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Fibonacci Length of Automorphism Groups Involving Tribonacci Numbers

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Abstract. The Fibonacci length of a finitely generated finite group $G = \langle a, b\rangle$ is the least integer n such that, for the sequences $x_1 = a, x_2 = b, x_{i+2} = x_i x_{i+1}, (i \geq 1)$ of the elements of G, $x_{n+1} = x_1$ and $x_{n+2} = x_2$.
The groups D_{2n}, Q_{2^n} and the simple groups of order $\leq 10^5$ are the only known groups that their Fibonacci lengths have been known. In this paper we shall generalize this notion for the 3-generated groups and whereby we calculate the Fibonacci lengths of the groups $\mathrm{Aut}(D_{2n})$ and $\mathrm{Aut}(Q_{2^n})$ which involve certain sequences of Tribonacci numbers.

1. Introduction

Many authors have studied the periodic sequences of elements of finite fields and groups (for example, see [2, 3, 5, 6, 9, 10]). Most of these investigations consider the periodic sequences modulo n. However, Campbell, Doostie and Robertson [6] considered the periodic sequences for an abstract and finitely presented groups by defining two parameters LEN and BLEN, computing them for D_{2n} ,Q_{2^n} and simple groups of orders less than 10^6.

In this paper by generalizing these notions, we study the Fibonacci length and basic Fibonacci length of $\mathrm{Aut}(D_{2n})$ and $\mathrm{Aut}(Q_{2^n})$. By considering a sequence of Tribonacci numbers we are able to calculate LEN and BLEN. An explicit formula for LEN is also determined in one case. Moreover, we show that if $n = p$ or $n = 2p$ ($p \geq 5$ is a prime), then LEN $= p\times$ BLEN, and if $n = 3.2^k (k \geq 2)$, then LEN $= 2\times$ BLEN.

2. Preliminaries

Let $G = \langle x, y\rangle$ be a finite non-abelian group. Then the sequence

$$a_1 = x\,,\;\; a_2 = y\,,\;\; a_{i+2} = a_i a_{i+1}\,,\;\; i \geq 1 \qquad (1)$$

of elements of G is called the Fibonacci orbit, and the least integer n where $a_{n+1} = a_1$ and $a_{n+2} = a_2$, denoted by LEN(x, y), is called the Fibonacci length of the generating pair (x, y). The basic Fibonacci orbit of length m is also defined to be the sequence (1) such that m is the least integer, where $a_1\theta = a_{m+1}$ and $a_2\theta = a_{m+2}$ for some $\theta \in \text{Aut}(G)$.

It is proved in [6] that m divides n and there are n/m elements of Aut(G) that map the Fibonacci orbit into itself.

For a non-abelian and 3-generated group $G = \langle a, b, c\rangle$, we define the sequence

$$x_1 = a\,,\;\; x_2 = b\,,\;\; x_3 = c\,,\;\; x_{i+3} = x_i x_{i+1} x_{i+2}\,,\;\; i \geq 1 \qquad (2)$$

of elements of G as the Tribonacci orbit and the least integer n such that $x_{n+1} = a, x_{n+2} = b, x_{n+3} = c$, as the Tribonacci length of the generating triple (a, b, c). We denote this length by LEN too. The definition of basic Tribonacci length is similar to the 2-generator case. We use the notation BLEN for basic length (Fibonacci or Tribonacci). For n-generator groups, (2) may be generalized (see [2]).

We also need some notions of group theory to optain the necessary presentations for Aut(D_{2n}) and Aut(Q_{2^n}). Let $G = G(n, m, r)$, where

$$G(n, m, r) = \langle a, b \,|\, a^n = b^m = 1,\, a^{b^{-1}} = a^r\rangle, \quad 1 < r < n-1.$$

Then it is easy to see that $|G| = md$ where d is the highest common factor of n and $r^m - 1$. We define the following sequences of numbers where r is the integer in the definition of G :

$f_0 = f_1 = 1, \quad f_n = f_{n-1} + f_{n-2},\ n \geq 2,$

$s_2 = 0,\ s_3 = 1, \quad s_k = s_{k-2} + r^{f_{k-4}}.s_{k-1},\ k \geq 4,$

$t_1 = t_2 = 1,\ t_3 = -2, \quad t_i = t_{i-1} + t_{i-2} + t_{i-3},\ i \geq 4,$

$t'_1 = 1, t'_2 = 2,\ t'_3 = 3, \quad t'_i = t'_{i-1} + t'_{i-2} + t'_{i-3},\ i \geq 4,$

$t''_1 = 1,\ t''_2 = 2,\ t''_3 = 4, \quad t''_i = t''_{i-1} + t''_{i-2} + t''_{i-3},\ i \geq 4.$

Then we have

Lemma 2.1. *The elements of the sequence* (1) *of the group* G *are of the form:*

$$a_1 = a,\;\; a_2 = b,\quad a_k = a^{s_k} \cdot b^{f_{k-2}},\; k \geq 2,$$

where s_k *is reduced modulo* n *and* f_{k-2} *is reduced modulo* m.

Proof. Let $a_k = a^{s_k} \cdot b^{f_{k-2}}$ and $a_{k+1} = a^{s_{k+1}} \cdot b^{f_{k-1}}$. Then

$$a_{k+2} = a_k \cdot a_{k+1} = a^{s_k} \cdot b^{f_{k-2}} \cdot a^{s_{k+1}} \cdot b^{f_{k-1}}\,.$$

On the other hand, $ba = a^r b$ gives that

$$b^y \cdot a^x = a^{x \cdot r^y} \cdot b^y,$$

for every non-negative integers x and y. Therefore,

$$a_{k+2} = a^{s_k} \cdot (a^{s_{k+1}r^{f_{k-2}}}) \cdot b^{f_{k-2}} \cdot b^{f_{k-1}} = a^{s_{k-2}} \cdot b^{f_k}. \quad \blacksquare$$

Lemma 2.2. *For every positive integer t and every $k \geq 2$, $(a_k)^t = a^\alpha \cdot b^\beta$, where*

$$\alpha = s_k(1 + r^{f_{k-2}} + r^{2f_{k-2}} + \cdots + r^{(t-1)f_{k-2}}), \;\; \beta = tf_{k-2}.$$

Proof. The proof follows by induction on t and using the relation $b^y a^x = a^{xr^y} b^y$. ■

Lemma 2.3. *For every $k \geq 2$, $t''_k = 1 + \sum_{i=1}^{k-1} t'_i$ and for every $k \geq 3$,*

$$t_{2^{2k-2}-2} \equiv 1 \,(\mathrm{mod}\, 2^k), \qquad t'_{2^{2k-2}-2} \equiv 0 \,(\mathrm{mod}\, 2),$$
$$t_{2^{2k-2}-1} \equiv 0 \,(\mathrm{mod}\, 2^k), \qquad t'_{2^{2k-2}-1} \equiv 1 \,(\mathrm{mod}\, 2),$$
$$t_{2^{2k-2}} \equiv 0 \,(\mathrm{mod}\, 2^k), \qquad t'_{2^{2k-2}} \equiv 0 \,(\mathrm{mod}\, 2).$$

Proof. For every $i \geq 1$, $t''_{i+1} - t''_i = t'_i$ and the first relation follows immediately for every $k \geq 2$. To complete the proof we may use induction on k. ■

We use the following lemma to optain the necessary presentations for $\mathrm{Aut}(D_{2n})$ and $\mathrm{Aut}(Q_{2^n})$.

Lemma 2.4.

(i) *For every $n \geq 3$, $\mathrm{Aut}(D_{2n}) \cong \mathrm{Hol}(Z_n)$,*

(ii) *for every $n \geq 4$, $\mathrm{Aut}(Q_{2^n}) \cong \mathrm{Hol}(Z_{2^{n-1}})$,*

where, $\mathrm{Hol}(Z_n)$ is the holomorph of the cyclic group Z_n.

Proof. See [8] and [11], respectively. ■

In computing $\mathrm{Hol}(Z_n)$ we consider two cases for $\mathrm{Aut}(Z_n)$: cyclic and non-cyclic. If it is cyclic, then $\mathrm{Aut}(D_{2n})$ is a 2-generated group, otherwise $\mathrm{Aut}(D_{2n})$ is 3-generated. The following lemma gives all the possible cases for n and the respective presentations for $\mathrm{Aut}(D_{2n})$ and $\mathrm{Aut}(Q_{2^n})$.

Lemma 2.5. $\mathrm{Aut}(Z_2) = 1$, $\mathrm{Aut}(Z_4) = Z_2$ *and,*

(i) $\mathrm{Aut}(Z_n) \cong Z_{\varphi(n)}$ *if $p > 2$ is a prime and $n = p^k$, $k \geq 1$, or $n = 2m$ for every odd positive integer m. (φ is the Eulerian function).*

(ii) $\mathrm{Aut}(Z_n) \cong Z_2 \times Z_{\varphi(n)/2}$ *if either $n = 2^k$, $k \geq 3$ or n is the product of two coprime odd numbers.*

(iii) $\mathrm{Aut}(Z_n) \cong Z_{2^{k-1}} \times Z_{\varphi(m)}$ *if $n = 2^k.m$, $k \geq 2$ and m is odd.*

Proof. $\mathrm{Aut}(Z_n)$ is abelian (see for example, 1.5.5. of [7]), and getting generating sets for $\mathrm{Aut}(Z_n)$ is possible in each case. ■

Lemma 2.6. *Let $G_1 = \mathrm{Aut}(D_{2n})$ and $G_2 = \mathrm{Aut}(Q_{2^n})$. Then G_1 may be presented by*

(i) *If $n = p^k$, $k \geq 1$ or $n = 2m$ (m odd), then*

$$G_1 = \langle a, b \,|\, a^n = 1, \, b^{\varphi(n)} = 1, \, a^{b^{-1}} = a^r \rangle, \;\; (r, n) = 1, \;\; 2 \leq r \leq n-1.$$

(ii) *If* $n = 2^k$, $k \geq 3$, *or* $n = m_1 \cdot m_2$ *where* m_1 *and* m_2 *are coprime odd integers, then*

$$G_1 = \langle a, b, c \mid a^n = b^2 = c^{\varphi(n)/2} = [b, c] = 1,\ a^{b^{-1}} = a^{-1},\ a^{c^{-1}} = a^{-1} \rangle.$$

(iii) *If* $n = 2^k.m$, $k \geq 2$, *and* $m \geq 3$ *is an odd integer, then*

$$G_1 = \langle a, b, c \mid a^n = b^{2^{k-1}} = c^{\varphi(m)} = [b, c] = 1,\ a^{b^{-1}} = a^{-1},\ a^{c^{-1}} = a^{-1} \rangle.$$

(iv) $\mathrm{Aut}(Q_8) = S_3$ *and for every* $k \geq 4$, G_2 *may be presented by*

$$G_2 = \langle a, b, c \mid a^{2^{k-1}} = b^2 = c^{2^{k-3}} = [b, c] = 1,\ a^{b^{-1}} = a^{-1},\ a^{c^{-1}} = a^{-1} \rangle.$$

Proof. Consider Lemma 2.5 and the corresponding presentations for the semi-direct products $Z_n : (Z_p \times Z_q)$ in the cases (ii) - (iv), and the presentation of $Hol(Z_n)$ in the case (i). ■

Theorem 2.7. *If* $\mathrm{Aut}(Z_n)$ *is cyclic, then the* LEN *of* $\mathrm{Aut}(D_{2n})$ *is the least integer* k *such that all the conditions* $s_{k+1} \equiv 1 \pmod n$, $s_{k+2} \equiv 0 \pmod n$, $f_k \equiv 1 \pmod{\varphi(n)}$, *and* $f_{k-1} \equiv 0 \pmod{\varphi(n)}$ *hold.*

Proof. In this case, $n = p^\alpha$ ($p \geq 3$, $\alpha \geq 1$) or $n = 2m$ (m is odd). Then consider Lemmas 2.6(i) and 2.1. So, $k =$ LEN is the least integer such that $a^{s_{k+1}} b^{f_{k-1}} = a$ and $a^{s_{k+2}} b^{f_k} = b$, and the result follows immediately. ■

By the definition the least integer t is such that $a_{t+1} = a_1\theta$ and $a_{t+2} = a_2\theta$ hold for some automorphism θ of a group, we see that k/t is the order of θ and the orders of a_1 and a_{t+1} are equal; similarly the orders of a_2 and a_{t+2} are equal. ($t =$BLEN, $k =$ LEN). Using this fact, we get

Corollary 2.8. *If* $\mathrm{Aut}(Z_n)$ *is cyclic, the integer* t *is the* BLEN *of* $\mathrm{Aut}(D_{2n})$ *if and only if all of the following conditions hold*

$s_{t+1}(r^{nf_{t-1}-1})/(r^{f_{t-1}-1}) \equiv 0 \pmod n$,

$s_{t+2}(r^{\varphi(n)f_t} - 1) \equiv 0 \pmod n$,

$nf_{t-1} \equiv 0 \pmod{\varphi(n)}$,

where $(r, n) = 1$ *and* $2 \leq r \leq n - 1$.

Proof. The result follows by Lemmas 2.6(i) and 2.2. ■

3. Results

Let $G = \mathrm{Aut}(D_{2n})$. Then using the preliminary results of Sec. 2, we get

Theorem A. *If* $n = 2^k$, $k \geq 3$, *then* LEN$= 2^{2k-2}$ *and* BLEN$= 4$.

Theorem B. *If* $n = p$ *or* $n = 2p$ *(for every prime* $p > 3$*), then* LEN$=$ $p\times$ BLEN*if* $n = 3$ *or* $n = 6$, *then* LEN$= 6$ *and* BLEN$= 3$.

Theorem C. *If* $n = 2^k.3$, $k \geq 2$ *then* LEN$= 2\times$ BLEN.

Corollary A.1. *If* $G = \mathrm{Aut}(Q_{2^k})$, $k \geq 4$, *then* LEN $= 2^{2k-4}$ *and* BLEN $= 4$.

4. Proofs

Proof of Theorem A. If $n = 2^k$ then G is 3-generated and consider 2.6(ii). For every $i \geq 5$, every element a_i of the sequence

$$a_1 = a,\ a_2 = b,\ a_3 = c,\ a_4 = abc,\ a_5 = bcabc,\ a_6 = cabcbcabc, \ldots$$

can be written as follows:

$$a_i = a^{t_{i-3}}.b^{t'_{i-3}}.c^{1+\sum_{j=1}^{i-4} t'_j}$$

where $\{t_i\}$ and $\{t'_i\}$ are the sequences of numbers defined in Section 2. This may be proved by induction on i and by considering the relations $a^{2^k} = 1$, $b^2 = 1$, $[b, c] = 1$, $ba = a^{-1}b$ and $ca = a^{-1}c$. It is also obvious that the powers of a, b and c reduce modulo 2^k, 2 and 2^{k-2}, respectively. Now let $l =$ LEN. Then, $a_{l+1} = a$, $b_{l+1} = b$, and $c_{l+1} = c$. Considering 2.3 yields LEN $= 2^{2(k-1)}$.

To show that BLEN $= 4$ we see that for every $i \geq 1$,

$$\mathrm{order}(a_i) = \begin{cases} 2^k, & i \equiv 1 \pmod 4 \\ 2, & i \equiv 2 \pmod 4 \\ 2^{k-2}, & i \equiv -1 \pmod 4 \\ 2^k, & i \equiv 0 \pmod 4. \end{cases}$$

because, $(abc)^2 = a^2c^2$, $(abc)^{2^{k-2}} = a^{2^{k-2}} \cdot c^{2^{k-2}} = a^{2^{k-2}}$ and then, $(abc)^{2^k} = a^{2^k} = 1$, i.e., a, b, c and abc have orders 2^k, 2, 2^{k-2} and 2^k, respectively. Using induction on i we get the orders of elements of the Fibonacci orbit $F_{a,b,c}$ as follows:

$$2^k, 2, , 2^{k-2}, 2^k, 2^k, 2, 2^{k-2}, 2^k, \ldots$$

So, BLEN $= 4$. This completes the proof. ∎

Proof of Theorem B. If $n = p$ or $n = 2p$, G is a 2-generated group. Since LEN/BLEN is the order of some automorphism of G such that $a\theta = a_{t+1}$ and $b\theta = a_{t+2}$ ($t =$ BLEN), it is sufficient to find this automorphism which should be of order p. Define $\theta \in \mathrm{Aut}\,(\mathrm{Aut}(D_{2p}))$ as follows:

$$\theta : \begin{cases} a \to a \\ b \to a^{2r}b \end{cases}, \quad 2 \leq r \leq p-1,$$

θ is of order p, for, $a\theta^p = a$ and

$$\begin{aligned} b\theta^p &= (a^{2r}b)\theta^{p-1} = a^{2r}((a^{2r}b)\theta^{p-2}) \\ &= a^{4r}(b\theta^{p-2}) = a^{4r}((a^{2r}b)\theta^{p-3}) \\ &= a^{6r}(b\theta^{p-3}) = \cdots = a^{2(p-1)r}(a^{2r}b) \\ &= a^{2pr}b = b. \end{aligned}$$

Let k = LEN and $m = k/t$. For every $i \geq 1$, we get $a\theta^i = a_{it+1}$ and $b\theta^i = a_{it+2}$ (by the action of θ on the Fibonacci orbit). Since $F_{a,b} = F_{a\theta,b\theta}$, then $a\theta^m = a_{mt+1}\theta = a\theta = a$ and $b\theta^m = a_{mt+2}\theta = b$, i.e., θ is of order m, so $k = pt$.

If $n = 2p$ we proceed in the similar way and define $\phi \in Aut(Aut(D_{4p}))$ as follows:

$$\phi : \begin{cases} a \to a \\ b \to a^{2r}b \end{cases}, \quad (r, 2p) = 1,\ 3 \leq r \leq p-1.$$

Then ϕ is also of order p, for $\phi(2p) = \phi(p) = p - 1$. Then $k/t = p$ holds in this case.

To complete the proof, let $p = 3$. Then by Lemma 2.6(i) we get

$$\mathrm{Aut}(D_6) = \langle a, b |\, a^3 = b^2 = 1,\ bab^{-1} = a^2\rangle,$$

$$\mathrm{Aut}(D_{12}) = \langle a, b |\, a^5 = b^4 = 1,\ bab^{-1} = a^4\rangle,$$

and the Fibonacci orbits are

$$a,\ b,\ ab,\ a^2,\ a^2b,\ ab,$$

and

$$a,\ b,\ ab,\ a^{-1}b^2,\ a^2b^3,\ ab,$$

respectively. Then LEN = 6 and BLEN = 3 hold for each group. ∎

Proof of Theorem C. In this case $G = \mathrm{Aut}(D_{2n})$ has a presentation isomorphic to

$$G_2 = \langle a, b, c |\, a^{3.2^k} = b^{2^{k-1}} = c^2 = [b, c] = 1,\ ba = a^{-1}b,\ ca = a^{-1}c\rangle.$$

Consider the sequence

$$a_1 = a,\ a_2 = b,\ a_3 = c,\ a_4 = abc,\ a_5 = bcabc, \ldots$$

and define $\theta \in \mathrm{Aut}\ (\mathrm{Aut}(D_{2n}))$ as follows:

$$\theta : \begin{cases} a \to a^{3.2^{k-1}+1} \\ b \to a^{3.2^{k-1}} \cdot b \\ c \to c. \end{cases}$$

Since $F_{a,b,c} = F_{a\theta,b\theta,c\theta}$, it is sufficient to show that $\theta^2 = 1$, i.e., LEN = 2× BLEN. We have

$$a\theta^2 = (a\theta)\theta = (a^{3.2^{k-1}+1})\theta = a^{(3.2^{k-1}+1)^2}.$$

However, for every $k \geq 2$,

$$3.2^k \,|\, (3.2^{k-1} + 1)^2 - 1 = 3.2^k(1 + 3.2^{k-2}).$$

Then $a\theta^2 = a$. Similarly, $c\theta^2 = c$ and

$$b\theta^2 = (b\theta)\theta = (a^{3.2^{k-1}}b)\theta = (a^{3.2^{k-1}})\theta.(b\theta) = a^{(3.2^{k-1})(3.2^{k-1}+1)} \cdot a^{3.2^{k-1}} \cdot b$$
$$= a^{3.2^k(3.2^{k-2}+1)}.b.$$

This completes the proof. ■

5. Computations

When $\mathrm{Aut}(Z_n)$ is cyclic we have formulated the results of Theorem 2.7 and Corollary 2.8, and with a simple procedure, FLAUT [1], it is possible to compute LEN and BLEN. If $\mathrm{Aut}(Z_n)$ is not cyclic we have the results of Sec. 3 to get LEN and BLEN.

In the definition of BLEN we see that there are some automorphisms θ such that order(θ) = LEN/BLEN. In the Table 1 we may also consider the order of such authomorphisms θ, where we have called them the special automorphisms. The exact definition of θ is also given. Our computations show that θ is the identity in some cases.

6. Results and Conclusion

The presentation of $Aut(Q_{2^k})$, $k \geq 4$ is similar to that of $\mathrm{Aut}(D_{2n})$ which originated from Lemma 2.6(ii). So Corollary A1 may be proved in a similar way as Theorem A. The following remarks complete and generalize Theorems A and C.

Remark 1. If $n = 2^k$ $(k \geq 3)$, then $\theta \in$ Aut $(\mathrm{Aut}(D_{2n}))$ defined by

$$\theta \begin{cases} a \to abc \\ b \to ac^2 \\ c \to a^{-2}bc^4 \end{cases}$$

is of order LEN/BLEN $= 2^{2k-4}$.

Proof. The proof follows by using the result of Theorem A. ■

Remark 2. For every prime $p \geq 3$ and for every integer $k \geq 2$, if $n = p.2^k$ then for the group $\mathrm{Aut}(D_{2n})$, LEN $= p(p-1).2^k$ and BLEN $= p(p-1).2^{k-1}$.

Proof. We define $\theta \in$ Aut $(\mathrm{Aut}(D_{2n}))$ as follows:

$$\theta : \begin{cases} a \to a^{1+p.2^{k-1}} \\ b \to a^{p.2^{k-1}} \cdot b \\ c \to c. \end{cases}$$

Then we get $\theta^2 = 1$, and the rest of the proof is similar to the proof of Theorem C. ■

Table 1

G	LEN	BLEN	Special automorphism
$\mathrm{Aut}(D_6)$	6	3	$\theta: \begin{cases} a \to a^{-1} \\ b \to a^{-1}b \end{cases}$
$\mathrm{Aut}(D_8)$	6	3	$\theta: \begin{cases} a \to a^{-1} \\ b \to a^{2}b \end{cases}$
$\mathrm{Aut}(D_{10})$	30	6	$\theta: \begin{cases} a \to a \\ b \to a^{4}b \end{cases}$
$\mathrm{Aut}(D_{12})$	6	3	$\theta: \begin{cases} a \to a^{-1}b^2 \\ b \to a^{2}b^3 \end{cases}$
$\mathrm{Aut}(D_{14})$	168	24	$\theta: \begin{cases} a \to a \\ b \to a^{4}b \end{cases}$
$\mathrm{Aut}(D_{16})$	16	4	$\theta: \begin{cases} a \to abc \\ b \to a \\ c \to a^{-2}b \end{cases}$
$\mathrm{Aut}(D_{18})$	24	24	$\theta = \mathrm{id}_{\mathrm{Aut}(D_{18})}$
$\mathrm{Aut}(D_{20})$	24	24	$\theta = \mathrm{id}_{\mathrm{Aut}(D_{20})}$
$\mathrm{Aut}(D_{22})$	660	60	$\theta: \begin{cases} a \to a \\ b \to a^{4}b \end{cases}$
$\mathrm{Aut}(D_{24})$	24	12	$\theta: \begin{cases} a \to a^{7} \\ b \to a^{6}b \\ c \to c \end{cases}$
$\mathrm{Aut}(D_{26})$	312	24	$\theta: \begin{cases} a \to a \\ b \to a^{4}b \end{cases}$
$\mathrm{Aut}(D_{28})$	168	24	$\theta: \begin{cases} a \to a \\ b \to a^{6}b \end{cases}$
$\mathrm{Aut}(D_{30})$	240	240	$\theta = \mathrm{id}_{\mathrm{Aut}(D_{30})}$
$\mathrm{Aut}(D_{32})$	64	4	$\theta: \begin{cases} a \to abc \\ b \to ac^2 \\ c \to a^{-2}b \end{cases}$
$\mathrm{Aut}(D_{34})$	406	24	$\theta: \begin{cases} a \to a \\ b \to a^{4}b \end{cases}$
$\mathrm{Aut}(D_{36})$	24	24	$\theta = \mathrm{id}_{\mathrm{Aut}(D_{36})}$
$\mathrm{Aut}(D_{38})$	456	24	$\theta: \begin{cases} a \to a \\ b \to a^{4}b \end{cases}$
$\mathrm{Aut}(D_{40})$	80	40	$\theta: \begin{cases} a \to a^{11} \\ b \to a^{10}b \\ c \to c \end{cases}$

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