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Slenderness for Rings

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Abstract. By analogy with the well-known abelian group concept, we define a ring A to be slender if every ring homomorphism $\mathbb{Z}^N \to A$ depends only on finitely many components of $\mathbb{Z}^N$, the direct product of $\aleph_0$ copies of $\mathbb{Z}$. The relationships between this notion, slenderness for abelian groups, and ultraconnected rings are explored. A topological characterization of slenderness for rings is also obtained.

1. Introduction

The class of slender abelian groups is an interesting and well-studied class and has topological as well as purely algebraic characterizations. The characteristic property of a slender group G that every homomorphism from a direct product into G be determined by its effect on finitely many components has obvious analogs for other kinds of algebraic structures, so it seems worthwhile to seek out versions of slenderness in other such settings. A fair amount of work has been done on slender modules (see, e.g., [12, 15] and the references therein). A related concept has been investigated by Börger and Rajagopalan [2] and subsequently by Henriksen and Smith [10]. A ring A with identity is said to be *ultraconnected* if every unital homomorphism from the direct product $\mathbb{Z}^N$ of $\aleph_0$ copies of the integers to A depends on a single component.

We introduce a concept of slenderness for rings based on the requirement that every homomorphism from $\mathbb{Z}^N$ be determined by its effect on finitely many e_i, where $e_1 = (1, 0, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, etc. This is a stronger condition than is obtained by the requirement that $f(e_i) = 0$ for all but finitely many values of i (in contrast to the abelian group case). Non-unital rings are thus drawn into the picture, but slenderness is only really interesting when there *are* homomorphisms from $\mathbb{Z}^N$ to a ring, so the ring needs some non-zero idempotents. Rings, whose additive groups are slender, are slender rings and ultraconnected rings are slender. These are, however, not the only sources of slender rings. We obtain many examples of slender rings and some characterizations of slenderness for rings and compare these with known characterizations of slender abelian groups. We also shed a little light on the structure of ultraconnected rings.

Our notation is fairly standard: N, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ represent the natural numbers, integers, rationals, reals, and complex numbers with or without structure, "group" always means "abelian group", the conventions of [7] are generally adhered to, all rings are associative, but no identity is assumed. It should be noted that elsewhere in the literature, "slender rings" means "rings which are slender as a module over itself".

2. Results

In what follows, ΠA_i will denote the direct product of a set $\{A_i : i \in I\}$ of rings or abelian groups, a typical element of ΠA_i will be called (a_i), and $\pi_j : \Pi A_i \to A_j$ will denote the jth projection. When $I = N$, the set of natural numbers and $A_i = A$ for each i, ΠA_i will also be called A^N, while the direct sum $\oplus A_i$ will also be called $A^{(N)}$. When A has an identity 1, the element of A^N which has 1 in position n and 0 elsewhere will be called e_n. Finally, in any ΠA_i, with any (a_i), we associate elements $\widehat{a_i}$ defined as follows: the ith entry of $\widehat{a_i}$ is a_i and all others are 0. Thus, for rings with identity, $\widehat{a_j} = (a_i)\, e_j$.

Definition 1. (Fuchs, see [6]) *A torsion-free abelian group G is slender if, for every homomorphism $f : \mathbb{Z}^N \to G$, we have $f(e_i) = 0$ for all but finitely many values of i.*

Definition 2. [2] *A ring A with identity is ultraconnected if, for every set $\{R_i : i \in N\}$ of rings with identity and every identity-preserving homomorphism $f : \Pi R_i \to A$, there exists an index j such that*

$$f((a_i)) = f(\widehat{a_j})$$

for all $(a_i) \in \Pi R_i$.

A group G is slender if and only if, for every homomorphism $f : \mathbb{Z}^N \to G$, there exists a finite subset $\{i_1, i_2, ..., i_n\}$ of N such that

$$f((a_i)) = \sum_{j=1}^{n} f(\widehat{a}_{i_j}) = \sum_{j=1}^{n} f(a_{i_j} e_{i_j}) = \sum_{j=1}^{n} a_{i_j} f(e_{i_j})$$

[7, Vol. II, p. 159]. Since a ring is ultraconnected if and only if it satisfies the condition of Definition 2 with each $R_i = \mathbb{Z}$ [2, 1.2], it seems reasonable to seek a generalization of ultraconnectedness based on slenderness which can usefully be applied to rings without identity, whose elucidation can be assisted by the substantial established theory of slender groups. Accordingly, we make the following definition.

Definition 3. *A ring A is slender if*

(i) *the additive group of A is torsion-free; and*
(ii) *for every ring homomorphism $f : \mathbb{Z}^N \to A$, there exist $i_1, i_2, ..., i_n$ such that*

$$f((a_i)) = \sum_{j=1}^{n} f(\widehat{a}_{i_j}) = \sum_{j=1}^{n} f((a_{i_j}) e_{i_j}) = \sum_{j=1}^{n} a_{i_j} f(e_{i_j})$$

for all $(a_i) \in \mathbb{Z}^N$.

Condition (ii) is trivially satisfied when there are no homomorphisms $f : \mathbb{Z}^N \to A$ and so is interesting only in the presence of idempotents in A. As we shall see later, in the presence of an identity, slenderness can be characterized in terms of unital homomorphisms. We include (i) in the definition because (ii) is effectively never satisfied in the presence of additive torsion (see Theorem 3).

Later we shall examine topological aspects of slenderness. For now we note a characterization with a topological flavor which is an exact counterpart of one characterization of slender groups.

Proposition 1. *The following conditions are equivalent for a torsion-free ring A.*

(i) *A is slender.*

(ii) *For every homomorphism $f : \mathbb{Z}^N \to A$, there exists n such that*

$$\prod_{i>n} \mathbb{Z}_i \subseteq Ker(f) \quad (\mathbb{Z}_i = \mathbb{Z} \text{ for each } i).$$

(iii) Every homomorphism $f : \mathbb{Z}^N \to A$ *is continuous with respect to the discrete topology on A and the product of the discrete topologies on $\mathbb{Z}^N$.*

Proof. Clearly (i)$\Leftrightarrow$(ii).

(ii)$\Leftrightarrow$(iii) This follows from the fact that $\{\Pi_{i>1}\mathbb{Z}_i,\ \Pi_{i>2}\mathbb{Z}_i, ..., \Pi_{i>n}\mathbb{Z}_i, ...\}$ is a basis of neighborhoods of 0 for the product topology. ■

We now consider some examples of slender rings.

Proposition 2. *The following conditions are equivalent for a torsion-free ring A with identity.*

(i) *A is slender and has no idempotents $\neq 0, 1$.*

(ii) *A is ultraconnected.*

Proof. (i)$\Rightarrow$(ii) Let $f : \mathbb{Z}^N \to A$ be a unital ring homomorphism. Then there exist $i_1, i_2, ..., i_n$ such that $f((a_i)) = \Sigma_{j=1}^{n} f(\widehat{a}_{i_j})$ for all (a_i). In particular,

$$1_A = f(1_{\mathbb{Z}^N}) = \sum_{j=1}^{n} f(e_{i_j})$$

and we can clearly assume $f(e_{i_1}), f(e_{i_2}), \ldots, f(e_{i_n}) \neq 0$, and thus, $f(e_{i_1}) = \cdots = f(e_{i_n}) = 1$. But if $i_1 \neq i_2$, then $e_{i_1}e_{i_2} = 0$, so $f(e_{i_1})f(e_{i_2}) = 0$. We conclude that $n = 1$. But then A is ultraconnected.

(ii)$\Rightarrow$(i) By [2], 1.4, 0, and 1 are the only idempotents in A, so every non-zero homomorphism $g : \mathbb{Z}^N \to A$ is unital, and hence, for some j, we have $g((a_i)) = g(\widehat{a}_j)$ for all (a_i). Thus, A is slender. ■

Proposition 3. *If the additive group of A is slender, then A is slender.*

Proof. Every ring homomorphism $f : \mathbb{Z}^N \to A$ is also an abelian group homomorphism. ■

Now, $\mathbb{Z}$ is a slender group, and all direct sums of slender groups are slender. Thus, for instance, if A is a ring with a slender additive group, then all matrix rings over A (including various rings of infinite matrices) and all semigroup rings $A[S]$ are slender. In particular, integral semigroup rings are slender.

The semigroup ring examples can be extended by the use of the result, proved below (Corollary 1) that the class of slender rings is closed under extensions.

Note that we already have enough information to show that, for rings with identity, slenderness is more general than ultraconnectedness (e.g., $\mathbb{Z} \oplus \mathbb{Z}$ is slender by Proposition 3) while slenderness is not a purely additive property (e.g., $\mathbb{Q}$ and $\mathbb{R}$ are ultraconnected [2], whence slender, but their additive groups are not slender).

We now consider some further properties of slender rings, including closure properties for the class of such rings. The first result is obvious but useful.

Proposition 4. *Subrings of slender rings are slender.*

The following lemma will also prove useful.

Lemma 1. *Let A be a ring, $f : \mathbb{Z}^N \to A$ a homomorphism, $\{i_1, i_2, ..., i_n\}$ a finite subset of N. Define $\widetilde{f} : \mathbb{Z}^N \to A$ by setting*

$$\widetilde{f}((a_i)) = f((a_i)) - \sum_{j=1}^{n} f(\widehat{a_{i_j}}) \quad \textit{for all } (a_i).$$

Then $\widetilde{f}$ is a homomorphism.

Proof. It is notationally convenient, and involves no loss of generality, to assume that $\{i_1, i_2, ..., i_n\} = \{1, 2, ..., n\}$. Now, for $(a_i), (b_i) \in \mathbb{Z}^N$, we have

$$\widetilde{f}((a_i))\widetilde{f}((b_i)) = f((a_i))f((b_i)) - f((a_i))\sum_{j=1}^{n} f(\widehat{b_j})$$

$$-\sum_{j=1}^{n} f(\widehat{a_j})f((b_i)) + \sum_{k,j=1}^{n} f(\widehat{a_k})f(\widehat{b_j})$$

$$= f((a_i))f((b_i)) - \sum_{j=1}^{n} f((a_i))f(\widehat{b_j})$$

$$-\sum_{j=1}^{n} f(\widehat{a_j})f((b_i)) + \sum_{k,j=1}^{n} f(\widehat{a_k})f(\widehat{b_j})$$

$$= f((a_i)(b_i)) - \sum_{j=1}^{n} f((a_i)\widehat{b_j}) - \sum_{j=1}^{n} f(\widehat{a_j}(b_i)) + \sum_{k,j=1}^{n} f(\widehat{a_k}\widehat{b_j})$$

$$= f((a_i)(b_i)) - \sum_{j=1}^{n} f(\widehat{a_j b_j}) - \sum_{j=1}^{n} f(\widehat{a_j b_j}) + \sum_{j=1}^{n} f(\widehat{a_j b_j})$$

$$= \widetilde{f}((a_i)(b_i)).$$

Addition is easier. ■

Proposition 5. *If $J \lhd A$ and if J, A/J are both slender, then so is A.*

Proof. Let $f : \mathbb{Z}^N \to A$ be a homomorphism and let $g : A \to A/J$ be the natural map. Then there exist $i_1, i_2, \ldots, i_n \in N$ such that, for all $(a_i) \in \mathbb{Z}^N$, we have

$$gf((a_i)) = \sum_{j=1}^{n} gf(\widehat{a}_{i_j}).$$

Let $\widetilde{f} : \mathbb{Z}^N \to A$ be as in Lemma 1. Then the image of $\widetilde{f}$ is in J, so there exist $k_1, k_2, \ldots, k_m \in N$ such that $\widetilde{f}((a_i)) = \Sigma_{\ell=1}^{m} \widetilde{f}(\widehat{a}_{k_\ell})$ for every (a_i). But $\widetilde{f}(\widehat{a}_{k_\ell}) = 0$ if $k_\ell \in \{i_1, i_2, \ldots, i_n\}$ and $\widetilde{f}(\widehat{a}_{k_\ell}) = f(\widehat{a}_{k_\ell})$ otherwise. Hence,

$$f((a_i)) = \widetilde{f}((a_i)) + \sum_{j=1}^{n} f(\widehat{a}_{i_j})$$

$$= \sum_{\ell=1}^{m} \widetilde{f}(\widehat{a}_{k_\ell}) + \sum_{j=1}^{n} f(\widehat{a}_{i_j})$$

$$= \sum \{f(a_r) : r \in \{i_1, i_2, \ldots, i_n, k_1, k_2, \ldots, k_m\}\}.$$ ■

Corollary 1. *If A has an ideal J containing no non-zero idempotents and if A/J is slender, then A is slender.*

Corollary 2. *If R is slender, then the following are slender:*

(i) $R[X]$;
(ii) $R[S]$ *for any monoid S without idempotents apart from its identity;*
(iii) $R[[X]]$.

Proposition 6. *Direct sums of slender rings are slender.*

Proof. For finite direct sums, this follows from Proposition 5 by induction. But if $\{A_i : i \in I\}$ is a set of slender rings and $f : \mathbb{Z}^N \to \oplus A_i$ a homomorphism, then there is a finite set $\{i_1, i_2, \ldots, i_n\}$ such that $f(1) \in A_{i_1} \oplus A_{i_2} \oplus \cdots \oplus A_{i_n}$ and thus, the image of f is contained in this subring. The result follows. ■

From Propositions 4 and 6, we obtain

Proposition 7. *Finite subdirect products of slender rings are slender.*

Since clearly $\mathbb{Z}^N$ is not slender and $\mathbb{Z}$ is, finiteness cannot be removed in Proposition 7.

Proposition 8. *If A is a union of a chain of ideals each of which is a slender ring, then A is slender.*

Proof. Let $A = \cup J_\lambda$, where $\{J_\lambda : \lambda \in \Lambda\}$ is a chain of ideals which are slender. For every homomorphism $f : \mathbb{Z}^N \to A$, we have $f(1) \in J_\mu$ for some $\mu \in \Lambda$ and the result follows from the slenderness of J_μ. ■

The next result establishes a connection between slenderness for rings and the original definition of slender groups.

Proposition 9. *If A is slender, then, for every homomorphism $f : \mathbb{Z}^N \to A$, we have $f(e_n) = 0$ for all but finitely many n.*

Proof. If $f((a_i)) = \Sigma_{j=1}^{n} f(\widehat{a}_{i_j})$, then, for $i \notin \{i_1, i_2, ..., i_n\}$, we have $f(e_i) = 0$. ■

In contrast to the abelian group case, the converse to Proposition 8 is false. This follows from Proposition 10 below and the fact that the ring of p-adic integers is not slender (Example 1). A very important characterization of slender groups is that due to Nunke [14]: A torsion-free group G is slender if and only if it has no subgroup isomorphic to $\mathbb{Z}^N$, $\mathbb{Q}$ or the p-adic integers (for any prime p). We shall obtain a (less tidy) analog of this for slender rings in two stages: first an "exclusion characterization" of the rings satisfying the conclusion of Proposition 9, then a characterization of the slender rings within this class. We can extract a little more information, which may be useful, about rings with identity satisfying the conclusion of Proposition 9.

Lemma 2. *Let A be a ring with identity which is additively torsion-free. If A has a subring isomorphic to $\mathbb{Z}^N$, then it has a subring with identity isomorphic to $\mathbb{Z}^N$.*

Proof. Let $\mathbb{Z}^N \subseteq A$ and let $\langle 1_A \rangle$ denote the subgroup (subring) generated by the identity of A.

Suppose first that $\mathbb{Z}^N \cap \langle 1_A \rangle \neq 0$. Let $k = \min\{n \in \mathbb{Z}^+ : k1_A \in \mathbb{Z}^N\}$ and let $k1_A = (n_i)$. Then, for every $(m_i) \in \mathbb{Z}^N$, we have

$$(n_i m_i) = (n_i)(m_i) = k1_A(m_i) = (km_i),$$

so $n_i m_i = km_i$ for each i. It follows that $n_i = k$ for each i, so that $k1_A = (n_i) = k(1, 1, 1, ...)$ and thus $1_A = (1, 1, 1, ...)$, i.e., $\mathbb{Z}^N$ is a unital subring.

Now, consider the case $\mathbb{Z}^N \cap \langle 1_A \rangle = 0$. Here, the subgroup $\mathbb{Z}^N + \langle 1_A \rangle$ is a unital subring isomorphic to the standard unital extension of $\mathbb{Z}^N$. But $\mathbb{Z}^N$ has an identity so $\mathbb{Z}^N + \langle 1_A \rangle = \mathbb{Z}^N \oplus \langle 1_A \rangle \cong \mathbb{Z}^N \oplus \mathbb{Z}$ (ring direct sum) $\cong \mathbb{Z}^N$. ■

Proposition 10. *Let A be a ring with identity which is additively torsion-free. The following conditions are equivalent.*

(i) *A has no subring isomorphic to $\mathbb{Z}^N$.*
(ii) *A has no unital subring isomorphic to $\mathbb{Z}^N$.*
(iii) *Every homomorphism from $\mathbb{Z}^N$ to A takes all but finitely many e_i to 0.*
(iv) *Every identity-preserving homomorphism from $\mathbb{Z}^N$ to A takes all but finitely many e_i to 0.*

Proof. (i)⇒(ii) Obvious.

(ii)⇒(i) Lemma 2.

(iii)⇒(iv) Obvious.

(iv)⇒(ii) If A had a unital subring isomorphic to $\mathbb{Z}^N$, the embedding would take no e_i to 0.

¬(iii)⇒¬(i) Let $f : \mathbb{Z}^N \to A$ be a homomorphism for which $f(e_{i_1}), f(e_{i_2}), f(e_{i_3}), \dots$ (infinitely many terms) are non-zero. If (m_i) is any member of $\mathrm{Ker}(f)$, then

$$m_{i_j} e_{i_j} = \widehat{m_{i_j}} = (m_i) e_{i_j} \in \mathrm{Ker}(f) \text{ for all } j,$$

so $m_{i_j} f(e_{i_j}) = f(m_{i_j} e_{i_j}) = 0$. Since A is torsion-free, we thus have $m_{i_j} = 0$ for each j. In particular, $\mathrm{Ker}(f) \cap \Pi \mathbb{Z}_{i_j} = 0$ so

$$\mathbb{Z}^N \cong \Pi \mathbb{Z}_{i_j} \cong (\Pi \mathbb{Z}_{i_j} + \mathrm{Ker}(f))/\mathrm{Ker}(f) \cong f(\Pi \mathbb{Z}_{i_j}) \subseteq A. \qquad \blacksquare$$

Corollary 3. (To proof) *Let A be an additively torsion-free ring. Then every homomorphism $f : \mathbb{Z}^N \to A$ takes all but finitely many e_i to 0 if and only if A has no subring $\cong \mathbb{Z}^N$.*

Proposition 11. *Let A be a ring which is additively torsion-free and such that for every homomorphism $f : \mathbb{Z}^N \to A$, $f(e_i) = 0$ for all but finitely many i. Then A is slender if and only if it has no non-zero subring which is a homomorphic image of $\mathbb{Z}^N/\mathbb{Z}^{(N)}$.*

Proof. Suppose A has a subring $S \neq 0$ which is a homomorphic image of $\mathbb{Z}^N/\mathbb{Z}^{(N)}$. Let g be the resulting homomorphism

$$\mathbb{Z}^N \to \mathbb{Z}^N/\mathbb{Z}^{(N)} \to S \subseteq A.$$

Then $g(e_i) = 0$ for all i so certainly there is no set $\{e_{i_j} : j = 1, 2, \dots, n\}$ such that $g((a_i)) = \Sigma_{j=1}^n g(\widehat{a}_{i_j})$ and thus A is not slender.

Conversely, suppose A has no non-zero subring which is a homomorphic image of $\mathbb{Z}^N/\mathbb{Z}^{(N)}$, and consider any $f : \mathbb{Z}^N \to A$. Then, for some n, we have $f(e_i) = 0$ for all $i > n$. As in Lemma 1, let $\widetilde{f}((a_i)) = f((a_i)) - \Sigma_{j=1}^n f(\widehat{a}_j)$. Then for every k, we have

$$\widetilde{f}(e_k) = \begin{cases} f(e_k) - f(e_k) & \text{if } k \le n \\ f(e_k) - 0 & \text{if } k > n \end{cases}$$
$$= 0,$$

so there is a surjection $\mathbb{Z}^N/\mathbb{Z}^{(N)} \to \mathrm{Im}(\widetilde{f}) \subseteq A$ and thus, $\mathrm{Im}(\widetilde{f}) = 0$. This means that $f((a_i)) = \Sigma_{j=1}^n f(\widehat{a}_j)$, for all (a_i) and thus, A is slender. $\blacksquare$

Using Propositions 9–11 and Corollary 3, we now obtain

Theorem 1. *A ring A which is additively torsion-free is slender if and only if it has no subring isomorphic to $\mathbb{Z}^N$ or a non-zero homomorphic image of $\mathbb{Z}^N/\mathbb{Z}^{(N)}$.* $\blacksquare$

By 3.4 of [2], every non-zero, torsion-free homomorphic image of $\mathbb{Z}^N/\mathbb{Z}^{(N)}$ has cardinality $2^{\aleph_0}$ (the punch line of the proof requiring 12.24 of [4]) so from Theorem 1, we obtain

Corollary 4. *All countable torsion-free rings are slender.* ■

(Note that the case of Corollary 4 in which the additive group is reduced follows from Saşiada's result [17] that countable reduced torsion-free groups are slender.)

Contrasting with Proposition 10 and Corollary 3, Proposition 11 does not come in a purely unital version; specifically, a non-slender unital ring need not have a *unital* subring which is a non-zero homomorphic image of $\mathbb{Z}^N/\mathbb{Z}^{(N)}$, e.g., consider $\mathbb{Z}^N/\mathbb{Z}^{(N)} \oplus \mathbb{Z}$. The theorem just proved can however be used to obtain a characterization of unital slender rings in terms of unital homomorphisms.

Theorem 2. *Let A be a torsion-free ring with identity. Then A is slender if and only if, for every unital homomorphism $f : \mathbb{Z}^N \to A$, there exists a finite set $\{i_1, i_2, ..., i_n\}$ such that $f((a_i)) = \Sigma_{j=1}^n f(\widehat{a}_{i_j})$ for all (a_i).*

Proof. Suppose A is not slender. If A has a unital subring isomorphic to $\mathbb{Z}^N$, then clearly the embedding map does not satisfy the stated condition. If there is no such subring, then by Lemma 2 and Theorem 1, A has a subring, $B \neq 0$ which is a homomorphic image of $\mathbb{Z}^N/\mathbb{Z}^{(N)}$. If B is a unital subring there is a unital homomorphism $\mathbb{Z}^N \to \mathbb{Z}^N/\mathbb{Z}^{(N)} \to B \subseteq A$ which fails to satisfy the stated condition. Otherwise, let u be the identity of B and observe that $B + \langle 1_A - u\rangle$ is a ring direct sum. There is a unital homomorphism $\Pi_{i>1}\mathbb{Z}_i \to B$ which takes all $e_i (i > 1)$ to 0. Also, there is a unital homomorphism $\mathbb{Z}_1 \to \langle 1_A - u\rangle$ which takes e_1 to $1_A - u$. These combine to give a unital homomorphism $\mathbb{Z}^N \to B \oplus \langle 1_A - u\rangle$. But the identity of $B \oplus \langle 1_A - u\rangle$ is 1_A, so there is a unital homomorphism $g : \mathbb{Z}^N \to A$ such that $g(e_1) \neq 0$ and $g(e_i) = 0$ for all other i. But for this g we have $g(1 - e_1) \neq 0$ while all components of $1 - e_1$ are in the kernel of g. Again, g fails to satisfy our condition and the theorem is proved. ■

The following result is in effect in [2]. A more transparent proof is outlined by Henriksen and Smith towards the end of their paper [10]. We present this proof in detail to facilitate the description of Example 1.

Theorem 3. *Let A be a finite ring with identity. Then there is a unital homomorphism $f : \mathbb{Z}^N \to A$ such that $f(e_i) = 0$ for all i.*

Proof. Let Φ be a non-principal ultrafilter on N. For each $(a_i) \in A^N$, let

$$T_r = \{i \in N : a_i = r\} \text{ for all } r \in A.$$

Then $\{T_r : r \in A\}$ is a finite partition of N so there is a unique (non-empty) $T_r \in \Phi$. Define $g : A^N \to A$ by setting

$$g((a_i)) = \text{ that } r \text{ for which } T_r \in \Phi.$$

If $g((a_i)) = r$ and $g((b_i)) = s$, where $V_s = \{i : b_i = s\}$, then $T_r \cap V_s \in \Phi$ and for all $i \in T_r \cap V_s$, we have $a_i + b_i = r + s$, whence

$$g((a_i)(b_i)) = g((a_i + b_i)) = r + s = g((a_i)) + g((b_i)).$$

Similarly, $a_i b_i = rs$ for all $i \in T_r \cap V_s$ and

$$g((a_i) + (b_i)) = g((a_i b_i)) = rs = g((s_i))g((b_i)).$$

Thus, g is a homomorphism. Now the identity of A^N is $(1_A, 1_A, 1_A, \ldots)$ and $W_{1_A} = \{i : i\text{th component} = 1_A\} = N \in \Phi$. Thus, $g(1_A, 1_A, \ldots) = 1_A$. On the other hand, $\{i : i \neq 1\} \in \Phi$ (as $\{1\} \notin \Phi$) so $g(1_A, 0, 0, \ldots) = 0$ and similarly $g(0, 1_A, 0, 0, \ldots), g(0, 0, 1_A, 0, 0, \ldots), \ldots = 0$.

Define $f : \mathbb{Z}^N \to A$ by setting $f((n_i)) = g((n_i 1_A))$ for all (n_i). This is the required homomorphism. ■

Example 1. The ring I_p of p-adic integers is not slender.

Proof. For each n, we apply the preceding theorem to the ring $\mathbb{Z}_{p^n}$. For $m \geq n$, we have a homomorphism $\pi_m^n \mathbb{Z}_{p^m} \to \mathbb{Z}_{p^n}$ given by $\pi_m^n(\overline{a}) = \overline{a}$. Let $g_m : \mathbb{Z}_{p^m}^N \to \mathbb{Z}_{p^m}$ and $f_m : \mathbb{Z}^N \to \mathbb{Z}_{p^m}$ be the maps of the theorem for each m.

Let $(\overline{a}_i)$ be in $\mathbb{Z}_{p^m}^N$ and let $\{i \in N : \overline{a}_i = \overline{r}\}$ be in Φ (so that $g_m((\overline{a}_i)) = \overline{r}$). Now, consider the corresponding $(\overline{a}_i)$ in $\mathbb{Z}_{p^n}^N$ for $n \leq m$. If $\overline{a}_i = \overline{r}$ in $\mathbb{Z}_{p^m}$ (i.e., $a_i \equiv r(\bmod p^m)$), then $\overline{a}_i = \overline{r}$ in $\mathbb{Z}_{p^n}$. Thus,

$$\{i : \overline{a}_i = \overline{r} \text{ in } \mathbb{Z}_{p^n}\} \supseteq \{i : \overline{a} = \overline{r} \text{ in } \mathbb{Z}_{p^m}\},$$

so the former set is in Φ, and hence, $g_n((\overline{a}_i)) = \overline{r} = \pi_m^n(\overline{r}) = \pi_m^n g_m(\overline{a}_i)$ (where $\overline{a}_i$ has an appropriate interpretation in its two appearances). Now, we introduce $\mathbb{Z}^N$. From the above discussion, we have, whenever $m \geq n$,

$$\begin{aligned} f_n((x_i)) &= g_n((x_i 1_{\mathbb{Z}_{p^n}})) = g_n((\overline{x}_i)) = \pi_m^n g_m(\overline{x}_i) \\ &= \pi_m^n g_m((x_i 1_{\mathbb{Z}_{p^m}})) = \pi_m^n f_m((x_i)) \end{aligned}$$

for all (x_i), i.e., $\pi_m^n f_m = f_n$ whenever $m \geq n$. Thus, there is induced a homomorphism $f : \mathbb{Z}^N \to \varprojlim \mathbb{Z}_{p^n} \cong I_p$ which must be unital and takes all e_i to zero. (This example is in [2]. We have given it in some detail as we will need the argument later.)

By Proposition 4, any torsion-free ring with a subring isomorphic to I_p is nonslender. Thus, we obtain the following examples of non-slender rings.

(i) The field of p-adic numbers and its algebraic closure.
(ii) The field $\mathbb{C}$ of complex numbers. As observed in [2], $\mathbb{C}$ is isomorphic to the algebraic closure of the field of p-adic numbers as both are algebraically closed and have transcendence degree $2^{\aleph_0}$ over $\mathbb{Q}$. For a proof, see pp. 311–317 of [11]. For a proof that $\mathbb{C}$ is isomorphic to the completion (with respect to the p-adic valuation) of the algebraic closure of the p-adic field, see [16, p. 83]. This achieves the same effect.
(iii) $M_2(\mathbb{R})$ (as it contains a copy of $\mathbb{C}$).
(iv) The quaternions.
(v) $I_p[X]$.

Since $\mathbb{R}$ is (ultraconnected and hence) slender, Corollary 2 says that $\mathbb{R}[X]$ is slender. Since $\mathbb{C}$ is not slender, it therefore follows that the class of slender rings is not closed under (torsion-free (semi-) prime) homomorphic images.

It should be possible to refine Proposition 11 and Theorem 1 by obtaining further information about the kinds of homomorphic images of $\mathbb{Z}^N/\mathbb{Z}^{(N)}$ which must be present in a non-slender ring containing no copy of $\mathbb{Z}^N$. Dimitrić [5] has observed that as a group $\mathbb{Z}^N/\mathbb{Z}^{(N)}$ contains a copy of $\mathbb{Z}^N$. His proof works also for rings, but in that case we can say a bit more.

Proposition 12. *Let I be an ideal of $\mathbb{Z}^N$ such that $\mathbb{Z}^{(N)} \subseteq I$ and $\mathbb{Z}^N/I$ is torsion-free. Let $\Gamma = \{S \subseteq N : \mathbb{Z}^S \subseteq I\}$ (where we have identified $\mathbb{Z}^S$ with its obvious copy). Consider the following conditions.*

(i) *$\mathbb{Z}^N/I$ contains no copy of $\mathbb{Z}^N$.*

(ii) *If $S_1, S_2, ..., S_n, ...$ are pairwise disjoint subsets of N, then $S_i \in \Gamma$ for all but finitely many values of i.*

(iii) *If $S_1, S_2, ..., S_n, ...$ are pairwise disjoint subset of N, then $S_i \in \Gamma$ for at least one value of i.*

We have the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. We shall use some functional notation for elements of $\mathbb{Z}^N$. If $S \subseteq N$, let χ_S be the characteristic function of $S(\chi_S(x) = 1$ if $x \in S$, 0 if $x \notin S)$. Then $\mathbb{Z}^S$ (strictly $\chi_S\mathbb{Z}^N) \subseteq I$ if and only if $\chi_S \in I$.

$\neg$(ii)$\Rightarrow\neg$(i) Suppose firstly that $N = S_1 \dot\cup S_2 \dot\cup \cdots \dot\cup S_n \dot\cup \cdots$ and there are infinitely many indices $i_1, i_2, ..., i_n, ...$ with $\chi_{S_1}, \chi_{S_2}, ..., \chi_{S_n}, ... \notin I$. Let

$$A = \{\sigma \in \mathbb{Z}^N : \sigma \text{ is constant on each } S_{i_j} \text{ and } \sigma = 0 \text{ off } \cup S_{i_j}\}.$$

Then A is a subring of $\mathbb{Z}^N$ and $A \cong \mathbb{Z}^N$. If $\alpha \in A \cap I$, then for any k, we have $\chi_{S_{i_k}}\alpha \in I$. But α is constant on S_{i_k}, so there is an integer m such that $\chi_{S_{i_k}}\alpha = m\,\chi_{S_{i_k}}$. Since $\chi_{S_{i_k}}\alpha \in I$ and $\mathbb{Z}^N/I$ is torsion-free, it follows that $m = 0$. Hence, $\alpha(x) = 0$ for all $x \in S_{i_k}$, so $\alpha = 0$. We now have

$$\mathbb{Z}^N \cong A \cong (A + I)/I \subseteq \mathbb{Z}^N/I.$$

More generally, if we have

$$S_1 \dot\cup S_2 \dot\cup \dot\cup S_n \dot\cup = T \subseteq N,$$

then defining $A' \subseteq \mathbb{Z}^T$ by analogy with A above, we obtain

$$\begin{aligned}\mathbb{Z}^N &\cong \mathbb{Z}^T \cong A' \cong (A' + (I \cap \mathbb{Z}^T))/(I \cap \mathbb{Z}^T) \subseteq \mathbb{Z}^T/(I \cap \mathbb{Z}^T)\\ &\cong (\mathbb{Z}^T + I)/I \subseteq \mathbb{Z}^N/I.\end{aligned}$$

$\neg$(ii)$\Rightarrow\neg$(iii) If there exist pairwise disjoint $S_1, S_2, ..., S_n, ...$ such that infinitely many S_{i_j} fail to belong to Γ, then these S_{i_j} constitute a collection of subsets violating (iii). Obviously (ii) $\Rightarrow$ (iii). ■

If $V \subseteq S \in \Gamma$, then $\chi_V = \chi_V\chi_S \in I$ so $V \in \Gamma$, while if $S, T \in \Gamma$, then $\chi_{S\cup T} = \chi_S + \chi_T - \chi_S\chi_T \in I$, so $S \cup T \in \Gamma$. Hence, $\{N\backslash S : S \in \Gamma\}$ is a filter and it satisfies the condition:

If $E_1, E_2, ..., E_n, ...$ are subsets of N such that $E_i \cup E_j = N$ whenever $i \neq j$, then the filter contains at least one (so almost all) E_i. $(*)$

Let Ψ be an ultrafilter, $E_1, E_2, ..., E_n, ...$ subset of N with $E_i \cup E_j = N$ whenever $i \neq j$. Thus, Ψ excludes at most one E_i. It follows that any finite intersection of ultrafilters satisfies $(*)$.

Example 2. Let Φ be an intersection of finitely many non-principal ultrafilters $\Phi_1, \Phi_2, ..., \Phi_k$ on N, $\Gamma = \{N \backslash T : t \in \Phi\}$. Then Φ satisfies $(*)$ and Γ satisfies conditions (ii) and (iii) of Proposition 12. Moreover, there is an ideal $J \supseteq \mathbb{Z}^{(N)}$ such that $\mathbb{Z}^N/J$ contains no copy of $\mathbb{Z}^N$. Let

$$J = \{(x_i) \in \mathbb{Z}^N : \{i : x_i = 0\} \in \Phi\}.$$

Then Z^N/J is a subdirect product of the ultrapowers $\mathbb{Z}^N/\Phi_1, \mathbb{Z}^N/\Phi_2, ..., \mathbb{Z}^N/\Phi_k$. The latter are prime commutative rings. Suppose $\mathbb{Z}^N/J$ has orthogonal idempotents $u_1, u_2, ..., u_n, ...$. Under each natural map $\mathbb{Z}^N/J \to \mathbb{Z}^N/\Phi_j$ at most one u_i is taken to the identity and all others to zero. Thus, almost all the u_i are in $\cap_{j=1}^{k}\mathrm{Ker}(\mathbb{Z}^N/J \to \mathbb{Z}^N/\Phi_j) = 0$. Hence, $\mathbb{Z}^N/J$ has no subring isomorphic to $\mathbb{Z}^N$.

Govaerts, Delanghe, and Impens [8] have shown that when A is a ring of integral Clifford numbers, every unital A-homomorphism from A^N to A is a projection. The motivation for this comes from work on continuous functions wherein analogous results can be found for other rings in place of A. We can obtain many examples of rings behaving like the one treated in [8] by means of a generalization of slenderness.

For a commutative ring A with identity, we define an A-algebra R to be *A-slender* if it is additively torsion-free and satisfies the A-algebra homomorphism analog of the condition in Definition 3. An A-module is slender if it satisfies the A-homomorphism analog of the condition in Definition 1 (see, e.g., [12]). The following result is reminiscent of Proposition 2.

Proposition 13. *Let A be a commutative ring with an identity, and R an A-algebra. If R is either*

(i) *a slender ring; or*

(ii) *a slender A-module;*

then R is A-slender.

Proof. (i) Let $f : A^N \to R$ be an A-homomorphism. Define $g : \mathbb{Z}^N \to A^N$ by setting $g((n_i)) = (n_i 1_A)$ for all (n_i). If R is a slender ring, then there is a suitable set $\{i_1, i_2, ..., i_n\}$ for which $gf((n_i)) = \Sigma_{j=1}^{n} fg(\widehat{n}_{i_j})$. Then, for every $(a_i) \in A^N$, we have

$$\begin{aligned} f((a_i)) &= f(1_{A^N}(a_i)) = f(g(1_{Z^N})(a_i)) = fg(1_{Z^N})f((a_i)) \\ &= \sum_{j=1}^{n} fg(e_{i_j})f((a_i)) = \sum_{j=1}^{n} f(g(e_{i_j})(a_i)) = \sum_{j=1}^{n} f(\widehat{a}_{i_j}). \end{aligned}$$

(ii) is fairly obvious (cf. Proposition 2). ■

The next result provides the examples.

Proposition 14. *Let A be a commutative ring with identity. If A is A-slender and the only central idempotents of A are* 0 *and* 1_A, *then every unital A-homomorphism from* A^N *to A is a projection.*

Thus, for many commutative rings A, the conclusion of [8] is true because of ring, or even additive group, properties rather than algebra properties. Some examples (without central idempotents $\neq 0, 1$ in all cases) are: countable torsion-free rings (e.g., integral Clifford algebras) (Corollary 4 or [2]) $R[[X]]$ and $R[S]$ for slender R as in Corollary 2, $R[X]$ for any R with identity [15, Proposition 2.10]. Although matrix rings over slender rings need not be slender (see after Example 1), if A is slender as an A-module, then, for each n, the matrix ring $M_n(A)$ is slender as an $M_n(A)$-module [12, Corollary 4] so $M_n(A)$ satisfies the conclusion of Proposition 13. On the other hand, every field of characteristic 0 (in fact every infinite field) satisfies the conclusion of Proposition 14 [3] though some such fields, e.g., $\mathbb{C}$ (see after Example 1) are not slender rings and no field K is slender as a vector space. (More generally, a noetherian self-injective ring cannot be slender as a module over itself.) This result has connections with the problem of determining when a linear functional defined on a ring of continuous functions is defined by "evaluation at a point". See [8] for some pertinent references.

In [8] the authors remark that they know of no infinite indecomposable ring with identity for which the conclusion of Proposition 13 fails. The ring I_p of p-adic integers is such a ring. For, $I_p/p^nI_p \cong \mathbb{Z}_{p^n}$ for each n and the argument in Example 1 then shows that there is a non-zero unital homomorphism from $I_p^N/I_p^{(N)}$ to I_p.

De Marco and Orsatti [13] have shown that a reduced torsion-free group is slender if and only if it is not complete with respect to any non-discrete linear metric. We now seek some kind of analog for slender rings. The result cannot translate straightforwardly to the ring case, as linear topologies are additively determined and slender rings need not have slender additive groups. Nor can we simply require the linear metric to one for which multiplication is continuous. For instance, $\mathbb{Z}[[X]]$ is slender (Corollary 2) but is complete in the (X)-adic metric (and its additive group $\cong \mathbb{Z}^N$ is not slender).

A *linear topology* on an abelian group is one for which there is a basis of open neighborhoods of 0 consisting of subgroups. When this basis is countable, the topology is quasimetrizable; in addition, when the intersection of the subgroups in the basis is zero, the topology is metrizable. Let $\{F_1, F_2, ..., F_n ...\}$ be such a basis in such a case. Then

$$\{F_1, F_1 \cap F_2, ..., F_1 \cap F_2 \cap \cdots \cap F_n, ...\}$$

is also a basis. Thus, we can and will assume that the subgroups in our basis form a descending chain. An account of this can be found in [18, Chapter II]. Note that, for metrizable linear topologies, a defining metric d can be assumed to be uniform, i.e., to satisfy the condition $d(x, y) = d(x - y, 0)$ for all x, y.

We shall relate convergence and Cauchy sequences to the basis $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ rather than to a metric. A *neat* Cauchy sequence $\langle b_n \rangle$ is one which satisfies the condition:

$$b_n - b_{n+1} \in F_n, \ \forall n.$$

Let $\langle a_n \rangle$ be any Cauchy sequence. Let

$$\sigma(1) = \min\{k : a_\ell - a_m \in F_1 \text{ for all } \ell, m \geq k\},$$
$$\sigma(2) = \min\{k : \sigma(1) < k \text{ and } a_\ell - a_m \in F_2 \text{ for all } \ell, m \geq k\}$$

and in general

$$\sigma(n+1) = \min\{k : \sigma(n) < k \text{ and } a_\ell - a_m \in F_{n+1} \text{ for all } \ell, m \geq k\}.$$

Then $a_{\sigma(n)} - a_{\sigma(n+1)} \in F_n$ for all n. Hence, $\langle a_{\sigma(n)} \rangle$ is a neat Cauchy sequence. If $\langle a_{\sigma(n)} \rangle$ converges, then so must the original Cauchy sequence $\langle a_n \rangle$. Hence, a metrizable linear topology is complete if and only if every neat Cauchy sequence converges. (This seems to be a well-established folklore, but the author is not aware of any written account.)

Finally, in what follows, a *CLM topology* is a complete linear metrizable topology.

Proposition 15. *Let H be a subgroup of an abelian group G. If H has a CLM topology, this can be extended to a CLM topology on G.*

Proof. Let $\mathcal{B} = \{B_1, B_2, ...\}$ be a countable group basis of neighborhoods of 0 in H. Then $\mathcal{B}$ is a similar basis for a topology in G, so G is metrizable. Now, let $\langle a_n \rangle$ be a neat Cauchy sequence in G. Then $a_{n+1} - a_n \in B_n \subseteq H$ for all n. Now, $a_1 - a_1 = 0 \in H$ and $a_{n+1} - a_1 = (a_{n+1} - a_n) + (a_n - a_1)$, so by induction, $a_n - a_1 \in H$ for all n. Clearly, $\langle a_n - a_1 \rangle$ is a Cauchy sequence, so by completeness of H, there exists $b \in H$ with $\lim(a_n - a_1) = b$. But then $b + a_1 = \lim a_n$ so G is complete. ■

The characterization theorem of [13, Sec. 2 and its proof] show that if a torsion-free group G contains no copy of I_p for any p, then in any complete linear metric on G, the completion (closure) of an infinite cyclic group is discrete. It then follows that G contains an infinite direct product of infinite cyclic groups. For slender *rings* we have to exclude infinite direct products of copies of $\mathbb{Z}$, and therefore are interested in cyclic groups (subrings) generated by idempotents.

Theorem 4. *A torsion-free ring A has a subring isomorphic to $\mathbb{Z}^N$ if and only if A^+ has a non-discrete CLM topology and a closed subring B in which*

(i) *multiplication is continuous*; and

(ii) *there is a convergent (whence null) sequence of non-zero orthogonal idempotents.*

Proof. Let the subring B be as described, and let $\langle u_n \rangle$ be a convergent sequence of orthogonal idempotents in B. If $a = \lim u_n$, then

$$a = \lim u_n = \lim u_n^2 = a^2 = \lim u_n \lim u_{n+1} = \lim u_n u_{n+1} = 0.$$

Clearly, we obtain an injective ring homomorphism $f : \mathbb{Z}^{(N)} \to A$ by defining $f(\Sigma a_i e_i) = \Sigma a_i u_i$. Now, let $\mathbb{Z}$ have the discrete topology in all its copies, $\mathbb{Z}^N$ the product topology, and $\mathbb{Z}^{(N)}$ the induced subspace topology.

Let $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_n \supseteq \cdots$ be a group basis of neighborhoods of 0 in A. Without loss of generality, we can assume that $u_n \in V_n$ for all n.

If in $\mathbb{Z}^{(N)}$, we have $\Sigma c_i e_i = \lim_{n\to\infty} \Sigma b_i^{(n)} e_i$. Then for every $m \in N$, there exists $k \in N$ such that $b_1^{(n)} = c_1, b_2^{(n)} = c_2, ..., b_m^{(n)} = c_m$ for all $n > k$. This means that

$$\sum c_i u_i - \sum b_i^{(n)} u_i = f\Big(\sum c_i e_i - \sum b_i^{(n)} e_i\Big) = f\Big(\sum_{i>m} c_i e_i - \sum_{i>m} b_i^{(n)} e_i\Big)$$
$$= \sum_{i>m} c_i u_i - \sum_{i>m} b_i^{(n)} u_i \in V_{m+1}$$

for all $n > k$. By invariance of the metric, we have

$$f\Big(\sum c_i e_i\Big) = \sum c_i u_i = \lim_{n\to\infty} \sum b_i^{(n)} u_i = \lim_{n\to\infty} f\Big(\sum b_i^{(n)} e_i\Big),$$

so f is continuous and in fact uniformly continuous. But $\mathbb{Z}^N$ is the completion of $\mathbb{Z}^{(N)}$ so there is a continuous group homomorphism $g: \mathbb{Z}^N \to A$ such that $g \mid \mathbb{Z}^{(N)} = f$. As f takes its values in B, we have $f(x)f(y) \to f(x_0y_0)$ as $x \to x_0, y \to y_0$.

Now, $\mathbb{Z}^N$ is actually a topological ring, and $\mathbb{Z}^{(N)}$ a topological subring. If $(r_i), (s_i) \in \mathbb{Z}^N$, then $(r_i) = \lim_{n\to\infty} \Sigma_{i=1}^n r_i e_i$ and $(s_i) = \lim_{n\to\infty} \Sigma_{i=1}^n s_i e_i$. Hence,

$$\begin{aligned}
g((r_i))g((s_i)) &= g\Big(\lim_{n\to\infty}\sum_{i=1}^n r_ie_i\Big)g\Big(\lim_{n\to\infty}\sum_{i=1}^n s_ie_i\Big)\\
&= \lim_{n\to\infty} g\Big(\sum_{i=1}^n r_ie_i\Big)\lim_{n\to\infty} g\Big(\sum_{i=1}^n s_ie_i\Big)\\
&= \lim_{n\to\infty} f\Big(\sum_{i=1}^n r_ie_i\Big)\lim_{n\to\infty} f\Big(\sum_{i=1}^n r_ie_i\Big)\\
&= \lim_{n\to\infty}\sum_{i=1}^n r_iu_i \lim_{n\to\infty}\sum_{i=1}^n s_iu_i\\
&= \lim_{n\to\infty}\Big(\sum_{i=1}^n r_iu_i\sum_{i=1}^n s_iu_i\Big)\\
&= \lim_{n\to\infty}\sum_{i=1}^n r_is_iu_i = g((r_is_i)) = g((r_i)(s_i)).
\end{aligned}$$

Thus, g is a ring homomorphism. If $(t_i) \in \mathrm{Ker}(g)$, then for every j, we have $t_je_j = (t_i)e_j \in \mathrm{Ker}(g) \cap \mathbb{Z}^{(N)} = \mathrm{Ker}(f) \cap \mathbb{Z}^{(N)} = 0$. Thus, $\mathrm{Ker}(g) = 0$ and $\mathbb{Z}^N \cong \mathrm{Im}(g) \subseteq A$.

Conversely, suppose (without loss of generality) $\mathbb{Z}^N \subseteq A$. Then the product on $\mathbb{Z}^N$ of the discrete topologies on the copies of $\mathbb{Z}$ is a non-discrete CLM topology which extends to A. But $\mathbb{Z}^N$ is a topological ring with respect to this topology and, in $\mathbb{Z}^N$, we have $\lim_{n\to\infty} e_n = 0$. ■

Theorem 5. *Let R be a torsion-free ring which is the completion of $\mathbb{Z}$ in some non-discrete linear metric topology. Then for some prime p, R has a subring isomorphic to the ring I_p of p-adic integers.*

Proof. First, note that as all its subgroups are ideals, $\mathbb{Z}$ is a topological ring with respect to any of its linear topologies. Let $\{B_1, B_2, ..., B_n, ...\}$ be a group basis of neighborhoods of 0 in $\mathbb{Z}$. Then (as a group and as a ring) we have $R \cong \varprojlim \mathbb{Z}/B_n$ (where the set of $\mathbb{Z}/B_m$ is directed by the embeddings $B_m \subseteq B_n$). Now, $\varprojlim \mathbb{Z}/B_n$ is closed in $\Pi\mathbb{Z}/B_n$, where the latter has the product topology from the discrete topologies in the $\mathbb{Z}/B_n$. As the $\mathbb{Z}/B_n$ are finite, $R(\cong \varprojlim \mathbb{Z}/B_n)$ is compact. Now the topology of R is also linear (see, e.g., [7, Vol. I, p. 68]). If $\{V_1, V_2, ..., V_n, ...\}$ is a group basis of neighborhoods of 0 in R and $x \in R\backslash\{0\}$, then there exists k such that $x \notin V_k$. But then $x \in x + V_k$ and $0 \in \cup\{x \notin y + V_k : y + V_k\}$. It follows that R is totally disconnected. Thus, the character group $\mathrm{Char}(R)$ of R is a discrete torsion group and $R \cong \mathrm{Char}(\mathrm{Char}(R))$. But $\mathrm{Char}(\mathrm{Char}(R)) = \mathrm{Hom}(\mathrm{Char}(R), \mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}(\mathrm{Char}(R), \Pi_p\mathbb{Z}(p^\infty)) \cong \Pi_p\mathrm{Hom}(\mathrm{Char}(R), \mathbb{Z}(p^\infty))$.

Let $\mathrm{Char}(R) = \oplus_p G_p$, where G_p is a p-group. Then we have (additively) $R \cong \mathrm{Char}(\mathrm{Char}(R)) \cong \Pi_p \mathrm{Hom}(\oplus_q G_q, \mathbb{Z}(p^\infty)) \cong \Pi_p \mathrm{Hom}(G_p, \mathbb{Z}(p^\infty))$. If some G_p were non-divisible, it would have a cyclic summand $\langle g \rangle$, whence R would have a summand $\cong \mathrm{Hom}(\langle g \rangle, \mathbb{Z}(p^\infty))$, also cyclic. But R is torsion-free. We conclude that each G_p is divisible, hence a direct sum of copies of $\mathbb{Z}(p^\infty)$. But then $\mathrm{Hom}(G_p, \mathbb{Z}(p^\infty))$ is isomorphic to a product of copies of $\mathrm{Hom}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) \cong I_p$. Then $\mathrm{Hom}(G_p, \mathbb{Z}(p^\infty))$ is a reduced I_p-module in a natural way (a similar argument was used by Harrison [9]). Write $R^+ = \Pi_p S_{(p)}$, where $S_{(p)} \cong \mathrm{Hom}(G_p, \mathbb{Z}(p^\infty))$. For every p, $S_{(p)} \Pi_{q \neq p} S_{(q)}$ is divisible and hence zero. Thus, each $S_{(p)} \lhd R$ and $R = \Pi_p S_{(p)}$ as a ring direct product. Now, $\mathbb{Z}$ has an identity, whence so does R and finally so do the $S_{(p)}$. Thus, every non-zero $S_{(p)}$ is a unital I_p-algebra and so contains a subring isomorphic to I_p, and therefore, so does R. ∎

Theorem 6. *A torsion-free ring A for which A^+ is reduced is slender if and only if, for every CLM on A^+,*

(i) *(the closure of) $\langle e \rangle$ is discrete for all idempotents e; and*

(ii) *there is no convergent sequence of non-zero orthogonal idempotents.*

(Here, $\langle e \rangle$ means the cyclic subgroup generated by e.)

Note that, in $\mathbb{Z}[[X]]$, $\langle 1 \rangle$ is indeed discrete in the (X)-adic metric (cf. our earlier remark).

Proof. First, suppose A is slender. If e is a non-zero idempotent, then the closure of $\langle e \rangle$ is a subring which is slender (Proposition 4) and thus has no subring $\cong I_p$ for any p (Example 1). By Theorem 5, $\langle e \rangle$ is closed and discrete.

Assuming there exists a convergent (whence null as in Theorem 4) sequence $\langle u_n \rangle$ of non-zero orthogonal idempotents, we can also assume that A^+ has a group basis $U_1 \supseteq U_2 \supseteq \cdots$ of neighborhoods of 0 such that $u_n \in U_n$ for each n. As $\langle u_1 \rangle$ is dense, there exists $\tau(1) > 1$ such that $\langle u_1 \rangle \cap U_{\tau(1)} = \{0\}$. In the same way, there exists $\tau(2) > \tau(1)$ such that $\langle u_{\tau(1)} \rangle \cap U_{\tau(2)} = \{0\}$. If $x \in U_{\tau(2)} \cap \langle u_1 \rangle \oplus \langle u_{\tau(1)} \rangle$, say $x = ru_1 + su_{\tau(1)}$, $r, s \in \mathbb{Z}$, then $ru_1 = x - su_{\tau(1)} \in U_{\tau(2)} + U_{\tau(1)} = U_{\tau(1)}$ so $r = 0$ and then $su_{\tau(1)} = x \in U_{\tau(2)}$, so $s = 0$. But then $x = 0$. Proceeding thus we obtain a subsequence $(u_1, u_{\tau(1)}, u_{\tau(2)}, \ldots, u_{\tau(n)}, \ldots)$ such that if $S = \langle u_1 \rangle \oplus \langle u_{\tau(1)} \rangle \oplus \cdots \oplus \langle u_{\tau(n)} \rangle \oplus \cdots$, then the relative topology on S has a basis

$$
\begin{aligned}
S \cap U_1 \supseteq S \cap U_{\tau(1)} \supseteq S \cap U_{\tau(2)} \supseteq \cdots \supseteq S \cap U_{\tau(n)}, \ldots, \text{i.e.,}\\
S \supseteq \langle u_{\tau_{(1)}} \rangle \oplus \langle u_{\tau_{(2)}} \rangle \oplus \cdots \oplus \langle u_{\tau_{(n)}} \rangle \oplus \cdots\\
\supseteq \langle u_{\tau_{(2)}} \rangle \oplus \cdots \oplus \langle u_{\tau_{(n)}} \rangle \oplus \cdots\\
\supseteq \cdots \supseteq \langle u_{\tau_{(n)}} \rangle \oplus \cdots \supseteq \cdots
\end{aligned}
$$

of group neighborhoods of 0. The closure of S is thus isomorphic to $\mathbb{Z}^N$. (In this part of the argument, we are imitating [13, p. 159]. But S is a ring so its closure is too. But then the slender ring A contains a copy of $\mathbb{Z}^N$. We have a contradiction, so (ii) holds as well as (i).

Now, suppose (i) and (ii) are valid. By (ii) and Theorem 4, A contains no copy of $\mathbb{Z}^N$. If A were to contain a non-zero homomorphic image of $\mathbb{Z}^N/\mathbb{Z}^{(N)}$, the latter would be additively algebraically compact, torsion-free, and reduced, whence a product of p-adic algebras with identities. But then as in the proof of Theorem 5, A would contain a subring $\cong I_p$ for some p.

Let e be the identity of a copy of $I_p \subseteq A$. As in Proposition 15, the p-adic metric on this copy of I_p extends to a CLM on A^+ in which $\langle e \rangle$ is not discrete. This cannot happen, so A cannot contain a non-zero homomorphic image of $\mathbb{Z}^N/\mathbb{Z}^{(N)}$. Hence, by Theorem 1, A is slender. ∎

When A^+ is reduced, we have the following analog of Nunke's characterization [14] of slender groups.

Corollary 5. (To proof) *A torsion-free ring A for which A^+ is reduced is slender if and only if it contains no copy of $\mathbb{Z}^N$ or any I_p.*

Referring to Proposition 1, we obtain

Corollary 6. *Let A be a torsion-free ring with identity whose only idempotents are* 0 *and* 1 *and with A^+ reduced. The following conditions are equivalent:*

(i) $\langle 1 \rangle$ *is discrete under any CLM topology on A^+;*
(ii) *A has no subring isomorphic to any I_p;*
(iii) *A is ultraconnected.*

No doubt there are generalizations of Theorem 6 and its corollaries in which more ring-theoretic substitutes can be found for the overtly additive condition of being reduced. Note however that something is needed—the mere absence of copies of $\mathbb{Z}^N$ and I_p does not ensure slenderness.

Proposition 16. *Let Φ be a non-principal ultrafilter on N. Then $\mathbb{Z}^N/\Phi$ is not slender, though it contains no copy of any I_p (and, of course, no copy of $\mathbb{Z}^N$).*

Proof. As $\mathbb{Z}/\Phi$ is prime, its only idempotents are 0 and 1. Thus, if $\mathbb{Z}^N/\Phi$ has a subring isomorphic to I_p, the latter must have the same identity as $\mathbb{Z}^N/\Phi$. Since I_p has lots of units, we are finished if we can show that $\mathbb{Z}^N/\Phi$ has ± 1 as its only units. Let $(a_i) + \Phi$ be a unit of $\mathbb{Z}^N/\Phi$. Then there exists $(b_i) \in \mathbb{Z}^N$ such that $\{i : a_i b_i = 1\} \in \Phi$. Hence, $\{i : a_i = 1\} \cup \{i : a_i = -1\} = \{i : a_i = \pm 1\} \supseteq \{i : a_i b_i = 1\} \in \Phi$, so as Φ is an ultrafilter, it contains either $\{i : a_i = 1\}$ or $\{a : a_i = -1\}$, and accordingly $(a_i) + \Phi = 1$ or -1. ∎

Note that this argument also shows that $\mathbb{Z}/\Phi$ does not contain any unital subring with divisible additive group. The fact that $\mathbb{Z}^N/\Phi$ has only ± 1 as units also follows (as does much else) from the fact that $\mathbb{Z}^N/\Phi$ is a non-standard model of arithmetic [1], but it seems preferable to give a self-contained argument. Proposition 16 also shows that, in Theorem 6, we cannot replace the condition "A^+ is reduced" by the condition "the divisible ideal of A contains no non-zero idempotents". $\mathbb{Z}^N/\Phi$ cannot be reduced, otherwise, it has a unital subring isomorphic to some I_p as in the proof of Theorem 6. On the other hand, the divisible ideal does not contain the one and only non-zero idempotent.

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