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Stability of the Karush - Kuhn - Tucker Point Set in a General Quadratic Programming Problem

Nguyen Nang Tam[1] **and Nguyen Dong Yen**[2]

[1] *Department of Mathematics, Hanoi Pedagogical Institute II Xuan Hoa, Me Linh, Vinh Phuc, Vietnam*

[2] *Hanoi Institute of Mathematics, P.O. Box 631, Bo Ho, Hanoi, Vietnam*

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Abstract. Necessary and sufficient conditions for the stability of the Karush - Kuhn - Tucker point set in a general indefinite quadratic programming problem are obtained in this paper.

1. Introduction

Given matrices $A \in \mathbb{R}^{m\times n}$, $F \in \mathbb{R}^{s\times n}$, $D \in \mathbb{R}^{n\times n}$, with D being symmetric, and vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $d \in \mathbb{R}^s$, we consider the following general indefinite quadratic programming (QP for brevity) problem $QP(D, A, c, b, F, d)$:

$$\begin{cases} \text{Minimize } f(x) := \frac{1}{2}x^T Dx + c^T x \\ \text{subject to } x \in \mathbb{R}^n,\ Ax \geq b,\ Fx \geq d \end{cases} \tag{1.1}$$

Here the superscript ^{T} denotes transposition. In what follows the pair (F, d) is not subject to change. So the set $\Delta(F, d) := \{x \in \mathbb{R}^n : Fx \geq d\}$ is fixed. Define $\Delta(A, b) = \{x \in \mathbb{R}^n : Ax \geq b\}$ and recall that $\bar{x} \in \Delta(A, b) \cap \Delta(F, d)$ is said to be a *Karush - Kuhn - Tucker point* of $QP(D, A, c, b, F, d)$ if there exists a pair of Lagrange multipliers $(\bar{u}, \bar{v}) \in \mathbb{R}^m \times \mathbb{R}^s$ such that

$$\begin{aligned} &D\bar{x} - A^T\bar{u} - F^T\bar{v} + c = 0, \\ &A\bar{x} \geq b, \quad \bar{u} \geq 0, \\ &F\bar{x} \geq d, \quad \bar{v} \geq 0, \\ &\bar{u}^T(A\bar{x} - b) + \bar{v}^T(F\bar{x} - d) = 0. \end{aligned}$$

The set of the Karush - Kuhn - Tucker points and the set of the solutions of problem $QP(D, A, c, b, F, d)$ are denoted, respectively, by $S(D, A, c, b, F, d)$ and sol (D, A, c, b, F, d). It is well known [5] that sol $(D, A, c, b, F, d) \subseteq S(D, A, c, b, F, d)$ and, moreover, every local solution of (1.1) is a Karush - Kuhn - Tucker point.

If $s = n$, $d = 0$, and F is the unit matrix in $\mathbb{R}^{n\times n}$, then problem (1.1) has the following *canonical form:*

$$\begin{cases} \text{Minimize } f(x) := \frac{1}{2}x^T Dx + c^T x \\ \text{subject to } x \in \mathbb{R}^n,\ Ax \geq b,\ x \geq 0 \end{cases} \tag{1.2}$$

For simplicity of notation, in the case of the canonical problem, we write $S(D, A, c, b)$ instead of $S(D, A, c, b, F, d)$, and sol (D, A, c, b) instead of sol (D, A, c, b, F, d). The upper semicontinuity of the multifunction

$$p' \mapsto S(p'),\ p' = (D', A', c', b') \in \mathbb{R}_0^{n\times n} \times \mathbb{R}^{m\times n} \times \mathbb{R}^n \times \mathbb{R}^m, \tag{1.3}$$

where $\mathbb{R}_0^{n\times n} \subset \mathbb{R}^{n\times n}$ denotes the subspace of all the symmetric matrices of order n, has been studied in [10] and [12]. This property can be interpreted as the *stability* of the Karush - Kuhn - Tucker point set $S(D, A, c, b)$ with respect to the change in the problem parameters. In this paper we are interested in finding out how the results in [10] and [12] can be extended to the case of problem (1.1). Namely, we wish to obtain some necessary and sufficient conditions for the upper semicontinuity of the multifunction

$$p' \mapsto S(p', F, d),\ p' = (D', A', c', b') \in \mathbb{R}_0^{n\times n} \times \mathbb{R}^{m\times n} \times \mathbb{R}^n \times \mathbb{R}^m, \tag{1.4}$$

which include the corresponding results of [10] and [12] as a special case. As in the canonical problem, the obtained results can be interpreted as the necessary and sufficient conditions for the *stability* of the Karush-Kuhn-Tucker point set $S(D, A, c, b, F, d)$ with respect to the change in the problem parameters.

Our proofs are based on several observations concerning the system of equalities and inequalities defining the Karush - Kuhn - Tucker point set. We wish to stress that the proofs in [10] and [12] cannot be applied directly to the case of problem (1.1). This is because, unlike the case of the canonical problem (1.2), $\Delta(F, d)$ may fail to be a cone with nonempty interior and the vertex 0. In order to deal with the general problem (1.1) we have to use some new arguments. However, the proof schemes proposed in [12] and [10] also work for the case of problem (1.1).

The paper is organized as follows. In Sec. 2 we will establish two necessary conditions for the u.s.c. property of the multifunction (1.4). Theorem 2.1 can be used for the case where $\Delta(F, d)$ is a polyhedral cone with a vertex x_0, where $x_0 \in \mathbb{R}^n$ is an arbitrarily given vector. Theorem 2.2 works for the case where $\Delta(F, d)$ is an arbitrary polyhedral set, but the conclusion is weaker than that of Theorem 2.1. Several sufficient conditions for the upper semicontinuity of the multifunction (1.4) are given in Sec. 3. The obtained results are then compared with the corresponding ones in [10], and two illustrative examples with non-convex QP problems are considered.

The reader is referred to [12] for a detailed review on the research on stability of quadratic programs. To our knowledge, up to now [1] - [7] and [9] are among the key references in the field. Some recent results on continuity properties of the solution map and the local-solution map in indefinite quadratic programming can be found in [8] and [11].

Now we explain some notations which will be used throughout the paper.

For any positive integer r, $\mathbb{R}^r$ is equipped with the Euclidean norm $\|\cdot\|$; vectors in $\mathbb{R}^r$ are understood as columns of r real numbers. The norms in the space of matrices $\mathbb{R}^{r\times n}$ and in the product space $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^s$, respectively, are defined by the following formulas:

$$\|M\| = \max\{\|Mx\| : x \in \mathbb{R}^n,\ \|x\| = 1\},$$
$$\|(x,u,v)\| = \|x\| + \|u\| + \|v\| \quad \forall\,(x,u,v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^s.$$

For any $M \in \mathbb{R}^{r\times n}$ and $q \in \mathbb{R}^r$, the set $\{x \in \mathbb{R}^n : Mx \geq q\}$ is denoted by $\Delta(M,q)$. For $F \in \mathbb{R}^{s\times n}$ and $A \in \mathbb{R}^{m\times n}$, we abbreviate the set

$$\{(u,v) \in \mathbb{R}^m \times \mathbb{R}^s : A^T u + F^T v = 0,\ u \geq 0,\ v \geq 0\}$$

to $\Lambda[A,F]$, and the set

$$\{(\xi,\eta) \in \mathbb{R}^m \times \mathbb{R}^s : \xi^T u + \eta^T v < 0\ \ \forall\,(u,v) \in \Lambda[A,F] \setminus \{(0,0)\}\}$$

to $\operatorname{int}(\Lambda[A,F])^*$. (The second set is nothing but the interior of the dual cone to the first one.)

2. Necessary Conditions for the Stability

Two sufficient conditions for the upper semicontinuity of the multifunction (1.4) will be obtained in this section.

Definition 2.1. *Let $p = (D,A,c,b) \in \mathbb{R}_0^{n\times n} \times \mathbb{R}^{m\times n} \times \mathbb{R}^n \times \mathbb{R}^m$. The multifunction (1.4) is said to be upper semicontinuous (u.s.c. for short) at p if, for any open subset $\Omega \subset \mathbb{R}^n$ containing $S(p,F,d)$, there exists $\delta > 0$ such that $S(p',F,d) \subset \Omega$ for every $p' = (D',A',c',b') \in \mathbb{R}_0^{n\times n} \times \mathbb{R}^{m\times n} \times \mathbb{R}^n \times \mathbb{R}^m$ satisfying*

$$\max\{\|D'-D\|, \|A'-A\|, \|c'-c\|, \|b'-b\|\} < \delta.$$

The following two remarks clarify some points in the assumption and conclusion of Theorem 2.1 below.

Remark 1. If there is a point $x_0 \in \mathbb{R}^n$ such that $Fx_0 = d$. Then $\Delta(F,d) = x_0 + \Delta(F,0)$, hence $\Delta(F,d)$ is a polyhedral convex cone with the vertex x_0. Conversely, for any $x_0 \in \mathbb{R}^n$ and any polyhedral cone K with the vertex 0, there exists a positive integer s and a matrix $F \in \mathbb{R}^{s\times n}$ such that $x_0 + K = \Delta(F,d)$, where $d := Fx_0$.

Remark 2. If $\Delta(F,d)$ and $\Delta(A,b)$ are nonempty, then $\Delta(F,0)$ and $\Delta(A,0)$, respectively, are the recession cones of $\Delta(F,d)$ and $\Delta(A,b)$. By definition,

$S(D, A, 0, 0, F, 0)$ is the Karush-Kuhn-Tucker point set of the following QP problem:

$$\text{minimize } x^T Dx \text{ subject to } x \in \mathbb{R}^n, \ Ax \geq 0, \ Fx \geq 0,$$

whose constraint set is the intersection $\Delta(A, 0) \cap \Delta(F, 0)$.

Theorem 2.1. *Assume that the set $S(p, F, d)$, $p = (D, A, c, b)$, is bounded and there exists $x_0 \in \mathbb{R}^n$ such that $Fx_0 = d$. If the multifunction (1.4) is upper semicontinuous at p, then*

$$S(D, A, 0, 0, F, 0) = \{0\}. \tag{2.1}$$

Proof. This proof follows a scheme given in the proof of Theorem 2.3 in [12]. Suppose, contrary to our claim, that there is a non-zero vector $\bar{x} \in S(D, A, 0, 0, F, 0)$. By definition, there exists a pair $(\bar{u}, \bar{v}) \in \mathbb{R}^m \times \mathbb{R}^s$ such that

$$D\bar{x} - A^T\bar{u} - F^T\bar{v} = 0, \tag{2.2}$$

$$A\bar{x} \geq 0, \quad \bar{u} \geq 0, \tag{2.3}$$

$$F\bar{x} \geq 0, \quad \bar{v} \geq 0, \tag{2.4}$$

$$\bar{u}^T A\bar{x} + \bar{v}^T F\bar{x} = 0. \tag{2.5}$$

For every $t \in (0, 1)$, we set

$$x_t = x_0 + \frac{1}{t}\bar{x}, \quad u_t = \frac{1}{t}\bar{u}, \quad v_t = \frac{1}{t}\bar{v}, \tag{2.6}$$

where x_0 is given by our assumptions. We claim that there exist matrices $D_t \in \mathbb{R}_0^{n\times n}$, $A_t \in \mathbb{R}^{m\times n}$ and vectors $c_t \in \mathbb{R}^n$, $b_t \in \mathbb{R}^m$ such that

$$\max\{||D_t - D||, ||A_t - A||, ||c_t - c||, ||b_t - b||\} \to 0 \quad \text{as } t \to 0,$$

and

$$D_t x_t - A_t^T u_t - F^T v_t + c_t = 0, \tag{2.7}$$

$$A_t x_t \geq b_t, \quad u_t \geq 0, \tag{2.8}$$

$$Fx_t \geq d, \quad v_t \geq 0, \tag{2.9}$$

$$u_t^T(A_t x_t - b_t) + v_t^T(Fx_t - d) = 0. \tag{2.10}$$

The matrices D_t, A_t and the vectors c_t, b_t will have the following representations

$$D_t = D + tD_0, \quad A_t = A + tA_0 \tag{2.11}$$

$$c_t = c + tc_0, \quad b_t = b + tb_0, \tag{2.12}$$

where the matrices D_0, A_0 and the vectors c_0, b_0 are to be constructed. First we observe that, due to (2.4) and (2.6), (2.9) holds automatically. Clearly,

$$\begin{aligned} A_t x_t - b_t &= (A + tA_0)\left(x_0 + \frac{\bar{x}}{t}\right) - (b + tb_0) \\ &= t(A_0 x_0 - b_0) + \frac{1}{t}A\bar{x} + A_0\bar{x} + Ax_0 - b, \end{aligned}$$

and

$$
\begin{aligned}
&u_t^T(A_t x_t - b_t) + v_t^T(F x_t - d) \\
&= \frac{\bar{u}^T}{t}\left[t(A_0 x_0 - b_0) + \frac{1}{t}A\bar{x} + A_0\bar{x} + A x_0 - b\right] + \frac{\bar{v}^T}{t}\left[F\left(x_0 + \frac{\bar{x}}{t}\right) - d\right] \\
&= \bar{u}^T(A_0 x_0 - b_0) + \frac{1}{t^2}(\bar{u}^T A\bar{x} + \bar{v}^T F\bar{x}) + \frac{\bar{u}^T}{t}(A_0\bar{x} + A x_0 - b).
\end{aligned}
$$

So, by (2.3) and (2.5), if we have

$$A_0\bar{x} + A x_0 - b = 0 \tag{2.13}$$

and

$$A_0 x_0 - b_0 = 0, \tag{2.14}$$

then (2.8) and (2.10) will be fulfilled. By (2.2),

$$
\begin{aligned}
&D_t x_t - A_t^T u_t - F^T v_t + c_t \\
&\quad = (D + tD_0)\left(x_0 + \frac{\bar{x}}{t}\right) - (A + tA_0)^T\frac{\bar{u}}{t} - F^T\frac{\bar{v}}{t} + c + tc_0 \\
&\quad = \frac{1}{t}(D\bar{x} - A^T\bar{u} - F^T\bar{v}) + t(D_0 x_0 + c_0) + D x_0 + D_0\bar{x} - A_0^T\bar{u} + c, \\
&\quad = t(D_0 x_0 + c_0) + D x_0 + D_0\bar{x} - A_0^T\bar{u} + c.
\end{aligned}
$$

Therefore, if we have

$$D x_0 + D_0\bar{x} - A_0^T\bar{u} + c = 0 \tag{2.15}$$

and

$$D_0 x_0 + c_0 = 0, \tag{2.16}$$

then (2.7) will be fulfilled.

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T$, where $\bar{x}^i \neq 0$ for $i \in I$ and $\bar{x}_i = 0$ for $i \notin I$, $I \subseteq \{1, \ldots, n\}$. Since $\bar{x} \neq 0$, I is nonempty. Fixing an index $i_0 \in I$, we define A_0 as the $m \times n$ matrix in which the i_0th column is $\bar{x}_{i_0}^{-1}(b - A x_0)$, and the other columns consist solely of zeros. Let $b_0 = A_0 x_0$. One can verify immediately that (2.13) and (2.14) are satisfied; hence conditions (2.8) and (2.10) are fulfilled. From what has been said it follows that our claim will be proved if we can construct a matrix $D_0 \in \mathbb{R}_0^{n\times n}$ and a vector c_0 satisfying (2.15) and (2.16). Let $D_0 = (d_{ij})$, where d_{ij} $(1 \leq i, j \leq n)$ are defined by the following formulas:

$$
\begin{aligned}
d_{ii} &= \bar{x}_i^{-1}\left(A_0^T\bar{u} - Dx_0 - c\right)_i \quad \forall i \in I, \\
d_{i_0 j} = d_{j i_0} &= \bar{x}_{i_0}^{-1}\left(A_0^T\bar{u} - Dx_0 - c\right)_j \quad \forall j \in \{1, \ldots, n\} \setminus I,
\end{aligned}
$$

and $d_{ij} = 0$ for other pairs (i, j), $1 \leq i, j \leq n$. Here $(A_0^T\bar{u} - Dx_0 - c)_k$ denotes the kth component of the vector $A_0^T\bar{u} - Dx_0 - c$. It is clear that D_0 is a symmetric matrix, hence $D_0 \in \mathbb{R}_0^{n\times n}$. If we define $c_0 = -D_0x_0$ then (2.16) is satisfied. A direct computation shows that (2.15) is also satisfied.

We have thus constructed matrices D_0, A_0 and vectors c_0, b_0 such that for x_t, u_t, v_t, D_t, A_t, c_t, b_t defined by (2.6), (2.11) and (2.12), conditions (2.7)-(2.10) are satisfied. Consequently, $x_t \in S(D_t, A_t, c_t, b_t, F, d)$. Since $S(p, F, d)$ is bounded, there is a bounded open set $\Omega \subset \mathbb{R}^n$ such that $S(p, F, d) \subset \Omega$. Since $\max\{\|D_t - D\|, \|A_t - A\|, \|c_t - c\|, \|b_t - b\|\} \to 0$ as $t \to 0$ and the multifunction $p' \mapsto S(p', F, d)$ is u.s.c. at $p = (D, A, c, b)$, $x_t \in \Omega$ for all sufficiently small t. This is impossible because $\|x_t\| = \|x_0 + \bar{x}/t\| \to \infty$ as $t \to 0$. The proof is complete. ∎

Remark 3. If $d = 0$, then $\Delta(F, d)$ is a cone with the vertex 0. In order to verify the assumptions of Theorem 2.1, one can choose $x_0 = 0$. In particular, this is the case of the canonical problem (1.2). Applying Theorem 2.1 we obtain the following necessary condition for the upper semicontinuity of the multifunction (1.3): *If $S(p)$, $p = (D, A, c, b)$ is bounded and if the multifunction $p' \mapsto S(p')$, $p' = (D', A', c', b')$, is u.s.c. at p, then $S(D, A, 0, 0) = \{0\}$.* Thus, Theorem 2.1 above extends Theorem 2.3 in [12] to the case where $\Delta(F, d)$ can be any polyhedral cone in $\mathbb{R}^n$, merely the standard cone $\mathbb{R}^n_+$.

In the sequel, $S(D, A)$ denotes the set of all $x \in \mathbb{R}^n$ such that there exists $u = u(x) \in \mathbb{R}^m$ satisfying the following system:

$$\begin{aligned} &Dx - A^T u = 0,\\ &Ax \geq 0, \quad u \geq 0,\\ &u^T Ax = 0. \end{aligned}$$

Remark 4. From the definition it follows that $S(D, A) = S(D, A, 0, 0, F, 0)$, where $s = n$ and $F = 0 \in \mathbb{R}^{n\times n}$.

Theorem 2.2. *Assume that $\Delta(F, d)$ is non-empty and $S(p, F, d)$, $p = (D, A, c, b)$, is bounded. If the multifunction (1.4) is upper semicontinuous at p, then*

$$S(D, A) \cap \Delta(F, 0) = \{0\}. \tag{2.17}$$

Remark 5. Observe that (2.1) implies (2.17). Indeed, suppose (2.1) holds. The fact that $0 \in S(D, A) \cap \Delta(F, 0)$ is obvious. So, if (2.17) does not hold, then there exists $\bar{x} \in S(D, A) \cap \Delta(F, 0)$, $\bar{x} \neq 0$. Taking $\bar{u} = u(\bar{x})$, $\bar{v} = 0 \in \mathbb{R}^s$, we see at once that the system (2.2) - (2.5) is satisfied. This means that $\bar{x} \in S(D, A, 0, 0, F, 0) \setminus \{0\}$, contrary to (2.1). Note that, in general, (2.17) does not imply (2.1).

Remark 6. If there exists x_0 such that $Fx_0 = d$. Then $x_0 \in \Delta(F, d) = \{x \in \mathbb{R}^n : Fx \geq d\}$. In particular, $\Delta(F, d) \neq \emptyset$. Thus, Theorem 2.2 can be applied to a larger class of problems than Theorem 2.1. However, Remark 5 shows that the conclusion of Theorem 2.2 is weaker than that of Theorem 2.1. One question still unanswered is whether the assumptions of Theorem 2.2 always imply (2.1).

Proof of Theorem 2.2. Assume that $\Delta(F,d)$ is non-empty, $S(D,A,c,b,F,d)$ is bounded and the multifunction $S(\cdot,F,d)$ is u.s.c. at p but (2.17) is violated. Then there is a non-zero vector $\bar{x} \in S(D,A) \cap \Delta(F,0)$. Hence, there exists $\bar{u} \in \mathbb{R}^m$ such that

$$D\bar{x} - A^T\bar{u} = 0, \tag{2.18}$$
$$A\bar{x} \geq 0, \quad \bar{u} \geq 0, \tag{2.19}$$
$$\bar{u}^T A\bar{x} = 0, \tag{2.20}$$
$$F\bar{x} \geq 0. \tag{2.21}$$

Let x_0 be an arbitrary point of $\Delta(F,d)$. Setting

$$x_t = x_0 + \frac{1}{t}\bar{x}, \quad u_t = \frac{1}{t}\bar{u}$$

for every $t \in (0,1)$, we claim that there exist matrices $D_t \in \mathbb{R}_0^{n\times n}$, $A_t \in \mathbb{R}^{m\times n}$ and vectors $c_t \in \mathbb{R}^n$, $b_t \in \mathbb{R}^m$ such that $\max\{\|D_t - D\|, \|A_t - A\|, \|c_t - c\|, \|b_t - b\|\} \to 0$ as $t \to 0$, and

$$\begin{aligned} &D_t x_t - A_t^T u_t - F^T 0 + c_t = 0,\\ &A_t x_t \geq b_t, \quad u_t \geq 0,\\ &F x_t \geq d,\\ &u_t^T(A_t x_t - b_t) + 0^T(F x_t - d) = 0. \end{aligned}$$

The matrices D_t, A_t and vectors c_t, b_t are defined by (2.11) and (2.12), where D_0, A_0, c_0, b_0 are constructed as in the proof of Theorem 2.1. Arguing similarly as in the preceding proof, we shall arrive at a contradiction. ■

3. Sufficient Conditions for the Stability

The following theorem gives three sufficient conditions for the upper semicontinuity of the multifunction (1.4). These conditions express some requirements on the behavior of the quadratic form $x^T Dx$ on the cone $\Delta(A,0) \cap \Delta(F,0)$ and the position of the vector (b,d) relative to the set $\operatorname{int}(\Lambda[A,F])^*$.

Theorem 3.1. *Suppose that one of the following three pairs of conditions*

$$\operatorname{sol}(D,A,0,0,F,0) = \{0\}, \quad (b,d) \in \operatorname{int}(\Lambda[A,F])^*, \tag{3.1}$$

$$\operatorname{sol}(-D,A,0,0,F,0) = \{0\}, \quad (b,d) \in -\operatorname{int}(\Lambda[A,F])^*, \tag{3.2}$$

and

$$S(D, A, 0, 0, F, 0) = \{0\}, \quad \text{int } (\Lambda[A, F])^* = \mathbb{R}^m \times \mathbb{R}^s, \tag{3.3}$$

is satisfied. Then, for any $c \in \mathbb{R}^n$ (and also for any $b \in \mathbb{R}^m$ if (3.3) *takes place), the multifunction $p' \mapsto S(p', F, d)$, $p' = (D', A', c', b')$, is upper semicontinuous at $p = (D, A, c, b)$.*

Proof. On the contrary, suppose that one of the three pairs of conditions **(3.1)**-(3.3) is satisfied but, for some $c \in \mathbb{R}^n$ (and also for some $b \in \mathbb{R}^m$ if **(3.3)** takes place), the multifunction $p' \mapsto S(p', F, d)$ is not u.s.c. at $p = (D, A, c, b)$. Then there exist an open subset $\Omega \subset \mathbb{R}^n$ containing $S(p, F, d)$, a sequence $p_k = (D_k, A_k, c_k, b_k)$ converging to p in $\mathbb{R}_0^{n \times n} \times \mathbb{R}^{m \times n} \times \mathbb{R}^n \times \mathbb{R}^m$, and a sequence $\{x_k\}$ such that, for each k, $x_k \in S(p_k, F, d)$ and $x_k \notin \Omega$. By the definition of Karush-Kuhn-Tucker point, for each k, there exists a pair of multipliers $(u_k, v_k) \in \mathbb{R}^m \times \mathbb{R}^s$ such that

$$D_k x_k - A_k^T u_k - F^T v_k + c_k = 0, \tag{3.4}$$

$$A_k x_k \geq b_k, \quad u_k \geq 0, \tag{3.5}$$

$$F x_k \geq d, \quad v_k \geq 0, \tag{3.6}$$

$$u_k^T (A_k x_k - b_k) + v_k^T (F x_k - d) = 0. \tag{3.7}$$

If the sequence $\{(x_k, u_k, v_k)\}$ is bounded, then the sequences $\{x_k\}$, $\{u_k\}$, $\{v_k\}$ are also bounded. Therefore, without loss of generality, we can assume that the sequences $\{x_k\}$, $\{u_k\}$ and $\{v_k\}$ converge, respectively, to some points $x_0 \in \mathbb{R}^n$, $u_0 \in \mathbb{R}^m$ and $v_0 \in \mathbb{R}^s$, as $k \to \infty$. Letting $k \to \infty$, from (3.4) - **(3.7)**, we get

$$\begin{aligned} &Dx_0 - A^T u - F^T v + c = 0, \\ &Ax_0 \geq b, \quad u_0 \geq 0, \\ &Fx_0 \geq d, \quad v_0 \geq 0, \\ &u_0^T (Ax_0 - b) + v_0^T (Fx_0 - d) = 0. \end{aligned}$$

Hence, $x_0 \in S(p, F, d) \subset \Omega$. On the other hand, since $x_k \notin \Omega$ for each k, we must have $x_0 \notin \Omega$, a contradiction. We have thus shown that the sequence $\{(x_k, u_k, v_k)\}$ must be unbounded. By considering a subsequence, if necessary, we can assume that $\|(x_k, u_k, v_k)\| \to \infty$ and, in addition, $\|(x_k, u_k, v_k)\| \neq 0$ for all k. Since the sequence of vectors

$$\frac{(x_k, u_k, v_k)}{\|(x_k, u_k, v_k)\|} = \left(\frac{x_k}{\|(x_k, u_k, v_k)\|}, \frac{u_k}{\|(x_k, u_k, v_k)\|}, \frac{v_k}{\|(x_k, u_k, v_k)\|} \right)$$

is bounded, it has a convergent subsequence. Without loss of generality, we can assume that

$$\frac{(x_k, u_k, v_k)}{\|(x_k, u_k, v_k)\|} \to (\bar{x}, \bar{u}, \bar{v}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^s, \quad \|(\bar{x}, \bar{u}, \bar{v})\| = 1. \tag{3.8}$$

Dividing both sides of (3.4) - (3.6) by $\|(x_k, u_k, v_k)\|$, both sides of (3.7) by $\|(x_k, u_k, v_k)\|^2$, and letting $k \to \infty$, by (3.8) we obtain

$$D\bar{x} - A^T\bar{u} - F^T\bar{v} = 0, \tag{3.9}$$

$$A\bar{x} \geq 0,\ \bar{u} \geq 0, \tag{3.10}$$

$$F\bar{x} \geq 0,\ \bar{v} \geq 0, \tag{3.11}$$

$$\bar{u}^T A\bar{x} + \bar{v}^T F\bar{x} = 0. \tag{3.12}$$

We first consider the case where (3.1) is fulfilled. It is evident that (3.9) - (3.12) imply

$$\bar{x}^T D\bar{x} = 0, \quad A\bar{x} \geq 0, \quad F\bar{x} \geq 0. \tag{3.13}$$

If $\bar{x} \neq 0$, then, by taking into account the fact that the constraint set $\Delta(A,0) \cap \Delta(F,0)$ of $QP(D,A,0,0,F,0)$ is a cone, one can deduce from (3.13) that either $\mathrm{sol}(D,A,0,0,F,0) = \emptyset$ or $\bar{x} \in \mathrm{sol}(D,A,0,0,F,0)$. This contradicts the first condition in (3.1). Thus $\bar{x} = 0$. Then it follows from (3.9) - (3.12) that $(\bar{u},\bar{v}) \in \Lambda[A,F] \setminus \{(0,0)\}$. (See Sec. 1 for the definition of the set $\Lambda[A,F]$). Since $(b,d) \in \mathrm{int}\,(\Lambda[A,F])^*$ by (3.1), we have

$$\bar{u}^T b + \bar{v}^T d < 0. \tag{3.14}$$

Consider the sequence $\{u_k^T b_k + v_k^T d\}$. By (3.4) and (3.7),

$$x_k^T D_k x_k + c_k^T x_k = u_k^T b_k + v_k^T d. \tag{3.15}$$

If, for each positive integer i, there exists an integer k_i such that $k_i > i$ and

$$u_{k_i}^T b_{k_i} + v_{k_i}^T d > 0, \tag{3.16}$$

then, by dividing both sides of (3.16) by $\|(x_{k_i}, u_{k_i}, v_{k_i})\|$ and letting $i \to \infty$, we have

$$\bar{u}^T b + \bar{v}^T d \geq 0,$$

contrary to (3.14). Consequently, there must exist a positive integer i_0 such that

$$u_k^T b_k + v_k^T d \leq 0 \quad \text{for every } k \geq i_0. \tag{3.17}$$

If the sequence $\{x_k\}$ is bounded, then, by dividing both sides of (3.15) by $\|(x_k, u_k, v_k)\|$ and letting $k \to \infty$, we get $\bar{u}^T b + v^T d = 0$, contrary to (3.14). Thus $\{x_k\}$ is unbounded. We can assume that $\|x_k\| \to \infty$ and $\|x_k\| \neq 0$ for each k. Then $\{x_k/\|x_k\|\}$ is bounded. We can assume that

$$\frac{x_k}{\|x_k\|} \to \hat{x} \quad \text{with } \|\hat{x}\| = 1.$$

Combining (3.15) with (3.17) gives

$$x_k^T D_k x_k + c_k^T x_k \leq 0 \quad \text{for every } k \geq i_0. \tag{3.18}$$

Dividing both sides of (3.18) by $\|x_k\|^2$ and letting $k \to \infty$, we obtain

$$\hat{x}^T D\hat{x} \le 0. \tag{3.19}$$

By (3.5) and (3.6),

$$A_k x_k \ge b_k, \quad Fx_k \ge d.$$

Dividing both sides of each of the last inequalities by $\|x_k\|$ and letting $k \to \infty$, one has

$$A\hat{x} \ge 0, \quad F\hat{x} \ge 0. \tag{3.20}$$

Combining (3.19) with (3.20), we assert that $\mathrm{sol}\,(D, A, 0, 0, F, 0) \neq \{0\}$, contrary to the first condition in (3.1). We have thus proved the theorem for the case where (3.1) is fulfilled.

Now we turn to the case where condition (3.2) is fulfilled. We deduce (3.13) from (3.9) - (3.12). If $\bar{x} \neq 0$, then, from (3.13), we get $\mathrm{sol}\,(-D, A, 0, 0, F, 0) \neq \{0\}$, which contradicts the first condition in (3.2). Thus, $\bar{x} = 0$. From (3.9) - (3.12) it follows that $(\bar{u}, \bar{v}) \in \Lambda[A, F] \setminus \{(0,0)\}$. By the second condition in (3.2),

$$\bar{u}^T b + v^T d > 0. \tag{3.21}$$

Consider the sequence $\{u_k^T b_k + v_k^T d\}$. We have (3.15). If there exists a positive integer i_0 such that (3.17) is valid, then, by dividing both sides of (3.17) by $\|(x_k, u_k, v_k)\|$ and letting $k \to \infty$, we obtain $\bar{u}^T b + \bar{v}^T d \le 0$, contrary to (3.21). Therefore, for each positive integer i, one can find an integer $k_i > i$ such that (3.16) holds. If the sequence $\{x_k\}$ is bounded, then, by dividing both sides of (3.15) by $\|(x_k, u_k, v_k)\|$ and letting $k \to \infty$, we have $\bar{u}^T b + \bar{v}^T d = 0$, contrary to (3.21). Thus, the sequence $\{x_k\}$ is unbounded. We can assume that $\|x_k\| \to \infty$ and $\|x_k\| \neq 0$ for all k. Since the sequence $\{x_k/\|x_k\|\}$ is well defined and bounded, without loss of generality, we can assume that

$$\frac{x_k}{\|x_k\|} \to \hat{x} \quad \text{with } \|\hat{x}\| = 1.$$

Combining (3.15) with (3.16) gives

$$x_{k_i}^T D_{k_i} x_{k_i} + c_{k_i}^T x_{k_i} > 0 \quad \text{for all } i. \tag{3.22}$$

Dividing both sides of (3.22) by $\|x_{k_i}\|^2$ and letting $i \to \infty$, we obtain $\hat{x}^T D\hat{x} \ge 0$ or, equivalently,

$$\hat{x}^T(-D)\hat{x} \le 0. \tag{3.23}$$

By (3.5) and (3.6),

$$A_{k_i} x_{k_i} \ge b_{k_i}, \quad Fx_{k_i} \ge d. \tag{3.24}$$

Dividing both sides of each of the inequalities in (3.24) by $\|x_{k_i}\|$ and letting $i \to \infty$, we have

$$A\hat{x} \ge 0, \quad F\hat{x} \ge 0. \tag{3.25}$$

Combining (3.23) with (3.25) yields $\mathrm{sol}(-D, A, 0, 0, F, 0) \neq \{0\}$, contrary to the first condition in (3.2). This proves the theorem in the case where (3.2) is fulfilled.

Now, let us consider the last case where (3.3) is assumed. From (3.9) - (3.12) we have $\bar{x} \in S(D, A, 0, 0, F, 0)$. By the first condition in (3.2), $\bar{x} = 0$. Then it follows from (3.9) - (3.12) that

$$A^T\bar{u} + F^T\bar{v} = 0, \quad \bar{u} \geq 0, \quad \bar{v} \geq 0, \quad \|(0, \bar{u}, \bar{v})\| = 1.$$

Therefore, $(\bar{u}, \bar{v}) \in \Lambda[A, F] \setminus \{(0,0)\}$. Since $\bar{u}^T\bar{u} + \bar{v}^T\bar{v} > 0$, then $(\bar{u}, \bar{v}) \notin$ int $(\Lambda[A, F])^*$. This contradicts the second condition in (3.3).

We have thus proved that if one of the pairs of conditions (3.1) - (3.3) is fulfilled, then the conclusion of the theorem must hold true. ■

We now proceed to show how the sufficient conditions (3.1) and (3.2) look in the case of the canonical problem (1.2). As in [10], for any $A \in \mathbb{R}^{n \times n}$, the dual of the cone $\Lambda[A] := \{\lambda \in \mathbb{R}^m : -A^T\lambda \geq 0, \ \lambda \geq 0\}$ is denoted by $(\Lambda[A])^*$. By definition, $(\Lambda[A])^* = \{\xi \in \mathbb{R}^m : \lambda^T\xi \leq 0 \ \ \forall \lambda \in \Lambda[A]\}$. The interior of $(\Lambda[A])^*$ is denoted by int $(\Lambda[A])^*$. One has

$$\mathrm{int}\,(\Lambda[A])^* = \{\xi \in \mathbb{R}^m : \lambda^T\xi < 0 \quad \forall \lambda \in \Lambda[A] \setminus \{0\}\}. \tag{3.26}$$

Lemma 3.1. *Suppose that, in problem* (1.1), $s = n$, $d = 0$, *and* F *is the unit matrix in* $\mathbb{R}^{n \times n}$. *Then the following statements hold:*

(a_1) *If* $b \in \mathrm{int}\,(\Lambda[A])^*$, *then* $(b, 0) \in \mathrm{int}\,(\Lambda[A, F])^*$;
(a_2) *If* $\mathrm{sol}\,(D, A, 0, 0) = \{0\}$, *then* $\mathrm{sol}\,(D, A, 0, 0, F, 0) = \{0\}$;
(a_3) *If* $b \in -\mathrm{int}\,(\Lambda[A])^*$, *then* $(b, 0) \in -\mathrm{int}\,(\Lambda[A, F])^*$;
(a_4) *If* $\mathrm{sol}\,(-D, A, 0, 0) = \{0\}$, *then* $\mathrm{sol}\,(-D, A, 0, 0, F, 0) = \{0\}$.

Proof. If $b \in \mathrm{int}\,(\Lambda[A])^*$, then, by (3.26),

$$\lambda^T b < 0 \quad \text{for all } \lambda \in \Lambda[A] \setminus \{0\}. \tag{3.27}$$

For any $(u, v) \in \Lambda[A, F] \setminus \{0\}$, we have

$$A^T u + F^T v = 0, \quad u \geq 0, \quad v \geq 0.$$

This yields

$$-A^T u = v \geq 0, \quad u \geq 0, \quad u \neq 0,$$

hence $u \in \Lambda[A] \setminus \{0\}$. By (3.27), $b^T u + 0^T v = b^T u = u^T b < 0$. This shows that $(b, 0) \in \mathrm{int}\,(\Lambda[A, F])^*$. Statement ($a_1$) has been proved. It is clear that (a_3) follows from (a_1).

For proving (a_2) and (a_4) it suffices to note that, under our assumptions,

$$\mathrm{sol}\,(D, A, 0, 0) = \mathrm{sol}\,(D, A, 0, 0, F, 0)$$

$$\mathrm{sol}\,(-D, A, 0, 0) = \mathrm{sol}\,(-D, A, 0, 0, F, 0). \qquad ■$$

The following result follows directly from Theorem 3.1 and Lemma 3.1.

Theorem 3.2 (cf. [10, Theorems 2.2 and 2.3]). *For the canonical problem* (1.2), *the following statements hold:*

(α_1) *If* $\mathrm{sol}(D,A,0,0)=\{0\}$ *and if* $b\in \mathrm{int}(\Lambda[a])^*$ *then, for any* $c\in\mathbb{R}^n$, *the multifunction (1.3) is upper semicontinuous at* $p:=(D,A,c,b)$;

(α_2) *If* $\mathrm{sol}(-D,A,0,0)=\{0\}$ *and if* $b\in -\mathrm{int}(\Lambda[a])^*$ *then, for any* $c\in\mathbb{R}^n$, *the multifunction (1.3) is upper semicontinuous at* $p:=(D,A,c,b)$.

From what has been said we can conclude that Theorem 3.1 extends Theorems 2.2 and 2.3 in [10] to the case of the general problem (1.1).

Let us consider two illustrative examples which show that our results can be applied to some classes of *non-convex* QP problems.

Example 1. (cf. [10, Example 3.1]) Consider the problem $QP(D,A,c,b,F,d)$ where

$$D=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} \text{ or } D=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ A=\begin{bmatrix}-\frac{1}{2}, -1\end{bmatrix},\ b=(-1),\ c=\begin{pmatrix}0\\0\end{pmatrix},$$

$$F=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ d=\begin{pmatrix}0\\0\end{pmatrix},\ x=\begin{pmatrix}x_1\\x_2\end{pmatrix}\in\mathbb{R}^2.$$

We have $\Delta(A,0)\cap\Delta(F,0)=\{0\}$, hence $\mathrm{sol}(D,A,0,0,F,0)=\mathrm{sol}(D,A,0,0)=\{0\}$. Since $b\in\mathrm{int}(\Lambda[A])^*$ then, by Lemma 3.1, $(b,0)\in\mathrm{int}(\Lambda[A,F])^*$. By Theorem 3.1, the multifunction (1.4) is upper semicontinuous at $p:=(D,A,c,b)$. (Note that the objective functions $f(x)=1/2(-x_1^2-x_2^2)$ and $\tilde{f}(x)=(1/2)(x_1^2-x_2^2)$ of the corresponding QP problems are non-convex.)

Example 2. (cf. [10], Example 3.2) Consider the problem $QP(D,A,c,b,F,d)$ where

$$D=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ A=[0,-1],\ b=(-1),\ c=\begin{pmatrix}0\\0\end{pmatrix},$$

$$F=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ d=\begin{pmatrix}0\\0\end{pmatrix},\ x=\begin{pmatrix}x_1\\x_2\end{pmatrix}\in\mathbb{R}^2.$$

An easy computation shows that

$$S(D,A,0,0,F,0)=S(D,A,0,0)=\{0\},$$

$$\mathrm{sol}(-D,A,0,0)=\{0\},\quad \text{and}\quad b\in-\mathrm{int}(\Lambda[A])^*.$$

By Lemma 3.1,

$$\mathrm{sol}(-D,A,0,0,F,0)=\{0\},\quad (b,0)\in-\mathrm{int}(\Lambda[A,F])^*.$$

Then, by Theorem 3.1, the multifunction (1.4) is upper semicontinuous at $p:=(D,A,c,b)$. (Note that the objective function $f(x)=(1/2)(x_1^2-x_2^2)$ of the corresponding QP problem is nonconvex.)

Several other illustrative examples can be found in [10] (Sec. 3).

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