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The Bundle Structure of Non-Commutative Tori*

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Abstract. The non-commutative torus $A_\omega = C^*(\mathbb{Z}^n, \omega)$ may be realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\widehat{S_\omega}$ with fibres $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ for some totally skew multiplier ω_1 on $\mathbb{Z}^n/S_\omega$. We prove that $A_\omega \otimes M_l(\mathbb{C})$ has the trivial bundle structure if and only if $\mathbb{Z}^n/S_\omega$ is torsion-free. It is shown that every non-commutative torus is stably isomorphic to a non-commutative torus with trivial bundle structure.

1. Introduction

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$, which is the universal object for unitary ω-representations of G. Our problem is to understand the structure, especially the bundle structure, of such C^*-algebras.

The twisted group C^*-algebra $C^*(\mathbb{Z}^n, \omega)$ by a multiplier ω on $\mathbb{Z}^n$ is called a *non-commutative torus of rank n* and is denoted by A_ω. The simplest non-trivial non-commutative tori arise when $G = \mathbb{Z}^2$. In this case we may assume that ω is antisymmetric and $\omega((1, 0), (0, 1)) = e^{\pi i\theta}$. When θ is irrational, one obtains a simple C^*-algebra called an *irrational rotation algebra*, and is denoted by A_θ. When $\theta = m/k$, one obtains a *rational rotation algebra*, and is denoted by $A_{m/k}$.

Now, the multiplier ω determines a subgroup S_ω of G called *symmetry group*. A multiplier ω on an abelian group G is called *totally skew* if the symmetry group S_ω is trivial, and A_ω is called *completely irrational* if ω is totally skew. Baggett and Kleppner [1] showed that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G, \omega)$ is a simple C^*-algebra.

Baggett and Kleppner [1] also showed that even when ω is not totally skew on a locally compact abelian group G, the restriction of ω-representations from G to S_ω induces a canonical homomorphism of $\mathrm{Prim}(C^*(G, \omega))$ with $\widehat{S_\omega}$. It was shown in [1] that there is a totally skew multiplier ω_1 on $\mathbb{Z}^n/S_\omega$ such that ω is similar to the pull-back of ω_1.

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Furthermore, it is known (see [4,6,9]) that $C^*(G, \omega)$ may be realized as the C^*-algebra $\Gamma(\zeta)$ of sections of a locally trivial C^*-algebra bundle ζ over $\widehat{S_\omega} = \mathrm{Prim}(C^*(G, \omega))$ with fibres $C^*(G, \omega)/x$ for $x \in \mathrm{Prim}(C^*(G, \omega))$ and all $C^*(G, \omega)/x$ turn out to be the simple, twisted group C^*-algebra $C^*(G/S_\omega, \omega_1)$. So A_ω is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\mathrm{Prim}(A_\omega) = \widehat{S_\omega}$ with fibres $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ for ω_1 a suitable totally skew multiplier on $\mathbb{Z}^n/S_\omega$.

A natural question is when the locally trivial bundle ζ is trivial. Poguntke [9] proved that A_ω is stably isomorphic to $C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^n/S_\omega, \omega_1)$.

Poguntke [8] showed that any primitive quotient of the group C^*-algebra $C^*(G)$ of a locally compact two step nilpotent group G is isomorphic to the tensor product of a completely irrational, non-commutative torus A_φ and $\mathcal{K}(\mathcal{H})$ for some (possibly finite-dimensional) Hilbert space $\mathcal{H}$. Since $C^*(G/S_\omega, \omega_1)$ is the primitive quotient of $C^*(G/S_\omega(\omega_1))$, where $G/S_\omega(\omega_1)$ is the extension group of G/S_ω by $\mathbf{T}$ defined by ω_1, $C^*(G/S_\omega, \omega_1)$ is isomorphic to $A_\varphi \otimes \mathcal{K}(\mathcal{H})$.

In this paper, we investigate the structure of the fibre of A_ω. We are going to show that $A_\omega \otimes M_l(\mathbb{C})$ has the trivial bundle structure if and only if $\mathbb{Z}^n/S_\omega$ is torsion-free. Furthermore, we will give an easy proof of the result of Poguntke.

2. Preliminaries

To fix notations, let

$$\mathbb{Z} = \text{the set of integers},$$
$$\mathbb{C} = \text{the set of complex numbers},$$
$$\otimes = \text{the minimal tensor product}.$$

We start our investigations with a study of decomposition of multipliers on $\mathbb{Z}^n/S_\omega$. If ω is a multiplier on G and H a closed subgroup of G, then we denote by $\omega|_H$ the restriction of ω to H. Furthermore, if $G = G_1 \oplus G_2$, and if ω_1 and ω_2 are multipliers on G_1 and G_2, respectively, then we denote by $\omega_1 \oplus \omega_2$ the multiplier on G defined by

$$(\omega_1 \oplus \omega_2)((x_1, x_2), (y_1, y_2)) = \omega_1(x_1, y_1)\omega_2(x_2, y_2),$$

$x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$.

For some groups G, each multiplier on G turns out to be a bicharacter.

Proposition 1. [7, Theorem 7.1] *Let G be a finitely generated discrete abelian group. Then every multiplier on G is similar to a bicharacter.*

Let ω be a multiplier on a locally compact abelian group G. Define a homomorphism $h_\omega : G \to \widehat{G}$ by $h_\omega(x)(y) = \omega(x, y)\omega(y, x)^{-1}$, $x, y \in G$ and let $S_\omega := \ker(h_\omega)$ denote the symmetry group of ω.

Next, we introduce the concept of C^*-algebra bundle over a locally compact Hausdorff space. Let $\mathrm{Prim}(C^*(G, \omega))$ be the primitive ideal space of the twisted group C^*-algebra $C^*(G, \omega)$ of a locally compact abelian group G defined by a multiplier ω.

Proposition 2. [1,6] *Let G be a locally compact abelian group and ω a multiplier on G. Then*

(i) there is a multiplier ω_1 on G/S_ω such that $C^*(G,\omega)/P$ is isomorphic to $C^*(G/S_\omega,\omega_1)$ for any $P \in \mathrm{Prim}(C^*(G,\omega))$ and ω is similar to the pull-back of a totally skew multiplier ω_1;

(ii) the restriction of ω-representations from G to S_ω induces a canonical homomorphism of $\mathrm{Prim}(C^*(G,\omega))$ with $\widehat{S_\omega}$.

By a trick similar to the proof of Theorem 1 in [4], one can show that, for a multiplier ω on a locally compact abelian group G, $C^*(G,\omega)$ can be realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle. That is, if A is a twisted group C^*-algebra of a locally compact abelian group, its C^*-algebra bundle is locally trivial. In particular, $A_\omega \cong C^*(\mathbb{Z}^n,\omega)$ may be represented as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\widehat{S_\omega}$ with fibres $C^*(\mathbb{Z}^n/S_\omega,\omega_1)$ (see [4,6,9] for details).

A problem then is to decide when this locally trivial bundle is actually trivial. Brabanter [2] proved that the rational rotation algebra has a non-trivial bundle structure. We will present a new proof of this result in the next section.

Let G be a finitely generated discrete abelian group, e.g., $\mathbb{Z}^n/S_\omega$, ω_1 a totally skew multiplier on G, and T the maximal torsion subgroup of G. Then $G \cong T \oplus F$ where F is a torsion-free subgroup. Note that $\omega_1|_F$ is always totally skew, but $\omega_1|_T$ need not be totally skew. A multiplier ω on a group G is said to be *type I* if $C^*(G,\omega)$ is a type I C^*-algebra.

Lemma 1. [4, Lemma 1] *Let ω be a multiplier on a locally compact abelian group G. Suppose G has a closed subgroup H such that $\omega|_H$ is totally skew and type I, and such that the group extension*

$$\{0\} \to H \to G \to G/H \to \{0\}$$

splits. Then there is a complement L to H in G such that (after replacing ω by a similar multiplier) ω splits as $\omega|_H \oplus \omega|_L$.

3. The Bundle Structure of Non-Commutative Tori

Let A_ω be a non-commutative torus of rank n. A_ω is isomorphic to the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\widehat{S_\omega}$ with fibres, the simple twisted group C^*-algebra $C^*(\mathbb{Z}^n/S_\omega,\omega_1)$ of a finitely generated discrete abelian group $\mathbb{Z}^n/S_\omega$ defined by a totally skew multiplier ω_1 on $\mathbb{Z}^n/S_\omega$. Here, ω is similar to the pull-back of ω_1. Then $\mathbb{Z}^n/S_\omega \cong T \oplus F$ where T is the maximal torsion subgroup of $\mathbb{Z}^n/S_\omega$ and F is a maximal torsion-free subgroup of $\mathbb{Z}^n/S_\omega$.

Assume T is trivial. Then, by Lemma 1, after replacing ω_1 by a similar multiplier, we may write $\mathbb{Z}^n/S_\omega = F$ and $\omega_1 = \omega_1|_F$. Let $\tilde{F}$ be the pull-back of F under the canonical map of $\mathbb{Z}^n$ to $\mathbb{Z}^n/S_\omega$. Then there is a subgroup F' such that $\tilde{F} = F' \oplus S_\omega \cong \mathbb{Z}^n$. And so $C^*(\mathbb{Z}^n,\omega) \cong C^*(\tilde{F},\omega|_{\tilde{F}}) \cong C^*(F',\omega|_{F'}) \otimes C^*(S_\omega) \cong C^*(F,\omega_1|_F) \otimes C^*(S_\omega) \cong C^*(\mathbb{Z}^n/S_\omega,\omega_1) \otimes C^*(S_\omega)$. This implies that if $\mathbb{Z}^n/S_\omega$ is torsion-free, then A_ω has the trivial bundle structure.

Theorem 1. [5, Theorem 2.2] *Let $A_\omega = C^*(u_1, \ldots, u_n)$ be a non-commutative torus of rank n, where $u_1, \ldots, u_n$ are unitary generators satisfying the commutation relations $u_i u_j u_i^{-1} u_j^{-1} = \exp(2\pi i \theta_{ij})$ (here, θ is a skew-symmetric $n \times n$ matrix with real entries). Then $K_0(A_\omega) \cong K_1(A_\omega) \cong \mathbb{Z}^{2^{n-1}}$, and $[1_{A_\omega}] \in K_0(A_\omega)$ is primitive.*

Proof. The proof is by induction on n. If $n = 1$, $A_\omega = C(S^1)$ is abelian, and the result is obvious. So assume that the result is true for all non-commutative tori of rank $n-1$. Write $A_\omega = C^*(B, u_n)$, where $B = C^*(u_1, \ldots, u_{n-1})$. Then the inductive hypothesis applies to B. Also, we can think of A_ω as the crossed product of B by an action α of $\mathbb{Z}$, where the generator of $\mathbb{Z}$ corresponds to u_n and acts on B by conjugation (sending u_j to $u_n u_j u_n^{-1} = \lambda_j u_j$, $\lambda_j = \exp(2\pi i \theta_{nj})$). Note that this action is homotopic to the trivial action, since we can homotope θ_{nj} to 0. Hence, $\mathbb{Z}$ acts trivially on the K-theory of B. The Pimsner–Voiculescu exact sequence for a crossed product gives

$$K_0(B) \xrightarrow{1-\alpha_*} K_0(B) \xrightarrow{\Phi} K_0(A_\omega) \to K_1(B) \xrightarrow{1-\alpha_*} K_1(B)$$

and similarly for K_1, where the map Φ is induced by inclusion. Since $\alpha_* = 1$ and since the K-groups of B are free abelian, this reduces a split short exact sequence

$$\{0\} \to K_0(B) \xrightarrow{\Phi} K_0(A_\omega) \to K_1(B) \to \{0\}$$

and similarly for K_1. So $K_j(A_\omega)$ is free abelian of rank $2 \cdot 2^{n-2} = 2^{n-1}$. Furthermore, since the inclusion $B \to A_\omega$ sends 1_B to 1_{A_ω}, $[1_{A_\omega}]$ is the image of $[1_B]$, which is primitive in $K_0(B)$ by inductive hypothesis. Hence, the image is primitive since the Pimsner–Voiculescu exact sequence is a split short exact sequence of torsion-free groups.

Therefore, $K_0(A_\omega) \cong K_1(A_\omega) \cong \mathbb{Z}^{2^{n-1}}$, and $[1_{A_\omega}] \in K_0(A_\omega)$ is primitive. ■

Now, we investigate the structure of the fibre $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ of $C^*(\mathbb{Z}^n, \omega)$.

Let G be a compactly generated locally compact abelian group and ω_1 a totally skew multiplier on G. Then let $E := G(\omega_1)$ be the extension group of G by $\mathbf{T}^1$ defined by ω_1. The following result of Poguntke clarifies the structure of the fibre of A_ω.

Theorem 2. [8, Theorem 1] *Let G be a compactly generated locally compact abelian group and ω_1 a totally skew multiplier on G. Let K be the maximal compact subgroup of E and let E_ρ be the stabilizer of an irreducible unitary representation ρ of K restricting on $\mathbf{T}^1$ to the identity. Then*

$$C^*(G, \omega_1) \cong C^*(E_\rho/K, m) \otimes \mathcal{K}(\mathcal{L}^2(E/E_\rho)) \otimes M_{\dim(\rho)}(\mathbb{C}),$$

where m is the associated Mackey obstruction.

This theorem is applied to understand the structure of the twisted group C^*-algebra $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$. Let $G = \mathbb{Z}^n/S_\omega$, $E = (\mathbb{Z}^n/S_\omega)(\omega_1)$, and let E_ρ be the stabilizer of an irreducible unitary representation ρ of the extension $K := T(\omega_1|_T)$ of T by $\mathbf{T}^1$ defined by $\omega_1|_T$, which restricts to the identity on $\mathbf{T}^1$. The Mackey method says that $C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong C^*(F \oplus T, \omega_1)$ is isomorphic to the primitive quotient of $C^*(E)$ lying over ρ. Then, by Theorem 2,

$$C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong C^*(E_\rho/K, m) \otimes \mathcal{K}(\mathcal{L}^2(E/E_\rho)) \otimes M_{\dim(\rho)}(\mathbb{C}).$$

Now by definition, E_ρ is of index $|S_{\omega_1|_T}|$ in E, where $S_{\omega_1|_T}$ is the symmetry group, a subgroup of T, of $\omega_1|_T$. So

$$\begin{aligned}[E : E_\rho] &= \#\text{ of irreducible } \omega_1|_T\text{-representations of } T \\ &= |S_{\omega_1|_T}|,\end{aligned}$$

and $\dim(\rho)\sqrt{|T|/|S_{\omega_1|_T}|}$, and E_ρ/K is a subgroup of finite index $[E : E_\rho]$ in E/K. Let F_ρ be the isomorphic image of E_ρ/K under the natural map of E/K to F. Then $\{x \in F \mid h_{\omega_1}(x)(y) = 1, \forall y \in S_{\omega_1|_T}\}$ is exactly F_ρ, and F_ρ is a subgroup of finite index $[E : E_\rho]$ in F. Let $J_F = F/F_\rho$, $J = J_F \oplus S_{\omega_1|_T}$, and $T_t = T/S_{\omega_1|_T}$. Then $|J_F| = |S_{\omega_1|_T}|$. Since F_ρ is a subgroup of F, we can consider $J_F \oplus S_{\omega_1|_T}$ as a subgroup of $(F \oplus T)/F_\rho$. So $(\mathbb{Z}^n/S_\omega)/F_\rho$ is isomorphic to $J_F \oplus T$ and $((\mathbb{Z}^n/S_\omega)/F_\rho)/J$ is isomorphic to T_t.

Next, we show that $C^*(E_\rho/K, m)$ is isomorphic to $C^*(F_\rho, \omega_1|_{F_\rho})$. By Theorem 2, $C^*(F_\rho, \omega_1|_{F_\rho}) \cong C^*(F_\rho(\omega_1|_{F_\rho})/\mathbf{T}^1, m_1)$, where m_1 is the associated Mackey obstruction. Let ω_2 be a totally skew multiplier on T_t whose pull-back to T is similar to $\omega_1|_T$. It is sufficient to show that the Mackey obstruction m_2, in the isomorphism

$$\begin{aligned}&C^*(F_\rho \oplus T_t, \omega_1|_{F_\rho} \oplus \omega_2) \\ &\cong C^*((F_\rho \oplus T_t)(\omega_1|_{F_\rho} \oplus \omega_2)/T_t(\omega_2), m_2) \otimes C^*(T_t, \omega_2) \\ &\cong C^*(F_\rho, \omega_1|_{F_\rho}) \otimes C^*(T_t, \omega_2)\end{aligned}$$

is essentially the same as m_1. But for $h \in F_\rho$, the unitary operators E'_h in [3, XII.1.17] are the same for F_ρ and for $F_\rho \oplus T_t$ up to scalar. They give the same Mackey obstructions. So

$$\begin{aligned}C^*((F_\rho \oplus = T_t)(\omega_1|_{F_\rho} \oplus \omega_2)/T_t(\omega_2), m_2) &\cong C^*(F_\rho(\omega_1|_{F_\rho})/\mathbf{T}^1, m_1) \\ &\cong C^*(F_\rho, \omega_1|_{F_\rho}),\end{aligned}$$

and $C^*(E_\rho/K, m)$ is isomorphic to $C^*(F_\rho, \omega_1|_{F_\rho})$.

Corollary 1. $C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong C^*(F_\rho, \omega_1|_{F_\rho}) \otimes M_{[E:E_\rho]}(\mathbb{C}) \otimes M_{\dim(\rho)}(\mathbb{C})$.

Proof. By Theorem 2,

$$\begin{aligned}C^*(\mathbb{Z}^n/S_\omega, \omega_1) &\cong C^*(E_\rho/K, m) \otimes \mathcal{K}(\mathcal{L}^2(E/E_\rho)) \otimes = M_{\dim(\rho)}(\mathbb{C}) \\ &\cong C^*(F_\rho, \omega_1|_{F_\rho}) \otimes M_{[E:E_\rho]}(\mathbb{C}) \otimes M_{\dim(\rho)}(\mathbb{C}).\end{aligned}$$

Here, $M_{[E:E_\rho]}(\mathbb{C}) \cong M_{|J_F|}(\mathbb{C})$ and $M_{\dim(\rho)}(\mathbb{C}) \cong M_{\sqrt{|T_t|}}(\mathbb{C})$. Hence, one obtains the result. ∎

Note that $C^*(F_\rho, \omega_1|_{F_\rho})$ is a completely irrational, non-commutative torus.

Let A_ω be a non-commutative torus. It follows from Corollary that A_ω is isomorphic to the C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over $\widehat{S_\omega}$ with fibres $C^*(F_\rho, \omega_1|_{F_\rho}) \otimes M_{[E:E_\rho]}(\mathbb{C}) \otimes M_{\dim(\rho)}(\mathbb{C})$.

Theorem 3. *Let l be a positive integer. Then $A_\omega \otimes M_l(\mathbb{C})$ is not isomorphic to $A \otimes M_{kl}(\mathbb{C})$ for any C^*-algebra A if $k \neq 1$.*

Proof. Assume that $A_\omega \otimes M_l(\mathbb{C})$ is isomorphic to $A \otimes M_{kl}(\mathbb{C})$ for some integer k and some C^*-algebra A. Then the unit $1_{A_\omega} \otimes I_l$ maps to the unit $1_A \otimes I_{kl}$, where I_d denotes the $d \times d$ identity matrix. Since $[1_A \otimes I_{kl}] = kl[1_A]$, there is a projection e in $A_\omega \otimes M_l(\mathbb{C})$ such that

$$[1_{A_\omega} \otimes I_l] = kl[e].$$

Hence,

$$l[1_{A_\omega}] = [1_{A_\omega} \otimes I_l] = kl[e].$$

But, by Theorem 1, the K-groups of A_ω are torsion-free, so $[1_{A_\omega}] = k[e]$, which contradicts Theorem 1 if $k \neq 1$.

Therefore, $A_\omega \otimes M_l(\mathbb{C})$ is not isomorphic to $A \otimes M_{kl}(\mathbb{C})$ for any C^*-algebra A if $k \neq 1$. ■

In particular, one obtains that no non-trivial matrix algebra can be factored out of any rational rotation algebra $A_{m/k}$. So every rational rotation algebra has a non-trivial bundle structure. This gives an alternative proof of a result of Brabanter.

Theorem 3 implies that if $A_\omega \otimes M_p(\mathbb{C})$ is isomorphic to $A_\rho \otimes M_q(\mathbb{C})$, then $p = q$. However, there are non-isomorphic non-commutative tori A_ω and A_ρ such that $A_\omega \otimes M_p(\mathbb{C})$ is isomorphic to $A_\rho \otimes M_p(\mathbb{C})$ for some integer p.

Corollary 2. *Let l be a positive integer. Then $A_\omega \otimes M_l(\mathbb{C})$ has a non-trivial bundle structure unless $\mathbb{Z}^n/S_\omega$ is torsion-free.*

Proof. Assume $A_\omega \otimes M_l(\mathbb{C})$ has the trivial bundle structure, i.e., $A_\omega \otimes M_l(\mathbb{C})$ is isomorphic to $C^*(F_\rho, \omega_1|_{F_\rho}) \otimes C(\widehat{S_\omega}) \otimes M_l(\mathbb{C}) \otimes M_k(\mathbb{C})$, where $M_k(\mathbb{C}) := M_{[E:E_\rho]}(\mathbb{C}) \otimes M_{\dim(\rho)}(\mathbb{C})$. If $\mathbb{Z}^n/S_\omega$ is not torsion-free, then $M_k(\mathbb{C})$ is non-trivial. So $A_\omega \otimes M_l(\mathbb{C})$ is isomorphic to $A \otimes M_{kl}(\mathbb{C})$ where A is isomorphic to $C^*(F_\rho, \omega_1|_{F_\rho}) \otimes C(\widehat{S_\omega})$. This contradicts Theorem 3 if $\mathbb{Z}^n/S_\omega$ is not torsion-free.

Therefore, $A_\omega \otimes M_l(\mathbb{C})$ has a non-trivial bundle structure unless $\mathbb{Z}^n/S_\omega$ is torsion-free. ■

We have obtained that $A_\omega \otimes M_l(\mathbb{C})$ has the trivial bundle structure if and only if $\mathbb{Z}^n/S_\omega$ is torsion-free.

4. Stable Isomorphism of Non-Commutative Tori

The non-commutative torus A_ω of rank n is obtained by an iteration of $n - 1$ crossed products by actions of $\mathbb{Z}$, the first action on $C(\mathbf{T}^1)$ (see [5]). When A_ω is not simple, by a change of basis, A_ω can be obtained by an iteration of $n-2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{m/k}$, where the actions on the fibre $M_k(\mathbb{C})$ of $A_{m/k}$ are trivial, since $M_k(\mathbb{C})$ is a factor of the fibre of A_ω.

Theorem 4. [2, Theorem 3] *The rational rotation algebra $A_{m/k}$ is stably isomorphic to $C^*(k\mathbb{Z} \times k\mathbb{Z})$.*

Poguntke proved that every non-commutative torus A_ω is stably isomorphic to $C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^n/S_\omega, \omega_1)$. The Mackey machine for a twisted crossed product says that $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ is isomorphic to the tensor product of a completely irrational, non-commutative torus A_ρ with a matrix algebra $M_{kd}(\mathbb{C})$.

Theorem 5. [9] *A_ω is stably isomorphic to $C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^n/S_\omega, \omega_1)$.*

Proof. By Theorem 4, $A_{m/k} \otimes \mathcal{K}(\mathcal{H})$ is isomorphic to $C^*(k\mathbb{Z} \times k\mathbb{Z}) \otimes M_k(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})$. The non-simple, non-commutative torus A_ω of rank n may be realized as the crossed product

$$A_{m/k} \times_{\alpha_1} \mathbb{Z} \times_{\alpha_2} \cdots \times_{\alpha_{n-2}} \mathbb{Z},$$

where α_i act trivially on the fibre $M_k(\mathbb{C})$ of $A_{m/k}$. So

$$\begin{aligned} A_\omega \otimes \mathcal{K}(\mathcal{H}) &\cong (A_{m/k} \times_{\alpha_1} \mathbb{Z} \times_{\alpha_2} \cdots \times_{\alpha_{n-2}} \mathbb{Z}) \otimes \mathcal{K}(\mathcal{H}) \\ &\cong (A_{m/k} \otimes \mathcal{K}(\mathcal{H})) \times_{\tilde{\alpha_1}} \mathbb{Z} \times_{\tilde{\alpha_2}} \cdots \times_{\tilde{\alpha_{n-2}}} \mathbb{Z}, \end{aligned}$$

where $\tilde{\alpha_i}$ are the canonical extensions of α_i such that $\tilde{\alpha_i}$ act trivially on $M_k(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})$. Thus,

$$\begin{aligned} A_\omega \otimes \mathcal{K}(\mathcal{H}) &\cong (C(\widehat{k\mathbb{Z} \times k\mathbb{Z}}) \otimes M_k(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H})) \times_{\tilde{\alpha_1}} \mathbb{Z} \times_{\tilde{\alpha_2}} \cdots \times_{\tilde{\alpha_{n-2}}} \mathbb{Z} \\ &\cong (C(\widehat{k\mathbb{Z} \times k\mathbb{Z}}) \times_{\alpha_1} \mathbb{Z} \times_{\alpha_2} \cdots \times_{\alpha_{n-2}} \mathbb{Z}) \otimes M_k(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H}). \end{aligned}$$

Thus, A_ω is stably isomorphic to $(C(\widehat{k\mathbb{Z} \times k\mathbb{Z}}) \times_{\alpha_1} \mathbb{Z} \times_{\alpha_2} \cdots \times_{\alpha_{n-2}} \mathbb{Z}) \otimes M_k(\mathbb{C})$. But $C(\widehat{k\mathbb{Z} \times k\mathbb{Z}}) \times_{\alpha_1} \mathbb{Z} \times_{\alpha_2} \cdots \times_{\alpha_{n-2}} \mathbb{Z}$ is a non-commutative torus with fibres $A_\rho \otimes M_d(\mathbb{C})$. So by a finite step of the above process, one can obtain that $A_\omega \otimes \mathcal{K}(\mathcal{H})$ is isomorphic to $C(\widehat{S_\omega}) \otimes A_\rho \otimes M_{kd}(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H}) \cong C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^n/S_\omega, \omega_1) \otimes \mathcal{K}(\mathcal{H})$.

Therefore, A_ω is stably isomorphic to $C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^n/S_\omega, \omega_1)$. ∎

We have obtained that the non-commutative torus A_ω is stably isomorphic to $C(\widehat{S_\omega}) \otimes A_\rho \otimes M_{kd}(\mathbb{C}) \cong C(\widehat{S_\omega}) \otimes C^*(\mathbb{Z}^n/S_\omega, \omega_1)$. Hence, A_ω is stably isomorphic to the non-commutative torus $C(\widehat{S_\omega}) \otimes A_\rho$, which has the trivial bundle structure.

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