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# Non-Existence of Homoclinic Orbits and Global Asymptotic Stability of FitzHugh–Nagumo System

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**Abstract.** The result that the FitzHugh–Nagumo system has no homoclinic orbits is given. By virtue of the result, the global asymptotic stability of an equilibrium point of the system is discussed.

## 1. Introduction

Our purpose in this paper is to consider the existence of the homoclinic orbit of the FitzHugh–Nagumo system [1, 9], which is obtained by simplifying some nerve model. The result for the homoclinic orbit of the system will be given and proved in Sec. 2. In Sec. 3, a sufficient condition for the non-existence of the closed orbit of the system will be given by using some plane curve. By virtue of these results, the global asymptotic stability of an equilibrium point of the system will be stated in Sec. 4. Finally, the phase portrait as an example illustrating our results will be given in Sec. 5.

To explicate the ion mechanism for the excitation and the conduction of nerve, Hodgkin and Huxley [6] developed the system of four non-linear ordinary differential equations as a model of nerve conduction in the squid giant axon (Loligo). FitzHugh [1] and Nagumo et al. [9] simplified the system by introducing the two-dimensional model

$$\begin{cases} \dot{w} = v - \dfrac{1}{3}w^3 + w + I, \\ \dot{v} = \rho(a - w - bv), \end{cases} \tag{FHN}$$

where the dot $(\cdot)$ denotes differentiation and $a$, $\rho$, $b$ are real constants such that

$$\text{(C1)} \quad a \in \mathbb{R}, \quad \rho > 0, \quad 0 < b < 1.$$

The variable $w$ corresponds to the potential difference through the axon membrane and $v$ represents the potassium activation (sodium in activation). The quantity $I$ is the current through the membrane. The system (FHN) for special values of $I$ has been investigated by using numerical methods and phase space analysis in [1] or [9].

The system (FHN) has a unique equilibrium point $(x_I, y_I)$ for each $I \in \mathbb{R}$, where

$$x_I = \sqrt[3]{\{3\Big(I + \frac{a}{b}\Big) + \sqrt{9\Big(I + \frac{a}{b}\Big)^2 + 4\Big(\frac{1}{b} - 1\Big)^3}\}/2} + \sqrt[3]{\{3\Big(I + \frac{a}{b}\Big) - \sqrt{9\Big(I + \frac{a}{b}\Big)^2 + 4\Big(\frac{1}{b} - 1\Big)^3}\}/2}$$

and

$$y_I = \frac{a - x_I}{b}.$$

Instead of the parameter $I$, we introduce a new parameter $\eta$. By the transformation $\eta = x_I$, $x = w - \eta$, and $y = v - a/b + \eta/b + \rho b(w - \eta)$, the system (FHN) is transformed to the following system:

$$\begin{cases} \dot{x} = y - \left\{\frac{1}{3}x^3 + \eta x^2 + (\eta^2 + \rho b - 1)x\right\} \\ \dot{y} = -\frac{\rho b}{3}\left\{x^3 + 3\eta x^2 + 3(\eta^2 + \frac{1}{b} - 1)x\right\}. \end{cases} \tag{FNS}$$

The system (FNS) is called the FitzHugh nerve system and has a unique equilibrium point $E(0, 0)$. Also, we note that this is a system of the Liénard type.

## 2. Non-Existence of Homoclinic Orbits

If the system (FNS) has the orbits whose $\alpha$- and $\omega$-limit sets are the equilibrium point, then we say that they are homoclinic orbits of the system. The first result is the following:

**Theorem 2.1.** *The system (FHN) has no homoclinic orbits.*

*Proof.* We prove Theorem 2.1 for the system (FNS) which is equivalent to the system (FHN). The theorem is proved by dividing into two cases $0 < \rho b < 1$ and $\rho b \geq 1$.

*Case 1.* $[0 < \rho b < 1]$. Let $\eta_0 = \sqrt{1 - \rho b}$. Then instead of (C1), we can assume the condition

$$\text{(C2)} \quad a \in \mathbb{R}, \quad 0 < b < 1, \quad 0 < \rho < \frac{1}{b}.$$

By considering the variational matrix about $E$, we see easily that,

(i) if $\eta^2 > \eta_0^2$, then $E$ is a stable focus or node;

(ii) if $\eta^2 < \eta_0^2$, then $E$ is an unstable focus or node.

Thus, in the cases (i) and (ii), the system (FNS) has no homoclinic orbits.

We set

$$\Gamma(x) = \frac{1}{3}x^2(x + 3\eta)$$

and

$$g(x) = \frac{\rho b}{3}\{x^3 + 3\eta x^2 + 3(\eta^2 + \frac{1}{b} - 1)x\}.$$

To prove the case

(iii) $\eta^2 = \eta_0^2$,

we need the following lemma for the system (FHS).

**Lemma 2.1.** [11] *Suppose there exists a* $\delta > 0$ *such that* $\Gamma(x) > 0$ *for* $0 < |x| < \delta$. *If*

$$\frac{1}{\Gamma(x)} \int_0^x \frac{g(\xi)}{\Gamma(\xi)} d\xi > \frac{1}{4} \quad for \quad 0 < x < \delta$$

*are satisfied, then the system (FNS) with* $\eta^2 = \eta_0^2$ *has no homoclinic orbits.*

Let

$$\delta = \frac{-3\eta + \sqrt{9\eta^2 + 12\gamma\rho b}}{2} > 0,$$

where $\gamma$ is a constant such that

$$0 < \gamma < \min\left\{\frac{6(1-\rho b)}{\rho b}, 3\right\}.$$

Then we have $0 < \delta < 3\eta$. Since $\eta^2 + (1/b) - 1 > 0$, we obtain

$$\begin{aligned} \frac{1}{\Gamma(x)} \int_0^x \frac{g(\xi)}{\Gamma(\xi)} d\xi &> \frac{3\rho b}{x^2(x+3\eta)} \int_0^x d\xi \\ \int_0^x d\xi &= \frac{3\rho b}{x(x+3\eta)} \\ &> \frac{\gamma\rho b}{\delta(\delta+3\eta)} \geq \frac{1}{4} \end{aligned}$$

for $0 < x < \delta$. Thus, from Lemma 2.1, the system (FNS) with $\eta^2 = \eta_0^2$ has no homoclinic orbits.

*Case 2.* $[\rho b \geq 1]$. By considering the variational matrix about $E$, we have that

(iv) if $\rho b \neq 1$ or $\eta \neq 0$, $E$ is a stable focus or node.

Thus, in this case the system has no homoclinic orbits. We consider the case

(v) $\rho b = 1$ and $\eta = 0$.

From the facts that the function $(1/3)x^3 + \eta x^2 + (\eta^2 + \rho b - 1)x$ with $\rho b \geq 1$ is monotone increasing for all $x \in \mathbb{R}$ and $xg(x) > 0$ $(x \neq 0)$, it follows that the following lemma holds (see, e.g., [5]).

**Lemma 2.2.** *If* $\rho b \geq 1$, *then the system* (FNS) *has no non-trivial closed orbits.*

Since the corresponding linear system for the case (v) has a pair of pure imaginary eigenvalues, $E$ is a focus or center. However, by Lemma 2.2, $E$ is not a center. Thus, in this case, the system has no homoclinic orbits, too.

We have completed the proof of Theorem 2.1. ■

*Remark.* The unique existence and non-existence of the closed orbit of the system (FNS) have been treated by [4, 7, 8, 10, etc.] However, the result for a homoclinic orbit of the system has not been given until now.

Let $F(x) = \Gamma(x) + (\eta^2 - \eta_0^2)x$. We denote

$$C^+ = \{(x, y) : x > 0 \text{ and } y = F(x)\} \text{ and } C^- = \{(x, y) : x < 0 \text{ and } y = F(x)\}.$$

By the above theorem and by checking the direction of the vector $(y - F(x), -g(x))$, we have the following:

- The positive (resp., negative) semitrajectory of (FNS) passing through any point on the curve $C^+$ (resp., $C^-$) meets the negative $y$-axis.
- The negative (resp., positive) semitrajectory of (FNS) passing through any point on the curve $C^+$ (resp., $C^-$) meets the positive $y$-axis.

Therefore, we conclude the following:

**Lemma 2.3.** *All positive semitrajectories of (FNS) near the equilibrium point $E$ keep on rotating around $E$.*

## 3. Non-Existence of Closed Orbits

In this section, we shall give a sufficient condition for which the system (FNS) has no non-trivial closed orbits. Our result is the following:

**Theorem 3.1.** *Assume that the condition*

$$\text{(C3)} \quad \rho b \geq 1$$

*or*

$$\text{(C4)} \quad \eta^2 \geq \eta_0^2 \text{ and } \eta^4 - 4\eta^2\eta_0^2 + \eta_0^4 + 2\Big(\frac{1}{b} - 1\Big)\eta^2 - 4\Big(\frac{1}{b} - 1\Big)\eta_0^2 + 4\Big(\frac{1}{b} - 1\Big)^2 \geq 0,$$

*or*

$$\text{(C5)} \quad 2\Big\{\eta_0^2 + \Big(\frac{1}{b} - 1\Big)\Big\}^3 < \eta^2\Big\{\eta^2 + 3\Big(\frac{1}{b} - 1\Big)\Big\}^2$$

*is satisfied. Then the system (FNS) has no non-trivial closed orbits.*

The pair $(\eta^2, \eta_0^2)$ satisfying the condition (C4) or (C5), is shown by the shaded area in Fig. 1.

Assume that the condition in Theorem 3.1 holds with $\eta \geq 0$ (the proof for the case $\eta < 0$ is essentially the same).

From Lemma 2.2, the system (FNS) with the condition (C3) has no non-trivial closed orbits. To prove the case (C4) or (C5), the following result is used.

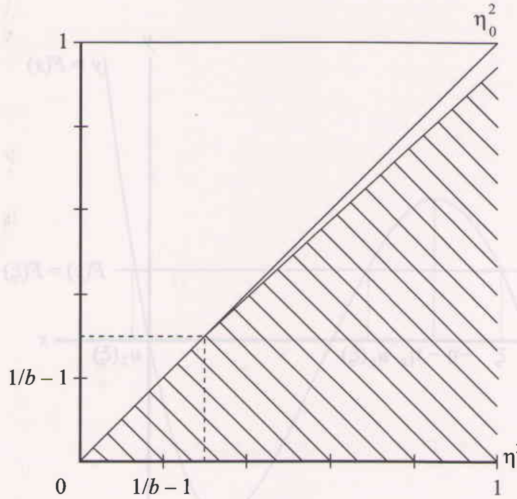


**Fig. 1.**

**Lemma 3.1.** [2] *Let $G(x) = \int_0^x g(\xi)d\xi$. If the curve $(F(x), G(x))$ has no intersecting points with itself, then the system (FNS) has no non-trivial closed orbits.*

We intend to show that the condition (C4) or (C5) implies that the curve $(F(x), G(x))$ has no intersecting points with itself. Let

$$\alpha(\eta) = \frac{-3\eta - \sqrt{3(4\eta_0^2 - \eta^2)}}{2} \quad \text{and} \quad \beta(\eta) = -\eta - \sqrt{2(\eta_0^2 + \frac{1}{b} - 1)}.$$

Now, we consider the equation

$$F(x) = F(\xi) \quad \text{for } \alpha(\eta) < \xi \leq -\eta - \eta_0.$$

This equation has two roots other than $x = \xi$. Let $u_1 = u_1(\xi)$ and $u_2 = u_2(\xi)$ denote these roots. Then we have $-\eta - \eta_0 \leq u_1 < \{-3\eta + \sqrt{3(4\eta_0^2 - \eta^2)}\}/2$ and $0 < u_2 < -\eta + 2\eta_0$ (see Fig. 2).

Let

$$\Phi(\xi) = \xi^3 + 3\eta\xi^2 + 3\Big(\eta^2 - 2\eta_0^2 - 2(\frac{1}{b} - 1)\Big)\xi - 3\eta\Big(\eta^2 + 2\eta_0^2 + 6(\frac{1}{b} - 1)\Big). \quad \blacksquare$$

We have the following lemmas for the function $\Phi(\xi)$.

**Lemma 3.2.** *If $\eta^2 \geq \eta_0^2$, then $\Phi(\alpha(\eta)) < 0$.*

*Proof.* Since $F(\alpha(\eta)) = 0$ and $\alpha(\eta) \geq -3\eta$ for $\eta^2 \geq \eta_0^2$, we have

$$\begin{aligned}\Phi(\alpha(\eta)) &= -3\Big(\eta_0^2 + 2(\frac{1}{b} - 1)\Big)\alpha(\eta) - 3\eta\Big(\eta^2 + 6(\eta_0^2 + \frac{1}{b} - 1)\Big)\\ &\leq -3\eta(\eta^2 + 3\eta_0^2) < 0.\end{aligned}$$

■

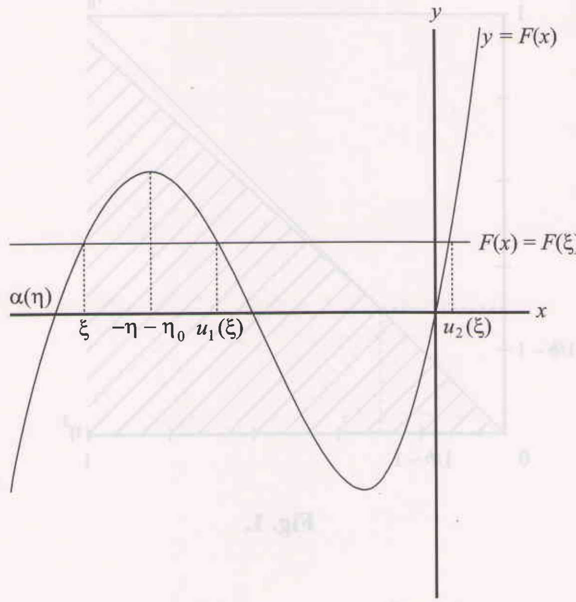


**Fig. 2.**

**Lemma 3.3.** *Assume that the condition [C5] is satisfied. Then* $\Phi(\beta(\eta)) < 0$.

*Proof.* From $\Phi'(\beta(\eta)) = 0$, we have

$$\begin{aligned}\Phi(\beta(\eta)) &= -4\Big(\eta_0^2 + \frac{1}{b} - 1\Big)\beta(\eta) - 4\eta\Big(\eta^2 + \eta_0^2 + 4\big(\frac{1}{b} - 1\big)\Big)\\ &= 4\Big\{\Big(\eta + \sqrt{2(\eta_0^2 + \frac{1}{b} - 1)}\Big)\Big(\eta_0^2 + \frac{1}{b} - 1\Big) - \eta\Big(\eta^2 + \eta_0^2 + 4\big(\frac{1}{b} - 1\big)\Big)\Big\} < 0. \quad \blacksquare\end{aligned}$$

From a property of the curve $(F(x), G(x))$, we shall show that, if $F(u_1) = F(u_2)$ for $\alpha(\eta) < \xi \leq -\eta - \eta_0$, then $G(u_2) - G(u_1) < 0$.

From $F(u_1) = F(u_2)$, we have

$$u_2^2 + u_1u_2 + u_1^2 = -3\{\eta(u_2 + u_1) + \eta^2 - \eta_0^2\}.$$

Thus, we obtain

$$\begin{aligned}G(u_2) - G(u_1) = \frac{\rho b}{12}(u_2 - u_1)\{&(u_2 + u_1)(u_2^2+u_1^2) + 4\eta(u_2^2 + u_1u_2 + u_1^2)\\ &+ 6(\eta^2 + \frac{1}{b} - 1)(u_2 + u_1)\}.\end{aligned}$$

Since $u_1$ and $u_2$ are solutions of the equation $F(x) = F(\xi)$, we obtain

$$u_1 + u_2 = -(3\eta + \xi) \quad \text{and} \quad u_1u_2 = 3\eta^2 - 3\eta_0^2 + 3\eta\xi + \xi^2.$$

Thus, we have

$$u_1^2 + u_2^2 = 3\eta^2 + 6\eta_0^2 - \xi^2.$$

By substituting $u_2 + u_1$ and $u_1^2 + u_2^2$ to $G(u_2) - G(u_1)$, we have

$$G(u_2) - G(u_1) = \frac{\rho b}{12}(u_2 - u_1)\Phi(\xi).$$

If the condition (C4) is satisfied, then from Lemma 3.2, we have $\Phi(\alpha(\eta)) < 0$ and $\beta(\eta) \leq \alpha(\eta)$. Since $\Phi$ is a function of the degree 3 and $\Phi(0) < 0$, we have $\Phi(\xi) < 0$ for $\alpha(\eta) < \xi \leq -\eta - \eta_0$.

If the condition (C5) is satisfied, then from Lemma 3.3, we have $\Phi(\xi) < 0$ for $\xi \leq 0$. From these facts and $u_2 - u_1 > 0$, if the condition (C4) or (C5) is satisfied, we conclude that $G(u_2) < G(u_1) < G(\xi)$ for $\alpha(\eta) < \xi \leq -\eta - \eta_0$. This means that the system (FNS) with the condition (C4) or (C5) has no non-trivial closed orbits. ■

*Remark 3.1.* The condition (C4) or (C5) is the same as that of [10]. However, the proof of Theorem 3.1 is easier than [10].

## 4. Global Asymptotic Stability

We say that the equilibrium point $E$ is globally asymptotically stable if $E$ is stable and if every orbit of the system (FNS) tends to $E$. By virtue of Theorem 3.1, we obtain the following:

**Theorem 4.1.** *Assume the condition (C3) or (C4) or (C5) in Theorem 3.1 is satisfied. Then the equilibrium point* $E$ (0, 0) *of the system (FNS) is globally asymptotically stable.*

*Proof.* We see from the uniqueness of orbits for the initial value problem and the Poincaré–Bendixon theorem that, if the system (FNS) satisfies the following conditions:

(iv) all orbits are bounded in the future;
(v) no closed orbits exists;
(vi) $E$ is asymptotically stable;

then the unique equilibrium point $E$ is globally asymptotically stable.

We shall check the above conditions to prove Theorem 4.1. Assume that the condition (C3) or (C4) or (C5) holds.

(iv) From [3], all orbits of the system (FNS) are bounded in the future.
(v) From Theorem 3.1, the system (FNS) has no non-trivial closed orbits.
(vi) Suppose $E$ is not stable. Then, from Lemma 2.3, every positive semitrajectories of (FNS) starting in the neighborhood of $E$ keeps on rotating around $E$ and going away from $E$. Hence, by the fact (iv) and the Poincaré–Bendixon theorem, the system has a closed orbit. This is a contradiction to the fact (v). Therefore, from Lemma 2.3, it follows that $E$ is asymptotically stable.

The proof of Theorem 4.1 is completed now. ■

In [7], we gave a result that the system (FNS) has a unique closed orbit if $\eta^2 = \eta_0^2 > (1/b) - 1$. Thus, from the above theorem, we have the following:

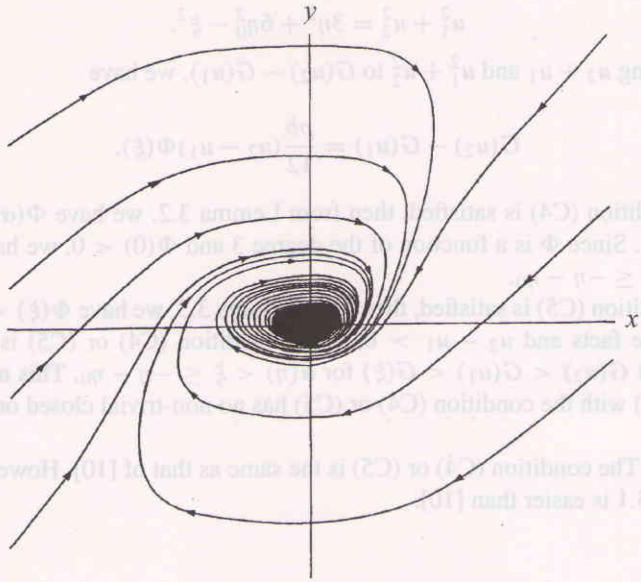


**Fig. 3.**

**Theorem 4.2.** *The equilibrium point E of the system (FNS) with $\eta^2 = \eta_0^2$ is globally asymptotically stable if and only if $\eta^2 \leq (1/b) - 1$.*

## 5. A Numerical Example

We shall present the phase portrait as an example illustrating an application of Theorem 4.2. Let $b = 4/5$ and $\rho = 1$. Then the system satisfies the condition $\eta^2 = \eta_0^2 \leq (1/b) - 1$. We see that the unique equilibrium point (0, 0) of the system is globally asymptotically stable as shown in Fig. 3.

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