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Short Communication

Stability Radii of Linear Differential Algebraic Equations

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The stability radius, introduced by Hinrichsen and Pritchart [2], is a measure for the robustness of a stable system. It is defined as the smallest value ρ of the norm of real or complex perturbations destabilizing the system.

In this article, we deal with the problem of stability radius for systems described by a differential algebraic equation of the form:

$$AX'(t) + BX(t) = 0, \tag{1}$$

with constant matrices A and B where the matrix A is degenerate and the pencil $\{A, B\}$ is regular.

Let the pencil $\{A, B\}$ have the index k, $k \geq 0$ and G, H be non-singular matrices such that

$$A = G\,\text{diag}\big(I_r, N\big)H; \qquad B = G\,\text{diag}\big(W, I_{m-r}\big)H, \tag{2}$$

where I_r and I_{m-r} are unit matrices in $R^{r\times r}$ and $R^{(m-r)\times(m-r)}$, respectively. Further, $W \in R^{r\times r}$ and N is a k - nilpotent matrix of the Jordan box form.

System (1) is equivalent to

$$\begin{aligned} &Y'(t) + WY(t) = 0, \\ &Z \equiv 0, \quad Y \in \mathbb{R}^r, \quad Z \in \mathbb{R}^{m-r}. \end{aligned} \tag{3}$$

We know that system (3) is asymptotically stable if and only if the eigenvalues of the matrix W lie within the positive complex half-plane. On the other hand, $\sigma(A, B) = \sigma(W)$. Thus, the trivial solution of system (1) is asymptotically stable if and only if the spectrum of the pencil $\{A, B\}$ lies within the negative complex half-plane $\mathbb{C}_- = \{z \in \mathbb{C} : \Re z \leq 0\}$.

We now study some simple properties of the perturbed system

$$AX'(t) + (B + \Delta)X(t) = 0, \tag{4}$$

where Δ, called disturbance matrix, is in $\mathbb{C}^{m\times m}$.

It is seen that the spectrum $\sigma(A, B+\Delta)$ of the pencil $\{A, B+\Delta\}$ does not converge to the spectrum $\sigma(A, B)$ of the pencil $\{A, B\}$ as $\Delta \to 0$. The continuity of the spectrum takes place only in the index 1 case. More precisely, we have the following theorem.

Theorem 1. *Suppose the pencil $\{A, B\}$ has the index k. Then*

(a) *if $k = 1$, the spectrum $\sigma(A, B+\Delta)$ of the pencil $\{A, B+\Delta\}$ converges to the spectrum $\sigma(A, B)$ of the pencil $\{A, B\}$ as $\Delta \to 0$;*

(b) *if $k \geq 2$, then for any $\varepsilon > 0$ and $\delta > 0$, there exists a disturbance Δ satisfying*

$$|\Delta| < \varepsilon$$

and there is $\lambda_0 \in \sigma(A, B+\Delta) \setminus \sigma(A, B)$ such that $\lambda_0 > \delta$.

Look again at the perturbed system (4). We suppose that the unperturbed system (1) is asymptotically stable. We denote by

$$\mathcal{U} = \left\{\Delta \in \mathbb{C}^{m\times m} : (3.1) \text{ is either irregular or unstable}\right\}$$

the set of "bad" disturbances. Let

$$d := \inf\{|\Delta| : \Delta \in \mathcal{U}\},$$

which is called the complex stability radius of the pencil $\{A, B\}$. From Lemma 2, it follows that if $\mathrm{ind}(A, B) \geq 2$, then for any $\varepsilon > 0$, we can choose a disturbance matrix Δ such that $|\Delta| < \varepsilon$ and (3.1) is unstable. This means that $d = 0$ and the problem becomes trivial. Thus, we study only the case of index 1. It is easy to see that in this case, d is a positive number.

Taking a sequence (Δ_n) in $\mathcal{U}$ such that $\lim_{n\to\infty} \Delta_n = \Delta$ and $\lim_{n\to\infty} |\Delta_n| = d$, we consider three cases:

(a) The pencil $\{A, B+\Delta\}$ has the index 1.

In this case, the continuity of the spectrum of the pencil $\{A, B+\Delta'\}$ shows that there exists $s \in i\mathbb{R}$ and a vector $0 \neq x \in \mathbb{C}^m$ such that

$$(\lambda A + B + \Delta)x = 0.$$

Hence,

$$d = |\Delta| \geq |G(\lambda)|^{-1} \geq \left(\sup_{s\in i\mathbb{R}} |G(s)|\right)^{-1}$$

(b) $\mathrm{ind}\{A, B+\Delta\} \geq 2$.

For any $\varepsilon > 0$, we choose Δ' such that $|\Delta'| < \varepsilon$. Then, in a similar way, we obtain

$$d + \varepsilon \geq |\Delta + \Delta'| \geq |G(\lambda)|^{-1} \geq \left(\sup_{s\in i\mathbb{R}} |G(s)|\right)^{-1}.$$

Because ε is arbitrary, then

$$d = |\Delta| \geq \left(\sup_{s\in i\mathbb{R}} |G(s)|\right)^{-1}.$$

(c) The pencil $\{A, B + \Delta\}$ is irregular. The proof is similar as in (a).

We now prove the inverse relation. We can show that if $\operatorname{ind}(A, B) = 1$, then the map $s \to |G(s)|$ attains the maximum over $i\mathbb{R}$. Let

$$s_0 := \left(\operatorname*{argmax}_{s\in i\mathbb{R}} |G(s)|\right)$$

and $u \in \mathcal{C}^m$ such that

$$|u| = 1; \quad |G(s_0)| = |G(s_0)u|.$$

A corollary of Hahn–Banach theorem shows that there is a linear functional y^* defined on $\mathbb{C}^m$ such that

$$|y^*| = 1; \quad y^*(G(s_0)u) = |G(s_0)u| = |G(s_0)|.$$

We put

$$D = -|G(s_0)|^{-1} u \cdot y^* \in \mathbb{C}^{m\times m}. \tag{5}$$

It is clear that

$$|D| = |G(s_0)|^{-1} = \left(\max_{s\in i\mathbb{R}} |G(s)|\right)^{-1}.$$

Then $D \in \mathcal{U}$ and

$$d \leq \left(\max_{s\in i\mathbb{R}} |G(s)|\right)^{-1}.$$

In order to have a general formula, we prove the following lemma.

Lemma 2. *If* $\operatorname{ind}(A, B) \geq 2$, *then the function* $s \to |G(s)| = |(sA + B)^{-1}|$ *is unbounded on* $i\mathbb{R}$.

Summing up, we have the following theorem:

Theorem 3. *The stability radius of (1) can be calculated by*

$$d = \left(\sup_{s\in i\mathbb{R}} |G(s)|\right)^{-1}.$$

In the case of the index 1, the function $|G(s)|$ *attains the maximum over* $i\mathbb{R}$ *and the matrix* D *given by (5) satisfies* $|D| = d$ *and* $D \in \mathcal{U}$.

Example 1. Let

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad B := \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have $\operatorname{ind}(A, B) = 1$; $\sigma(A, B) = \{-2, -1\} \subset \mathring{\mathbb{C}}_-$. Therefore, the pencil $\{A, B\}$ is asymptotically stable. It is easy to see that $d = 2/3$ and

$$D = \begin{pmatrix} 0 & \frac{2}{3} & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} & 0 \end{pmatrix}$$

with $\sigma(A, B + D) = \{-7/3, 0\}$.

Example 2. Let

$$A := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad B := \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

We have $\operatorname{ind}(A, B) = 2$; $\sigma(A, B) = \{-1/3\} \subset \overset{\circ}{\mathbb{C}}_-$. It is easy to see that

$$G(s) = (sA + B)^{-1} = \begin{pmatrix} \frac{s+1}{3s+1} & -\frac{1}{3s+1} & -\frac{s^2}{3s+1} \\ \frac{s-1}{(3s+1)} & \frac{2}{3s+1} & -\frac{s(s+1)}{3s+1} \\ -\frac{s+1}{3s+1} & \frac{1}{3s+1} & \frac{s^2+3s+1}{3s+1} \end{pmatrix}.$$

We see that $|G(s)|$ is unbounded on $i\mathbb{R}$. Thus, $d = 0$.

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