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Remarks on the Semilinear Wave Equations*

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Abstract. We study the semilinear wave equation of the form: $\Box u + u_t + |u_t|^{p-1}u_t = |u|^{q-1}u$. When $1 < q \le p$, a solution exists globally in all time and when $1 < p < q$, the local solution blows up in finite time for negative initial energy.

1. Introduction and Results

Let Ω be a bounded domain in $\mathbb{R}^N (N \ge 1)$ with a smooth boundary $\partial\Omega$. We are concerned with the following initial boundary value problem for semilinear wave equations with damping and source terms

$$\begin{cases} \Box u + au_t + b|u_t|^{p-1}u_t = c|u|^{q-1}u & \text{in } (0,\infty)\times\Omega, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) & \text{for } x \in \Omega, \\ u(t,x) = 0 & \text{for } t \ge 0, x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Box = \partial_t^2 - \sum_{i=1}^N \partial^2/\partial x_i^2$ is the D'Alembertian operator, $a, b, c > 0, p, q > 1$, and the initial data $u_0(x) \in H_0^1(\Omega)$ and $u_1(x) \in L^2(\Omega)$. In our case, we always put $a = b = c = 1$ in the equation of (1.1).

For problem (1.1), many authors have studied a variety of examples. More precisely, when $c = 0$, the decay properties of solutions of the Cauchy problem of (1.1) was studied by Matsumura [8]. In the case of $a = b = 0$, problem (1.1) was studied by many authors (see [1 - 3]), where the local solution blows up in finite time. When $b = 0$, the problem (1.1) was studied by Ikehata–Suzuki [6] and Ikehata [5], and they proved that the local solution blows up in finite time by the concepts of stable and unstable sets due to Sattinger [10].

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Recently, Georgiev–Todorova [4] treated the case when $a = 0$, $b = 1$, and $c = 1$, and proved that if $1 < q \leq p$, a solution exists globally in all time. On the other hand, they also proved that if $1 < p < q$, the local solution blows up in finite time for sufficiently negative initial energy

$$\mathcal{E}(0) = \frac{1}{2}\|u_1\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega)} - \frac{1}{q+1}\|u_0\|^{q+1}_{L^{q+1}(\Omega)}.$$

Ono [9] also treated the case when $a = 0$, $b > 0$, and $c = 1$, and he proved that if $1 < p < q$, the local solution blows up in finite time for negative initial energy. Thereby, our main objective is to combine linear damping term u_t and non-linear damping term $|u_t|^{p-1}u_t$ with the source term $|u|^{q-1}u$ to show that in some domain the solution exists globally in all time, and in some domain the local solution blows up in finite time.

In this paper, we mainly study the global existence of a solution to problem (1.1) for the case $1 < q \leq p$. On the other hand, the local solution blows up in finite time for the case $1 < p < q$ with negative initial energy.

Now we state our results.

Theorem 1.1. *Suppose $1 < q \leq N/(N-2)$ if $N \geq 3$ and $q > 1$ if $N \leq 2$. If $1 < q \leq p$, then problem* (1.1) *has unique global solution in the class*

$$u \in C\left([0,T); H^1_0(\Omega)\right), \quad u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^{p+1}((0,T)\times\Omega)$$

for any positive T.

Before we state another result, we first define the energy for the problem (1.1) by

$$\mathcal{E}(t) = \frac{1}{2}\|u_t(t,\cdot)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)} - \frac{1}{q+1}\|u(t,\cdot)\|^{q+1}_{L^{q+1}(\Omega)}. \qquad (1.2)$$

Theorem 1.2. *Suppose $1 < q \leq N/(N-2)$ if $N \geq 3$ and $q > 1$ if $N \leq 2$. If $1 < p < q$, then the local solution to problem* (1.1) *blows up in finite time for negative initial energy* ($\mathcal{E}(0) < 0$).

Remark 1. If $a = 0$, Theorem 1.1 coincides with the global existence result of Georgiev–Todorova [4]. So Theorem 1.1 will become a kind of extension of Georgiev–Todorova [4].

Remark 2. If $a = 0$, Theorem 1.2 improves the blow-up result of Georgiev–Todorova [4]. So when $a \neq 0$, our Theorem 1.2 is new.

The plan of this paper is as follows: In Sec. 2, we discuss the local existence and global existence of solutions to problem (1.1), and by combining the ideas of Georgiev–Todorova [4] and Ono [9] we will give the proof of Theorem 1.2 in Sec. 3.

2. Local and Global Existence of Solutions

Throughout this paper, the function spaces are the usual Lebesgue and Sobolev spaces over Ω. For our convenience, we use $\|\cdot\|_m$ instead of $\|\cdot\|_{L^m(\Omega)}$, where Ω is a bounded domain in $\mathbb{R}^N(N \geq 1)$ with a smooth boundary $\partial\Omega$. We introduce the natural energy space $H = H_0^1(\Omega) \times L^2(\Omega)$ for the problem (1.1).

First of all, we have the following local existence theorem introducing the linear damping term u_t in the well-known local existence theorem of Georgiev and Todorova [4].

Theorem 2.1. *Suppose* $1 < q \leq N/(N-2)$ *if* $N \geq 3$, $q > 1$ *if* $N \leq 2$ *and* $p > 1$. *Then, for any initial data*

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in H,$$

there exists some positive T *such that problem* (1.1) *admits a unique solution in the class*

$$U \in X_T = \left\{ U = \begin{pmatrix} u \\ u_t \end{pmatrix}; U \in C([0,T]; H),\, u_t \in L^{p+1}((0,T) \times \Omega) \right\}.$$

Proof of Theorems 2.1 and 1.1. The proof of Theorems 2.1 and 1.1 is almost the same as in [4], because $\int_\Omega |u_t|^2 dx \geq 0$ is well defined. Roughly speaking, for the uniqueness of a solution, by using the fact that $(w_{1t} - w_{2t} + |w_{1t}|^{p-1}w_{1t} - |w_{2t}|^{p-1}w_{2t}, w_{1t} - w_{2t}) \geq 0$, we can apply directly the well-known Georgiev–Todorova local existence theorem.

As mentioned above, $\int_\Omega |u_t|^2 dx \geq 0$ is well defined and the linear damping term u_t allows one to derive *a priori* estimates for the global existence purpose. Therefore, we omit the detailed proof of Theorems 2.1 and here. For the detailed proof of Theorems 2.1 and 1.1, we refer to Theorem 2.1 of [4] (see also Theorem 3.1 in [7]) and Theorem 1.1. ■

3. Blow-Ups of Solutions

Before proving Theorem 1.2, we first need the following two propositions.

By multiplying the equation in (1.1) by u_t, and integrating over $x \in \Omega$, we can get easily from (1.2)

$$\mathcal{E}'(t) + \|u_t\|_2^2 + \|u_t\|_{p+1}^{p+1} = 0. \tag{3.1}$$

Thus from (3.1), we get

$$\mathcal{E}(t) \leq \mathcal{E}(0). \tag{3.2}$$

Proposition 3.1. *Let* $\mathcal{A}(t) = \|u\|_2^2$. *Then it satisfies the following inequality:*

$$\mathcal{A}''(t) \geq 2\left\{\mathcal{K}(t) - M(-\mathcal{E}(t))^{\rho-1}(\|u_t\|_2^2 + \|u_t\|_{p+1}^{p+1})\right\}, \tag{3.3}$$

where M, ρ are positive constants which are defined later and $\mathcal{K}(t)$ is defined by

$$\mathcal{K}(t) = 2\|u_t\|_2^2 - 2\mathcal{E}(t) + \frac{q-1}{2(q+1)}\|u\|_{q+1}^{q+1} > 0. \tag{3.4}$$

Proof. From the definition of $\mathcal{A}(t)$ and using the equation multiplied by u, we get from (1.2)

$$\begin{aligned}\mathcal{A}''(t) &= 2\|u_t\|_2^2 + 2\int_\Omega (\Delta u u - u_t u - |u_t|^{p-1}u_t u + |u|^{q+1})dx \\ &= 4\|u_t\|_2^2 - 4\left(\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{q+1}\|u\|_{q+1}^{q+1}\right) \\ &\quad + \frac{2(q-1)}{q+1}\|u\|_{q+1}^{q+1} - 2\int_\Omega (u_t u + |u_t|^{p-1}u_t u)dx \\ &= 2\left\{2\|u_t\|_2^2 - 2\mathcal{E}(t) + \frac{q-1}{q+1}\|u\|_{q+1}^{q+1} - \int_\Omega (u_t u + |u_t|^{p-1}u_t u)dx\right\}.\end{aligned} \tag{3.5}$$

Now, we estimate the last two terms on the right-hand side of (3.5). By the Hölder inequality and the assumption $p < q$, we have

$$\begin{aligned}\left|\int_\Omega |u_t|^{p-1}u_t u dx\right| &\leq C\|u\|_{p+1}\|u_t\|_{p+1}^p \leq C\|u\|_{q+1}\|u_t\|_{p+1}^p \\ &= C\|u\|_{q+1}^{1-(q+1)/(p+1)}\|u_t\|_{p+1}^p\|u\|_{q+1}^{(q+1)/(p+1)},\end{aligned} \tag{3.6}$$

where C is a positive constant. Since $\mathcal{E}(0) < 0$, then from (1.2) and (3.2) we get

$$\|u\|_{q+1} \geq (-\mathcal{E}(t))^{1/(q+1)} \geq (-\mathcal{E}(0))^{1/(q+1)} > 0. \tag{3.7}$$

Since $1/(q+1) - 1/(p+1) < 0$, then from (3.7) we get

$$\begin{aligned}\|u\|_{q+1}^{q+1(1/(q+1)-1/(p+1))} = \|u\|_{q+1}^{1-(q+1)/(p+1)} &\leq (-\mathcal{E}(t))^{1/(q+1)-1/(p+1)} \\ &\leq (-\mathcal{E}(0))^{1/(q+1)-1/(p+1)}.\end{aligned} \tag{3.8}$$

The Young inequality implies

$$C\|u_t\|_{p+1}^p\|u\|_{q+1}^{(q+1)/(p+1)} \leq \left(\frac{C}{\varepsilon}\right)^{(p+1)/p} \|u_t\|_{p+1}^{p+1} + \varepsilon^{p+1}\|u\|_{q+1}^{q+1} \tag{3.9}$$

for any $\varepsilon > 0$.

Thus from (3.6), (3.8), and (3.9), we arrive at

$$\begin{aligned}\left|\int_\Omega |u_t|^{p-1}u_t u dx\right| \leq \left(\frac{C}{\varepsilon}\right)^{(p+1)/p} &(-\mathcal{E}(t))^{1/(q+1)-1/(p+1)} \|u_t\|_{p+1}^{p+1} \\ &+ \varepsilon^{p+1}(-\mathcal{E}(0))^{1/(q+1)-1/(p+1)} \|u\|_{q+1}^{q+1}.\end{aligned} \tag{3.10}$$

By the Schwarz inequality and Hölder inequality and by the assumption $1 < q$, we have

$$\left|\int_\Omega u_t u dx\right| \le C\|u_t\|_2 \|u\|_{q+1}^{1-(q+1)/2} \|u\|_{q+1}^{(q+1)/2}. \tag{3.11}$$

Since $1/(q+1) - 1/2 < 0$, then from (3.7) we have

$$\|u\|_{q+1}^{q+1(1/(q+1)-1/2)} = \|u\|_{q+1}^{1-(q+1)/2} \le (-\mathcal{E}(t))^{1/(q+1)-1/2} \le (-\mathcal{E}(0))^{1/(q+1)-1/2}. \tag{3.12}$$

The Young inequality implies

$$C\|u_t\|_2 \|u\|_{q+1}^{(q+1)/2} \le \left(\frac{C}{\varepsilon_0}\right)^2 \|u_t\|_2^2 + \varepsilon_0^2 \|u\|_{q+1}^{q+1} \tag{3.13}$$

for any $\varepsilon_0 > 0$.

Thus from (3.11) - (3.13), we arrive at

$$\left|\int_\Omega u_t u dx\right| \le \left(\frac{C}{\varepsilon_0}\right)^2 (-\mathcal{E}(t))^{1/(q+1)-1/2} \|u_t\|_2^2 + \varepsilon_0^2 (-\mathcal{E}(0))^{1/(q+1)-1/2} \|u\|_{q+1}^{q+1}. \tag{3.14}$$

Therefore, from (3.5), (3.10), and (3.14), we arrive at

$$\begin{aligned} A''(t) \ge 2\Big\{ & 2\|u_t\|_2^2 - 2\mathcal{E}(t) + \frac{q-1}{q+1}\|u\|_{q+1}^{q+1} \\ & - \left(\frac{C}{\varepsilon}\right)^{(p+1)/p} (-\mathcal{E}(t))^{1/(q+1)-1/(p+1)} \|u_t\|_{p+1}^{p+1} \\ & - \varepsilon^{p+1} (-\mathcal{E}(0))^{1/(q+1)-1/(p+1)} \|u\|_{q+1}^{q+1} \\ & - \left(\frac{C}{\varepsilon_0}\right)^2 (-\mathcal{E}(t))^{1/(q+1)-1/2} \|u_t\|_2^2 \\ & - \varepsilon_0^2 (-\mathcal{E}(0))^{1/(q+1)-1/2} \|u\|_{q+1}^{q+1} \Big\}. \end{aligned} \tag{3.15}$$

Now we define ($\mathcal{E}(0) < 0$)

$$\varepsilon^{p+1} = \frac{q-1}{4(q+1)} (-\mathcal{E}(0))^{1/(p+1)-1/(q+1)} > 0,$$

$$\varepsilon_0^2 = \frac{q-1}{4(q+1)} (-\mathcal{E}(0))^{1/2-1/(q+1)} > 0,$$

$$M_0 = \left(\frac{C}{\varepsilon}\right)^{(p+1)/p} > 0 \quad \text{and} \quad M_1 = \left(\frac{C}{\varepsilon_0}\right)^2 > 0.$$

Then (3.15) becomes

$$A''(t) \geq 2\Big\{ 2\|u_t\|_2^2 - 2\mathcal{E}(t) + \frac{q-1}{2(q+1)}\|u\|_{q+1}^{q+1} - M_0(-\mathcal{E}(t))^{1/(q+1)-1/(p+1)}\|u_t\|_{p+1}^{p+1} - M_1(-\mathcal{E}(t))^{1/(q+1)-1/2}\|u_t\|_2^2 \Big\}.$$

Define

$$\rho - 1 = \max\left\{ \frac{1}{q+1} - \frac{1}{p+1}, \frac{1}{q+1} - \frac{1}{2} \right\} \quad (1 < 2\rho < 2),$$
$$M = \max\{M_0, M_1\},$$

and then finally we arrive at

$$A''(t) \geq 2\left\{ \mathcal{K}(t) - M(-\mathcal{E}(t))^{\rho-1}\left(\|u_t\|_2^2 + \|u_t\|_{p+1}^{p+1}\right) \right\},$$

which is our desired inequality (3.3). ■

Proposition 3.2. *Let $1 < p < q$ and $\mathcal{E}(0) < 0$. Then we have the following two inequalities*

$$\mathcal{B}'(t) \geq \frac{\mathcal{K}(t)}{M}, \tag{3.16}$$
$$\mathcal{B}(t)^{1/\rho} \leq M'\mathcal{K}(t), \tag{3.17}$$

where $\mathcal{B}(t) = (-\mathcal{E}(t))^\rho + \rho\mathcal{A}'(t)/M$ and M' is a positive constant as defined later.

Proof. By the definition of $\mathcal{B}(t)$, and using (3.1), (3.3), and $1 < 2\rho < 2$, we have

$$\begin{aligned}\mathcal{B}'(t) &= \rho(-\mathcal{E}(t))^{\rho-1}(-\mathcal{E}'(t)) + \frac{\rho}{M}\mathcal{A}''(t) \\ &\geq \rho(-\mathcal{E}(t))^{\rho-1}(\|u_t\|_2^2 + \|u_t\|_{p+1}^{p+1}) \\ &\quad + \frac{\rho}{M}\left[2\left\{\mathcal{K}(t) - M(-\mathcal{E}(t))^{\rho-1}(\|u_t\|_2^2 + \|u_t\|_{p+1}^{p+1})\right\}\right] \\ &= \frac{2\rho}{M}\mathcal{K}(t) \geq \frac{\mathcal{K}(t)}{M},\end{aligned}$$

which is our desired inequality (3.16).

By the definition of $\mathcal{A}(t)$, Schwarz and Young inequalities imply

$$|\mathcal{A}'(t)| \leq C\|u\|_{q+1}\|u_t\|_2\,. \tag{3.18}$$

Owing to the definition of $\mathcal{B}(t)$, the Hölder inequality, assumption $1 < 2\rho < 2$, and inequality (3.18), we have

$$\mathcal{B}(t)^{1/\rho} \le 2\left\{-\mathcal{E}(t) + \left(\frac{|\mathcal{A}'(t)|}{M}\right)^{1/\rho}\right\} \le 2\left\{-\mathcal{E}(t) + (C\|u_t\|_2\|u\|_{q+1})^{1/\rho}\right\}$$
$$\le 2\left\{-\mathcal{E}(t) + 2\|u_t\|_2^2 + \frac{C}{2}\|u\|_{q+1}^{2/2\rho-1}\right\}, \tag{3.19}$$

where in the last line of the two terms we have used $\alpha^\gamma\beta^{1-\gamma} \le \gamma\alpha + (1-\gamma)\beta$ for all $\alpha, \beta \ge 0$ and $1 \ge \gamma \ge 0$, and $1 < 2\rho < 2$. For $1 < q$, we see that $2/(2\rho-1) < q+1$ and the assumption $\mathcal{E}(0) < 0$. Then we have from the inequality (3.7)

$$1 \le (-\mathcal{E}(0))^{-1/(q+1)}\|u\|_{q+1} \le (-\mathcal{E}(0))^{-[1-2/((2\rho-1)(q+1))]}\|u\|_{q+1}^{q+1-2/(2\rho-1)},$$

and then we have

$$\|u\|_{q+1}^{2/(2\rho-1)} \le (-\mathcal{E}(0))^{-[1-2/((2\rho-1)(q+1))]}\|u\|_{q+1}^{q+1}. \tag{3.20}$$

Thus from (3.19) and (3.20), we have

$$\mathcal{B}(t)^{\frac{1}{\rho}} \le 2\left\{2\|u_t\|_2^2 - \mathcal{E}(t) + \frac{C}{2}(-\mathcal{E}(0))^{-[1-2/((2\rho-1)(q+1))]}\|u\|_{q+1}^{q+1}\right\}$$
$$\le 2\left\{2\|u_t\|_2^2 - 2\mathcal{E}(t) + \frac{q-1}{2(q+1)}\|u\|_{q+1}^{q+1}\frac{C(q+1)}{q-1}(-\mathcal{E}(0))^{-[1-2/((2\rho-1)(q+1))]}\right\}$$
$$\le M'\left\{2\|u_t\|_2^2 - 2\mathcal{E}(t) + \frac{q-1}{2(q+1)}\|u\|_{q+1}^{q+1}\right\},$$

where

$$M' = 2\max\left\{1, \frac{C(q+1)}{q-1}(-\mathcal{E}(0))^{-[1-2/((2\rho-1)(q+1))]}\right\}.$$

This completes the proof of Proposition 3.2. ■

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. From (3.16) and (3.4), we have

$$\mathcal{B}'(t) \ge \frac{\mathcal{K}(t)}{M} \ge -C\mathcal{E}(t) \ge -C\mathcal{E}(0) > 0,$$

and there exists a $T > 0$ such that $\mathcal{B}(t) > 0$ for $t \ge T$. Then owing to the Proposition 3.2, we have for $t \ge T$

$$\frac{d}{dt}\left(\mathcal{B}(t)^{(\rho-1)/\rho}\right) = \frac{\rho-1}{\rho}\mathcal{B}(t)^{-1/\rho}\mathcal{B}'(t) \le \frac{\mathcal{K}(t)(\rho-1)}{\rho MM'\mathcal{K}(t)} \le \frac{C(\rho-1)}{\rho}. \tag{3.21}$$

Since $(\rho-1)/\rho < 0$, then (3.21) leads to

$$\mathcal{B}(t) \geq \left(\mathcal{B}(T)^{(\rho-1)/\rho} + \frac{C(\rho-1)}{\rho}(t-T)\right)^{\rho/(\rho-1)} \tag{3.22}$$

for any $t \geq T$. Thus, there exists a finite T^* so that the right side of (3.22) tends to infinity as $t \to T^*$. From (1.2), we have

$$-2\mathcal{E}(t) + \|u_t\|_2^2 \leq \frac{2}{q+1}\|u\|_{q+1}^{q+1} \quad \text{and} \quad \|u_t\|_2^2 \leq \frac{2}{q+1}\|u\|_{q+1}^{q+1},$$

and then from inequality (3.17), we have $\mathcal{B}(t)^{1/\rho} \leq C\|u\|_{q+1}^{q+1}$. Thus, we have $\|u\|_{q+1} \longrightarrow \infty$ as $t \to T^*$. Therefore, the local solution blows up in finite time. This completes the proof of Theorem 1.2. ■

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