in terms of the defining equations. Cutkosky and Herzog [8] studied conditions under which there is a positive integer t such that $R(I)_\Delta$ is Cohen-Macaulay for $c \geq te$. Lavila-Vidal [22] proved that if $R(I)$ is Cohen–Macaulay, then there exist Cohen–Macaulay diagonals $R(I)_\Delta$. But the culminate point was the following recent result of hers:

Theorem 5.3. [23] *Let $V = \operatorname{Biproj} R(I)$. There exists a Cohen–Macaulay diagonal $R(I)_\Delta$ if and only if the folllowing conditions are satisfied:*
(i) *V is locally Cohen–Macaulay.*
(ii) *$\Gamma(V, \mathcal{O}_V) = k$ and $H^i(V, \mathcal{O}_V) = 0$ for $i \geq 1$.*

Moreover, Lavila-Vidal and Zarzuela [24] also studied the Gorenstein property of $R(I)_\Delta$.

6. Koszul Property of $R(I)_\Delta$

In this section we will study the Koszul property of the diagonal subalgebra.

Let A be a positively graded algebra over a field k. Let M be a finitely generated graded A-module. Set

$$t_i(M) := \sup\{j|\ \operatorname{Tor}_i^A(L,k)_j \neq 0\}$$

with $t_i(M) = -\infty$ if $\operatorname{Tor}_i^A(L,k)_j = 0$. We call

$$\operatorname{reg} M := \sup\{t_i(M) - i|\ i \geq 0\}$$

the Castelnuovo–Mumford regularity of M, and M is said to have a linear A-resolution if $\operatorname{reg} M$ is equal the least non-vanishing degree of M. If k has a linear A-resolution, we call A a Koszul algebra [2].

A Koszul algebra is always defined by quadrics. Hence we will first study the problem when the diagonal algebra is defined by quadrics.

Let T be a standard bigraded algebra over k, that is, T is generated over k by the elements of $T_{(1,0)}$ and $T_{(0,1)}$. Let $T = S/J$ be a presentation of T, where $S = k[X,Y]$ is a bigraded polynomial ring in two sets of variables $X = \{x_i\}$ and $Y = \{y_j\}$ with $\deg x_i = (1,0)$ and $\deg y_j = (0,1)$. It is easy to see that S_Δ can be represented as the factor ring of a polynomial ring $k[T]$ by an ideal generated by quadrics and we can estimate the degree of the minimal generators of the kernel of the natural map $S_\Delta \to T_\Delta$. As a consequence of this estimation we obtain the following result.

Proposition 6.1. [7] *Let $\oplus S(-a_j,-b_j) \to S \to T$ be a finite free presentation of T as an S-module. Then T_Δ is defined by quadrics if*

$$c \geq \frac{1}{2}\max_j a_j,$$
$$e \geq \frac{1}{2}\max_j b_j.$$

To study the Koszul property of T_Δ we need to consider a finite free presentation of T as an S-module:

$$0 \to F_s \to \cdots \to F_1 \to F_0 = S \to T.$$

It can be shown that T_Δ is a Koszul algebra if

$$\operatorname{reg}(F_i)_\Delta \leq i+1$$

for $i = 1,\dots,s$. To estimate $\operatorname{reg}(F_i)_\Delta$ we note that F_i is a direct sum of modules of the form $S(a,b)$. The diagonal $S(a,b)_\Delta$ of each of the twisted module $S(a,b)$ can be written as the Segre product of twisted modules over the graded algebras $k[X]$ and $k[Y]$. The Castelnuovo–Mumford regularity of such a Segre product can be computed and we get

$$\operatorname{reg} S(a,b)_\Delta = \max\left\{\lceil\frac{a}{c}\rceil, \lceil\frac{b}{e}\rceil\right\},$$

where $\lceil p \rceil$ denotes the least integer $\geq p$. Therefore, we can compute $\operatorname{reg}(F_i)_\Delta$ in terms of the shifting bidegrees of the twisted modules of F_i. Summing up we obtain the following criterion for the Koszul property of the diagonal subalgebras.

Theorem 6.2. [7] *T_Δ is a Koszul algebra if, for $i = 1,\dots,s$,*

$$\max\left\{\lceil\frac{a}{c}\rceil, \lceil\frac{b}{e}\rceil\right\} \leq i+1,$$

where (a,b) runs over all shifting bidgrees of the twisted modules of F_i.

Corollary 6.3. *T_Δ is a Koszul algebra for c and e large enough.*

Since we know the minimal free resolution of the Rees algebra $R(I)$ of a homogeneous ideal I when I is generated by a regular sequence (see Sec. 5), we can deduce from Proposition 6.1 and Theorem 6.2 the following result. Note that $R(I)$ can be made a standard bigraded algebra if I is generated by forms of the same degree.

Corollary 6.4. *Let I be an ideal generated by a regular sequence of r forms of the same degree d. Then*
(i) *$R(I)_\Delta$ is defined by quadrics if $c \geq d/2 + de$.*
(ii) *$R(I)_\Delta$ is a Koszul algebra if $c \geq d(r-1)/r + de$.*

We believe that the bound given in (ii) can be improved and we have raised the following problem.

Problem. *Is $R(I)_\Delta$ a Koszul algebra if $c \geq d/2 + de$?*

It was proved by Backelin [1] that the Veronese subrings of a Koszul algebra are all Koszul algebras. Since any bigraded algebra can be made a (simply) graded algebra by considering the total degree, we have also raised the following problem.

Problem. *Let T be a bigraded Koszul algebra. Are all diagonal subalgebras of T Koszul?*

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