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Survey

Normal Sets, Polyblocks, and Monotonic Optimization

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Abstract. A normal set is a subset of the non-negative orthant R^n_+ such that, whenever it contains a point x, it contains all $x' \in R^n_+$ such that $x' \leq x$. We investigate properties of normal sets and elementary normal sets called polyblocks. These properties furnish the foundation for a new approach to the numerical study of systems of monotonic inequalities and optimization problems involving differences of monotone increasing functions (d.i. functions).

1. Introduction

The role of convexity in modern optimization theory is well known. Since any inequality $g(x) \leq 0$, where $g : R^n \to R$ is an arbitrary continuous function, can be converted into an equivalent inequality $u(x) - v(x) \leq 0$, with two convex functions $u(x), v(x)$ (see, e.g., [23]), it is natural that the difference convex (d.c.) structure underlies a wide variety of non-convex problems. In fact, convex analysis which was primarily developed for the needs of convex optimization has become in recent years an essential tool in non-convex optimization as well.

Aside from convexity, monotonicity is another very useful concept when dealing with mathematical models of systems in economics, engineering, and other fields. The simplest monotonicity property for a function $f(x)$ is that of being *increasing* (*decreasing*, resp.) on R^n_+, i.e., such that $f(x) \leq f(x')$ ($f(x) \geq f(x')$, resp.) whenever $0 \leq x \leq x'$. The analysis of monotonicity for the purpose of applications to engineering design problems was explored by Wilde et al. [1, 14, 15, 29], and subsequently, Hansen et al. [5]. Dealing with constrained optimization problems involving partial monotonicity, these authors focused on finding which constraints must be tight at the optimum in order to lower the dimension of the problem and reduce it to a form more amenable to an effective solution. A concept closely related to monotonicity is that of *normal set* which is defined as any set $G \subset R^n_+$ such that $x \in G$ whenever $0 \leq x \leq x'$, and $x' \in G$. Normal sets were first introduced in mathematical economics (see, e.g., [11, 13]) mostly from a conceptual point of view, in connection with the analysis of production activities within an economic system. Just as convex sets are essentially lower level sets of quasiconvex functions, normal sets are essentially lower level sets of increasing functions.

From the point of view of numerical optimization, the most basic property of an increasing function is that when seeking a minimizer of it over a constraint set D; once a solution $x^0 \in D$ is known, then all the solutions in the orthant $x^0 + R^n_+$ can be omitted because no better solution than x^0 can be found among the latter. Such information is very useful and may sometimes help simplify the problem drastically by limiting the search process to a restricted area. Likewise, if the constraint set is a normal set, then any infeasible solution z^0 can be strictly separated from the constraint set by an orthant $x^0 + R^n_+$, where $x^0 \leq z^0$ is some suitable feasible solution. It is well known that the classical separation property of convex sets is fundamental for many solution strategies in convex and non-convex optimization. This suggests that the specific separation property of normal sets should play an equally important role in the analysis and solution of monotonic optimization problems.

The aim of this paper is to present a systematic study of normal sets with a view of application to the theory of monotonic inequalities and monotonic optimization. We shall show that any closed normal set is the intersection of a decreasing sequence of elementary normal sets called *polyblocks*. This outer approximation of normal sets by polyblocks is similar to the outer approximation of convex sets by polyhedrons. It can be used to establish a characterization of the structure of the solution set of a monotonic system in such a way as to allow efficient numerical analysis of monotonic inequalities and monotonic optimization problems. More importantly, the polyblock approximation method leads to a general approach for solving optimization problems involving differences of increasing functions. The fact that any polynomial of several variables is a difference of two increasing functions on the non-negative orthant implies that the range of applicability of this approach includes polynomial programming, in particular, non-convex quadratic programming, whose importance in global and combinatorial optimization has very much increased in recent years.

This paper consists of six sections. After Sec. 1, we shall review in Sec. 2, the basic properties of normal sets and reverse normal sets. Aside from known properties [11, 19], we shall establish a number of new ones which seem to play a major role in monotonic optimization. Polyblocks and reverse polyblocks are introduced and studied in Sec. 3. Section 4 is devoted to systems of monotonic inequalities. Here, we shall introduce the concepts of upper and lower basic solutions and shall prove that any of these solutions can be characterized by a sequence of natural numbers between 1 and n. Based on this characterization of the solution set structure of a monotonic system, algorithms will be proposed in Sec. 5 for maximizing or minimizing an increasing function under monotonic constraints. Finally, in Sec. 6, the approach will be extended to d.i. optimization, i.e., optimization of differences of increasing functions.

2. Normal Sets

We begin by introducing some notations and concepts. For any two vectors $x', x \in R^n$, we write $x' \geq x$ and say that x' *dominates* x if $x'_i \geq x_i, \ \forall i = 1, \ldots, n$. We write $x' > x$ and say that x' *strictly dominates* x if $x'_i > x_i, \ \forall i = 1, \ldots, n$.

Let $R^n_+ = \{x \in R^n \mid x \geq 0\}$ and $R^n_{++} = \{x \in R^n \mid x > 0\}$. For $x \in R^n_+$, let $I(x) = \{i \mid x_i = 0\}$ and denote

$$K_x = \{x' \in R^n_+ \mid x'_i > x_i \ \forall i \notin I(x)\}, \quad \mathrm{cl}K_x = \{x' \in R^n_+ \mid x' \geq x\}.$$

For $a \leq b$, the box (hyper-rectangle) $[a, b]$ is defined to be the set of all x such that $a \leq x \leq b$. We also write $(a, b] := \{x \mid a < x \leq b\}$, $[a, b) := \{x \mid a \leq x < b\}$. As usual e is the vector of all ones and e^i the ith unit vector of R^n.

A set $G \subset R^n_+$ is called *normal* if, for any two points $x, x' \in R^n_+$ such that $x' \leq x$, if $x \in G$, then $x' \in G$, too. The empty set, the singleton $\{0\}$, and R^n_+ are special normal sets which we will refer to as *trivial subsets of* R^n_+. If G is a normal set, then $G \cup \{x \in R^n_+ \mid x_i = 0 \text{ for some } i = 1, \dots, n\}$ is still normal.

For any set $D \subset R^n_+$, the orthant R^n_+ is a normal set containing D. The intersection of all normal sets containing D, i.e., the smallest normal set containing D, is called the *normal hull* of D.

Proposition 1. *The normal hull of a set $D \subset R^n_+$ is the set $N[D] := (D - R^n_+) \cap R^n_+$. If D is compact, then so is $N[D]$.*

Proof. The set $N[D]$ is obviously normal and any normal set containing D obviously contains it. Therefore, $N[D]$ is the normal hull of D. Let D be compact and let $x^k \in N[D]$, $x^k \to x^0$ as $k \to +\infty$. Then $x^k = y^k - z^k$, with $y^k \in D, z^k \in R^n_+$. Since D is compact, we can assume, by passing to a subsequence if necessary, that $y^k \to y^0 \in D$. Hence, $z^k = y^k - x^k \to z^0 = y^0 - x^0 \geq 0$, i.e., $x^0 = y^0 - z^0$ with $y^0 \in D,\ z^0 \in R^n_+$, which implies that $x^0 \in N[D]$. Therefore, $N[D]$ is closed. If $D \subset [0, b]$, then $N[D] \subset [0, b]$, so $N[D]$ is bounded, and hence, compact. ■

Proposition 2. *The intersection and the union of a family of normal sets are normal sets.*

Proof. Immediate. ■

Proposition 3. *Every normal set is connected. A normal set G has a non-empty interior if and only if it contains a point $u \in R^n_{++}$.*

Proof. The first assertion is trivial because, for any two points x, x' in a normal set G, both segments joining 0 to x and 0 to x' belong to G. If there is $u \in G \cap R^n_{++}$, then since $[0, u] \subset G$ and $[0, u]$ has interior points, it follows that $\mathrm{int}G \neq \emptyset$. The converse is obvious because an interior point of a subset of R^n_+ must have positive coordinates. ■

A point $y \in R^n_+$ is called an *upper boundary point* of a normal set G if $y \in \mathrm{cl}G$ (hence, $[0, y] \subset \mathrm{cl}G$) while $K_y \subset R^n_+ \setminus G$. The set of upper boundary points of G is called the *upper boundary* of G and is denoted by $\partial^+ G$. If G is closed, then obviously $\partial^+ G \subset G$.

Proposition 4. *Let $G \subset [0, b]$ be a compact normal set with non-empty interior. For every $u \in G$ and $v \in R^n_+ \setminus \{0\}$ the halfline $\Gamma(u, v) := \{u + \alpha v \mid \alpha \geq 0\}$ meets the upper boundary of G at a unique point $\sigma_G(u, v)$ defined by*

$$\sigma_G(u, v) = u + \mu v, \quad \mu = \sup\{\alpha \mid u + \alpha v \in G\}. \tag{1}$$

Proof. Obviously, $u \in \Gamma(u, v) \cap G$, and whenever $x \in \Gamma(u, v) \cap G$, then the whole segment joining u and x belongs to $\Gamma(u, v) \cap G$. Hence, $\Gamma(u, v) \cap G$ is a segment. Let u and y be the endpoints of this segment. Clearly, $y = \sigma_G(u, v)$ and $y \in G$. If there were $x \in G \cap K_y$, then $[y, x] \subset G$, and since $y = u + \mu v$, we would have $u_i + \mu v_i < x_i$, $\forall i \notin I(y)$, while $u_i + \mu v_i = x_i = 0$, $\forall i \in I(y)$, hence there would exist $\alpha > \mu$ such that $u_i + \mu v_i < u_i + \alpha v_i < x_i$, $\forall i \notin I(y)$, i.e., such that $u + \alpha v \in [y, x] \subset G$, contradicting (1). Therefore, $K_y \subset R^n_+ \setminus G$, and so $y \in \partial^+ G$. For any $y' \in \partial^+ G \cap \Gamma(u, v)$, we have $y' \in \Gamma(u, v) \setminus K_y$, hence, $y' = u + \alpha v$ with $\alpha \leq \mu$, i.e., $y' \leq y$. On the other hand, since $y' \in \partial^+ G$, it follows that $K_{y'} \cap G = \emptyset$, i.e., $y \in \Gamma(u, v) \setminus K_{y'}$, and hence, $y \leq y'$. Therefore, $y' = y$, completing the proof of Proposition 4. ■

Corollary 1. *A compact normal set G is equal to the normal hull of its upper boundary* $\partial^+ G$.

Proof. For any $x \in G \setminus \{0\}$, we have $x \leq y := \sigma_G(0, x) \in \partial^+ G$, i.e., $x \in [0, y] \subset N[\partial^+ G]$. Therefore, $G \subset N[\partial^+ G]$. The converse is obvious. ■

Let D be a compact subset of R^n_+. A point $v \in D$ is called an *upper extreme point* of D if $x \in G, x \geq v \Rightarrow x = v$. Clearly, every upper extreme point v of a compact normal set $G \subset R^n_+$ satisfies $K_v \subset R^n_+ \setminus G$, and hence is an upper boundary point of G. In other words, if $V = V(G)$ denotes the set of upper extreme points of G, then $V \subset \partial^+ G$.

Proposition 5. *A compact normal set* $G \subset R^n_+$ *is equal to the normal hull of the set V of its upper extreme points.*

Proof. In view of Corollary 1, $N[V] \subset N[\partial^+ G] = G$, so it suffices to show that $\partial^+ G \subset N[V]$. Let $y \in \partial^+ G$. Define $x^1 \in \text{argmax}\{x_1 | \ x \in G, \ x \geq y\}$, and $x^i \in \text{argmax}\{x_i | \ x \in G, \ x \geq x^{i-1}\}$ for $i = 2, \ldots, n$. Then $v := x^n \in G$ and $v \geq x$ for all $x \in G$ satisfying $x \geq y$. Therefore, $x \in G, x \geq v \Rightarrow x = v$. This means that $y \leq v \in V$, hence $y \in N[V]$, as was to be proved. ■

Proposition 6. *The set of upper extreme points of the normal hull of a compact set* $D \subset R^n_+$ *is contained in the set of upper extreme points of D.*

Proof. If $v \in D$ but v is not an upper extreme point, then there exists a point $x \in D$ satisfying $x \geq v, x \neq v$. Since $D \subset N[D]$, this implies that v is not an upper extreme point of $N[D]$. ■

Remark 1. For normal sets, upper extreme points play a role analogous to that of extreme points for convex sets. In fact, Propositions 5 and 6 are analogous to well-known propositions in convex analysis, namely that a compact convex set is equal to the convex hull of the set of its extreme points, and any extreme point of the convex hull of a compact set is an extreme point of this set.

A function $f : R^n \to R$ is said to be *increasing* on R^n_+ if $f(x) \leq f(x')$ whenever $0 \leq x \leq x'$; it is said to be increasing on a box $[a, b] \subset R^n_+$ if $f(x) \leq f(x')$ whenever $a \leq x \leq x' \leq b$. Functions increasing in this sense abound in economics, engineering, and many other fields. Outstanding examples are production functions (e.g., the Cobb–Douglas function $f(x) = \Pi_i x_i^{\alpha_i}$, $\alpha_i \geq 0$), cost functions, and utility functions in Mathematical Economics, posynomials (in particular quadratic functions) with non-negative coefficients, posynomials $\Sigma_{j=1}^m c_j \Pi_{i=1}^n (x_i)^{a_{ij}} (c_j \geq 0, a_{ij} \geq 0)$ in engineering design problems, etc. Other non-trivial examples are functions of the form $f(x) = \sup\{g(u)|\ u \in D(x)\}$, where $g : R_+^n \to R$ is a continuous function and $D : R_+^n \to 2^{R_+^n}$ is a compact-valued multimapping such that $D(x') \supset D(x)$ for $x' \geq x$.

Proposition 7.
(i) *If f_1, f_2 are increasing functions, then for any non-negative numbers λ_1, λ_2, the function $\lambda_1 f_1 + \lambda_2 f_2$ is increasing.*
(ii) *The pointwise supremum of a bounded above family $(f_\alpha)_{\alpha \in A}$ of increasing functions and the pointwise infimum of a bounded below family $(f_\alpha)_{\alpha \in A}$ of increasing functions are increasing.*

Proof. Immediate. ■

It is well known that the maximum of a quasiconvex function over a compact set is equal to its maximum over the convex hull of this set and is attained at one extreme point. Analogously:

Proposition 8. *The maximum of an increasing function $f(x)$ over a compact set D is equal to its maximum over the normal hull of D and is attained at at least one upper extreme point.*

Proof. Let $\bar{x}$ be a maximizer of $f(x)$ on $G = N[D]$. Since, by Proposition 5, G is equal to the convex hull of the set V of its upper extreme points, there exists $v \in V$ such that $\bar{x} \leq v$. Then $f(v) \geq f(\bar{x})$, hence, v is also a maximizer of $f(x)$ on G. But, by Proposition 6, v is also an upper extreme point of D, hence, it is also a maximizer of $f(x)$ on D. ■

Just as convex sets are essentially lower level sets of quasiconvex functions, normal sets are essentially lower level sets of increasing functions, as shown by the next proposition.

Proposition 9. *For any increasing function $g(x)$ on R_+^n, the level set $G = \{x \in R_+^n|\ g(x) \leq 1\}$ is a normal set, closed if $g(x)$ is lower semi-continuous. Conversely, for any compact normal set $G \subset R_+^n$ with non-empty interior there exists a lower semicontinuous increasing function $g : R_+^n \to R_+$ such that $G = \{x \in R_+^n|\ g(x) \leq 1\}$.*

Proof. We need only prove the second assertion. Let G be a compact normal set with non-empty interior. For every $x \in R_+^n$, define $g(x) = \inf\{\lambda > 0|\ x \in \lambda G\}$. From the assumption $\text{int}G \neq \emptyset$, there is $u > 0$ such that $[0, u] \subset G$ (Proposition 3). Then, for any $x \in R_+^n$, the halfline $\{\alpha x|\ \alpha > 0\}$ intersects $[0, u] \subset G$, hence, $0 \leq g(x) < +\infty$. Since for every $\lambda > 0$ the set λG is normal, if $x \leq x' \in \lambda G$, then $x \in \lambda G$, too, so $g(x') \geq g(x)$, i.e., $g(x)$ is increasing. We show that $G = \{x|\ g(x) \leq 1\}$. In fact, if $x \in G$, then obviously $g(x) \leq 1$. Conversely, if $x \notin G$, then since G is compact, there exists $\alpha < 1$ such that $\alpha x \notin G$, i.e., $x \notin (1/\alpha)G$. Hence, since G is normal, $x \notin \lambda G$ for all $\lambda > 1/\alpha$, which in turn implies that $g(x) > 1/\alpha > 1$, and hence, $G = \{x \in R_+^n|\ g(x) \leq 1\}$. It remains to prove the lower semicontinuity of $g(x)$. Let $\{x^k\} \subset R_+^n$ be a sequence such that $g(x^k) \leq \alpha\ \forall k$. Then, for any given $\alpha' > \alpha$, we have

$x^k \in \alpha' G\ \forall k$, hence $x^0 \in \alpha' G$ in view of the closedness of the set $\alpha' G$. This implies that $g(x^0) \leq \alpha'$ and since α' can be taken arbitrarily near to α, we must have $g(x^0) \leq \alpha$. Therefore, the set $\{x \in R^n+ \mid g(x) \leq \alpha\}$ is closed, as was to be proved. ■

Note that if $G = \{x \mid g(x) \leq 1\}$, where $g(x)$ is a continuous increasing function, then obviously $\partial^+ G \subset \{y \in R^n_+ \mid g(y) = 1\}$, but the converse may not be true.

A set $H \subset R^n_+$ is said to be *reverse normal* if $x' \geq x \in H$ implies $x' \in H$. It is said to be *reverse normal* in a box $[0, b]$ if $b \geq x' \geq x \in H$ implies $x' \in H$ or equivalently, if $x' \notin H$ whenever $0 \leq x' \leq x \notin H$. Clearly, a set H is reverse normal if and only if the set $H^\flat = R^n_+ \setminus H$ is normal. For any set $D \subset R^n_+$, the set $D + R^n_+$ is obviously the smallest reverse normal set containing D. We call it the *reverse normal hull* of E and denote it by $rN[D]$.

It follows from Proposition 9 that, for any increasing function $h(x)$ on R^n_+, the set $H = \{x \in R^n_+ \mid h(x) \geq 1\}$ is reverse normal and this set is closed if $h(x)$ is upper semicontinuous.

Let H be a reverse normal set. A point $y \in R^n_+$ is said to be a *lower boundary point* of H if $y \in \mathrm{cl}H$ (hence, $y + R^n_+ \subset \mathrm{cl}H$) while $x \notin H\ \forall x < y$. The set of lower boundary points of H is called the *lower boundary* of H and is denoted by $\partial^- H$. If H is closed, then $\partial^- H \subset H$.

Proposition 10. *Let H be a reverse normal set. For every $u \in H$ and $v \in R^n_+ \setminus \{0\}$, the halfline $\{u - \alpha v \mid \alpha \geq 0\}$ meets $\partial^- H$ at a unique point $\omega_H(u, v)$ defined by*

$$\omega_H(u, v) = u - \lambda v, \quad \lambda = \sup\{\alpha \mid u - \alpha v \in H\}. \tag{2}$$

Proof. Similar to the proof of Proposition 4. ■

Corollary 2. *A closed reverse normal set H is equal to the reverse normal hull of its lower boundary $\partial^- H$.*

Proof. Since the fact is obvious when $0 \in H$ (i.e., $H = R^n_+$), we may assume that $0 \notin H$. For any $x \in H$, we have $x \geq y := \omega_H(x, x) \in \partial^- H$, i.e., $x \in rN[\partial^- H]$. Therefore, $H \subset rN[\partial^- H]$. The converse is obvious. ■

A point v of a compact set $D \subset R^n_+$ is called a *lower extreme point* if $x \in D, x \leq v \Rightarrow x = v$. Analogously to Propositions 5 and 8, we can prove that a closed reverse normal set is equal to the reverse normal hull of the set of its lower extreme points; the minimum of an increasing function over a compact set $D \subset R^n_+$ is attained at a lower extreme point.

Let $G \subset [0, b]$ be a normal set and $G^\flat = R^n_+ \setminus G$. It is easily verified that

$$(\partial^- G^\flat) \cap R^n_{++} \subset \partial^+ G \subset \partial^- G^\flat, \tag{3}$$

but in general $(G \cap \partial^- G^\flat) \setminus \partial^+ G \neq \emptyset$. A normal set G such that $G \cap (\partial^- G^\flat) \subset \partial^+ G$ is said to be *regular*. A set G is said to be *robust* if any point of G is the limit of a sequence of interior points of G.

Proposition 11. *A normal set G is regular if and only if it is robust.*

Proof. Let $H = G^\flat$. Suppose G is robust and let $y \in G \cap \partial^- H$. Then $y + R^n_+ \subset \mathrm{cl}H$ (because $y \in \partial^- H$). If $z \in G \cap K_y$, then, since $z \in G$, we have $z = \lim_{k \to +\infty} z^k$, $z^k \in G$, $z^k > 0$, and, since $z \in K_y$, i.e., $z_i > y_i \ \forall i \notin I(y)$, we must have, for k large enough, $z^k_i > y_i \ \forall i \notin I(y)$, i.e., $z^k \in y + R^n_{+}$, while $z^k \notin \mathrm{cl}H$, a contradiction. Therefore, $K_y \cap G = \emptyset$, and hence, $y \in \partial^+ G$. Conversely, suppose for some $y \in G$ there is no interior point of G in some neighborhood of y. Then, for $z = \lambda y$ with $0 < \lambda < 1$ and λ close enough to 1, one has $z \in \mathrm{cl}H$, and $x \in G \ \forall x < z$, hence, $z \in \partial^- H$. On the other hand, $z \notin \partial^+ G$ because $y \in G \cap K_z$. Therefore, $G \cap \partial^- H \setminus \partial^+ G \neq \emptyset$. ■

3. Polyblocks

The simplest non-empty normal set is a box $[0, y] \subset R^n_+$, determined by a point $y \in R^n_+$. By Proposition 2, the union of a family of boxes is a normal set. Conversely, it is obvious that

Proposition 12. *For any normal set G, we have*

$$G = \cup_{y \in G}[0, y]. \quad ■$$

This suggests that a compact normal set could be approximated by a finite union of boxes. An "elementary" normal set which is the union of finitely many boxes (i.e., the normal hull of a finite set in R^n_+) is called a polyblock. More precisely, a set P is called a *polyblock* in $[a, b]$ if $P = \cup_{z \in T}[a, z]$, where $T \subset [a, b]$ ($|T| < +\infty$). The set T is called the *vertex set* of the polyblock. A vertex $z \in T$ is said to be *improper* if it is dominated by some other $z' \in T$, i.e., if there is $z' \in T \setminus \{z\}$ such that $[0, z] \subset [0, z']$. Of course a polyblock is fully determined by its *proper* vertices.

Proposition 13. *Any polyblock is normal and compact. The union or intersection of finitely many polyblocks is a polyblock.*

Proof. The first assertion follows from the fact that any box $[a, z] \subset R^n_+$ is a normal compact set while the union of a finite family of normal compact sets is a normal compact set. The union of finitely many polyblocks is obviously a polyblock. To see that the intersection of finitely many polyblocks is a polyblock, it suffices to observe that $(\cup_i A_i) \cap (\cup_j B_j) = \cup_{i,j}(A_i \cap B_j)$ and $[a, p] \cap [a, q] = [a, u]$ with $u_i = \min\{p_i, q_i\}$. ■

The concept of polyblock is analogous to that of polytope in convex analysis. In fact, just as a polytope is the convex hull of finitely many points in R^n, a polyblock is the normal hull of finitely many points in R^n_+. We next show that, just as any convex compact set is the intersection of a nested family of polytopes and can be approximated, as closely as desired, by a polytope enclosing it, any normal compact set is the intersection of a nested family of polyblocks and can be approximated, as closely as desired, by a polyblock containing it.

Proposition 14. *Let $G \subset [0, b]$ be a normal closed set. For any $z \in [0, b] \setminus G$, there exists $y \in \partial^+ G$ such that the set K_y separates z strictly from G (i.e., contains z but is disjoint from G).*

Proof. Recall that $K_y := \{x \in R^n_+ | \, y_i < x_i \, \forall i \notin I(y)\}$, where $I(y) = \{i \,|\, y_i = 0\}$. Let y be the last point of G on the ray from 0 through z (i.e., $y = \sigma_G(0, z)$ as defined by (1)). Clearly, $z \in K_y$ and $y \in \partial^+ G$ by Proposition 4, hence, K_y is disjoint from G. ■

Proposition 15. *If* $0 \le \bar{x} < \bar{z} \le b$, *then* $P = [0, \bar{z}] \setminus K_{\bar{x}}$ *is a polyblock in* $[0, b]$ *with vertex set* $V = \{z^i | \, i \notin I(\bar{x})\} \subset R^n_{++}$, *where*

$$z^i = \bar{z} - (\bar{z}_i - \bar{x}_i)e^i.$$

Proof. Let $K_i = \{x \in R^n_+ | \, \bar{x}_i < x_i\}$. Since $K_{\bar{x}} = \cap_{i \notin I(\bar{x})} K_i$, we have $P = [0, \bar{z}] \setminus K_{\bar{x}} = \cup_{i \notin I(\bar{x})}([0, \bar{z}] \setminus K_i)$. But

$$[0, \bar{z}] \setminus K_i = \{x | \, 0 \le x_i \le \bar{x}_i, \; 0 \le x_j \le \bar{z}_j \; \forall j \ne i\} = [0, z^i],$$

where z^i denotes the vector such that $z^i_j = \bar{z}_j \; \forall j \ne i, \; z^i_i = \bar{x}_i$, i.e., $z^i = \bar{z} - (\bar{z}_i - \bar{x}_i)e^i$. To prove that $V \subset R^n_{++}$, consider any z^i with $i \notin I(\bar{x})$. Then for every $j \ne i$, we have, $z^i_j = \bar{z}_j > 0$, while $z^i_i = \bar{x}_i > 0$. ■

Proposition 16. *Let* G *be a compact set contained in a box* $[0, b] \subset R^n_+$. *Then the following assertions are equivalent:*

(i) *G is normal;*
(ii) *For any point* $z \in [0, b] \setminus G$, *there exists a polyblock in* $[0, b]$ *separating* z *from* G *(i.e., containing* G *but not* z*).*
(iii) *G is the intersection of a family of polyblocks in* $[0, b]$.

Proof. (i)⇒(ii) If $z \in [0, b] \setminus G$, then by Proposition 14, there exists $y \in \partial^+ G$ such that $z \in K_y$ but $K_y \cap G = \emptyset$, i.e., $[0, b] \setminus K_y$ (which is a polyblock by Proposition 15) separates z from G.

(ii)⇒(iii) Let E be the intersection of all polyblocks containing G. Clearly, $G \subset E$. If (ii) holds, then, for any $z \in E \setminus G$, there is a polyblock containing G but not z, so $E \subset G$.

(iii)⇒(i) Obvious by Proposition 3 because any polyblock is closed and normal. ■

A set $Q \subset [a, b] \subset R^n_+$, which is the union of boxes $[y, b]$, $y \in T \subset [a, b]$, $|T| < +\infty$, is called a *reverse polyblock* in $[a, b]$ with *vertex set* T. A vertex $y \in T$ is *improper* if there exists $y' \in T \setminus \{y\}$ such that $y' \le y$, i.e., $[y, b] \subset [y', b]$. Of course a reverse polyblock is fully determined by its *proper* vertices. The next propositions are analogous to Propositions 15 and 16.

Proposition 17. *If* $0 \le \bar{y} < \bar{x} \le b$, *then* $Q = [\bar{y}, b] \setminus [\bar{y}, \bar{x})$ *is a reverse polyblock with vertices*

$$y^i = \bar{y} + (\bar{x}_i - \bar{y}_i)e^i \quad i = 1, \ldots, n.$$

If $\bar{x} \in \partial^- H$, *where* H *is a reverse normal set and* $Q = [\bar{y}, b] \setminus [\bar{y}, \bar{x})$, *then* $H \cap Q = H \cap [\bar{y}, b]$.

Proof. Let $L_i = \{u | \, \bar{y}_i \le u_i < \bar{x}_i\}$. Since $[\bar{y}, \bar{x}) = \cap^n_{i=1} L_i$, we have $Q = [\bar{y}, b] \setminus [\bar{y}, \bar{x}) = \cup^n_{i=1}([\bar{y}, b] \setminus L_i) = \cup^n_{i=1}\{u | \, \bar{x}_i \le u_i \le b_i, \; \bar{y}_j \le u_j \le b_j \, \forall j \ne i\} = \cup^n_{i=1}[y^i, b]$. The second assertion is immediate because $[\bar{y}, \bar{x})$ is disjoint from H when $\bar{x} \in \partial^- H$. ■

Proposition 18. *Let H be a compact subset of $[a, b]$. Then the following assertions are equivalent:*
(i) *H is reverse normal in $[0, b]$;*
(ii) *for any $y \in [a, b] \setminus H$, there exists a reverse polyblock separating y from H;*
(iii) *H is the intersection of a family of reverse polyblocks in $[a, b]$.*

Proof. Similar to the proof of Proposition 16. ■

4. Systems of Monotonic Inequalities

By the system of monotonic inequalities (or monotonic system, for short), we mean a couple of inequalities of the form

$$\begin{cases} g(x) \leq 1, & (4) \\ h(x) \geq 1, & (5) \end{cases}$$

where $g(x), h(x)$ are increasing functions on R^n_+. Often $g(x) = \max_{i=1,\dots,m_1} g_i(x)$, $h(x) = \min_{j=m_1+1,\dots,m} h_j(x)$, where $g_i(x),\ h_j(x)$ are increasing functions on R^n_+, so a monotonic system may actually consist of finitely many inequalities:

$$g_i(x) \leq 1 \ (i = 1, \dots, m_1); \quad h_j(x) \geq 1 \ (j = m_1 + 1, \dots, m).$$

Setting

$$G = \{x \in R^n_+ |\, g(x) \leq 1\}, \quad H = \{x \in R^n_+ |\, h(x) \geq 1\},$$

we can rewrite the system as

$$x \in G \cap H, \tag{6}$$

where G is a normal set, and H a reverse normal set. We will make the following blanket assumption for this section:

$$\left| \begin{array}{l} G \text{ and } H \text{ are closed subsets of } R^n_+; \\ \text{int}G \neq \emptyset,\ \ G \subset [0, c],\ \ H^b := R^n_+ \setminus H \subset [a, b],\ \text{ where } 0 \leq a < c \leq b. \end{array} \right. \tag{7}$$

Conditions (7) can always be made to hold, provided $G \cap H$ is compact, say $G \cap H \subset [a, c]$. Indeed, it suffices to replace G, H by $G' := G \cap [0, c]$, $H' := H \cap \{x \in [0, b] |\, x \geq a\}$, respectively, where $b \geq c$ is selected so that $G' \cap H' = G \cap H$. Clearly, the new sets G', H' will satisfy (7).

To provide insight into the structure of the solution set of a monotonic system (4)–(5), we shall focus on characterizing particular solutions called upper basic and lower basic solutions. These concepts are motivated by the application to optimization problems under monotonic constraints.

4.1. Upper Basic Solutions

A point $x \in G \cap H$ is called an *upper basic solution* (ubs for short) of the system (4)–(5) if $x \leq x' \in G \cap H$ implies $x = x'$. Clearly, any ubs x must belong to $\partial^+ G$ (upper boundary of G) because, if $x \notin \partial^+ G$, then there is $y \in K_x \cap G$, and since H is reverse normal and $x \in H$, one must have $y \in H$, i.e., $y \in G \cap H$, but $y \neq x$ (because $y \in K_x$), conflicting with x being a ubs.

A ubs of (4) and (5) is nothing but an upper extreme point of the set $G \cap H$. Therefore, as we saw in the proof of Proposition 5, for any $y \in G \cap H$, there is a ubs $x \geq y$, namely $x = z^n$, where $z^1 \in \operatorname{argmax}\{z_1 \mid z \in G \cap H,\ z \geq y\}$, $z^i \in \operatorname{argmax}\{z_i \mid z \in G \cap H,\ z \geq z^{i-1}\}$ for $i = 2, \ldots, n$.

To describe a characterization of ubs's we will assume, additionally, $a > 0$ in (7), so that

$$G \cap H \subset [a, b] \subset (0, b]. \tag{8}$$

As usual, define $G^\flat := R^n_+ \setminus G$. Condition (8) implies that

$$K_x \cap (G \cap H) = \emptyset \quad \forall x \in \partial^- G^\flat. \tag{9}$$

Indeed, for any $x \in \partial^- G^\flat$, we have $\operatorname{int} K_x \subset G^\flat$, hence $K_x \cap G \subset \operatorname{cl} K_x \setminus \operatorname{int} K_x \subset \{x \mid \min_i x_i = 0\}$, and therefore, in view of (8), $K_x \cap (G \cap H) = \emptyset$.

Also, setting $H_a = \{x \in H \mid x \geq a\}$, we have from (8):

$$G \cap H \subset H_a. \tag{10}$$

Now, let us fix a vector $v \in R^n_{++}$, and for any $z \in [0, b] \setminus G$, define

$$\pi(z) = z - \lambda v, \quad \lambda = \sup\{\alpha \mid z - \lambda v \in [0, b] \setminus G\}, \tag{11}$$

i.e., $\pi(z) = \omega_{G^\flat}(z, v)$ (last point of $\operatorname{cl} G^\flat$ on the halfline $\{z - \alpha v \mid \alpha \geq 0\}$; see Proposition 7 and formula (2)). Clearly, $\pi(z) < z \in [0, b]$ because $\lambda v > 0$.

Proposition 19. *Every upper basic solution of the system (4)–(5) is the limit of a sequence $\{z^k\} \subset H_a$ such that $z^0 \geq z^1 \geq z^2 \geq \cdots$ and*

$$\begin{aligned} z^0 = b,\ z^{k+1} &= z^k - (z^k_{i_k} - x^k_{i_k}) e^{i_k}; \\ x^k = \pi(z^k), &\quad i_k \notin I(x^k), \ k = 0, 1, \ldots. \end{aligned} \tag{12}$$

For the proof, we need some auxiliary propositions.

Lemma 1. *Every sequence $z^0 = b \geq z^1 \geq z^2 \geq \cdots \geq 0$ has a limit.*

Proof. By compactness, the sequence z^k has at least an accumulation point $\tilde{x}$. This point satisfies $z^k \geq \tilde{x},\ \forall k$ because $z^0 \geq z^1 \geq \cdots$. Now, if $x = \lim_{q \to +\infty} z^{k_q}$ is an arbitrary accumulation point, then $z^{k_q} \geq \tilde{x},\ \forall q$, hence, $x \geq \tilde{x}$. By interchanging the roles of x and $\tilde{x}$, one also has $\tilde{x} \geq x$, hence, $x = \tilde{x}$. Therefore, $\tilde{x} = \lim_{k \to +\infty} z^k$. ■

Lemma 2. *The sequences $\{z^k\}$, $\{x^k\}$ in (12) satisfy $z^k - x^k \to 0$ as $k \to +\infty$.*

Proof. By (12), $z^{k+1}_{i_k} = x^k_{i_k}$, while by Lemma 1, $\lim_{k \to +\infty} \|z^{k+1} - z^k\| = 0$. Therefore,

$$z^k_{i_k} - x^k_{i_k} = z^k_{i_k} - z^{k+1}_{i_k} \leq \|z^k - z^{k+1}\| \to 0 \quad (k \to +\infty).$$

But by construction, $z^k - x^k = z^k - \pi(z^k) = \lambda_k v$, so $z^k_{i_k} - x^k_{i_k} = \lambda_k v_{i_k}$, hence, $\lambda_k = (z^k_{i_k} - x^k_{i_k})/v_{i_k}$. Since $v_{i_k} \geq \min_{i=1,\ldots,n} v_i > 0$, it follows that $\lambda_k \to 0$, and consequently, $z^k - x^k \to 0$. ■

Lemma 3. *If z^k, x^k satisfy (12), then $\tilde{x} \in (\partial^+ G) \cap H$, where $\tilde{x}$ is the common limit of z^k and x^k as $k \to +\infty$.*

Proof. Since $x^k \in \partial^- G^\flat\ \forall k$, one must have $\tilde{x} = \lim_{k\to+\infty} x^k \in \partial^- G^\flat$. On the other hand, since $z^k \in H_a\ \forall k$, one must have $\tilde{x} = \lim_{k\to+\infty} z^k \in H_a$. The latter implies that $\tilde{x} > 0$, and since $\tilde{x} \in \partial^- G^\flat$, it follows from (3) that $\tilde{x} \in \partial^+ G$. Thus, $\tilde{x} \in (\partial^+ G) \cap H$. ∎

Proof of Proposition 19. Let x be any upper basic solution. We shall construct a nested sequence of boxes $[0, z^0] \supset [0, z^1] \supset \cdots \supset [0, x]$ such that $z^k \in H_a$, and (12) holds. First, observe that, if $b \notin H$, then $b' \notin H$ for some $b' > b$, hence, $[0, b'] \subset R^n_+ \setminus H$, contradicting (7). Therefore, $b \in H_a$. If $b \in G$, then b is the only upper basic solution, hence, $x = b$ and the sequence $z^k = b$, $\forall k$ satisfies the desired conditions. Now, let $b \notin G$, and suppose that we have already defined $z^0, z^1, \ldots, z^h$ satisfying (12), where $z^k \in H_a$, and $[0, z^k] \supset [0, x]$ for $k = 0, 1, \ldots, h$. If $z^h \in G$, then since $z^h \in H$ and $z^h \geq x$, we have, $z^h = x$ (by the definition of an upper basic solution), so $z^k = x$ ($\forall k \geq h+1$) satisfies (12). Otherwise, since $z^h \in H_a \setminus G \subset R^n_{++} \setminus G$, we have $x^h < z^h$, $x^h \in \partial^- G^\flat$, so that, in view of (9), $K_{x^h} \cap (G \cap H) = \emptyset$. Therefore, the polyblock $P_{h+1} = [0, z^h] \setminus K_{x^h}$ still contains x. Let V_{h+1} be the set of proper vertices of P_{h+1} that belong to H_a. Since H_a is reverse normal, if $y \notin H_a$, then $[0, y] \cap H_a = \emptyset$, hence $V_{h+1} = \emptyset$ would imply that $P_{h+1} \cap H_a = \emptyset$, conflicting with $x \in P_{h+1}$. Therefore, $V_{h+1} \neq \emptyset$ and there exists $z^{h+1} \in V_{h+1}$ such that $x \in [0, z^{h+1}]$. By Proposition 15, $z^{h+1} = z^h - (z^h_i - x^h_i)e^i$ for some $i = i_h \notin I(x^h)$. Since $z^{h+1} \in H_a$, the sequence $z^0, z^1, \ldots, z^{h+1}$ satisfies (12). Thus, a sequence $\{z^k\} \subset H_a$ satisfying (12) can be constructed such that $z^0 \geq z^1 \geq z^2 \geq \cdots \geq x$. By Lemmas 1 and 2, the two sequences z^k, x^k tend to a common limit $\tilde{x}$ and by Lemma 3, $\tilde{x} \in G \cap H$. Since $z^k \geq x$, $\forall k$, it follows that $\tilde{x} \geq x$, and hence $\tilde{x} = x$ because x is a ubs. This completes the proof of Proposition 19. ∎

4.2. Lower Basic Solutions

A point $x \in G \cap H$ is called a *lower basic solution* (lbs for short) of the system (4)–(5) if $x \geq x' \in G \cap H$ implies $x = x'$. Clearly, any lbs x must belong to $\partial^- H$ (lower boundary of H) because, if $x \notin \partial^- H$, then, since $x \in H$, there must exist $x' \in H$ such that $x' < x$ and, since G is normal, $x' \in G$, i.e., $x' \in G \cap H$ and $x' < x$, conflicting with x being a lower basic solution.

An lbs can also be defined as a minimal element of the set $G \cap H$ with respect to the ordering $x \geq x' \Leftrightarrow x_i \geq x'_i\ \forall i$. By Zorn's Lemma, for any feasible solution of the system (4)–(5), there exists a lower basic solution dominated by it, namely a minimal element of the set of all $x \in G \cap H$ that are dominated by this solution.

To describe a characterization of lbs's, it is convenient to assume $c = b$ in (7), so that

$$\text{int}G \neq \emptyset, \quad G \subset [0, b], \quad H^\flat := R^n_+ \setminus H \subset [0, b], \quad G \cap H \subset [a, b].$$

Fix a vector $v \in R^n_{++}$, e.g., $v = b - a$, and for every $z \in H^\flat$, define

$$\rho(z) = z + \mu v, \quad \mu = \sup\{\alpha |\ z + \alpha v \in [a, b] \setminus H\}, \tag{13}$$

i.e., $\rho(z) = \sigma_{H^\flat}(z, v)$ (first point of H on the halfline $\{z + \alpha v |\ \alpha \geq 0\}$; see Proposition 4 and formula (1)).

Proposition 20. *Any lower basic solution of the system (4)–(5) is the limit of a sequence* $\{z^k\} \subset G$ *such that* $z^0 := a \leq z^1 \leq z^2 \leq \cdots$ *and*

$$z^0 = a, z^{k+1} = z^k + (x^k_{i_k} - z^k_{i_k})e^{i_k}$$
$$x^k = \rho(z^k), \quad k = 0, 1, \ldots. \tag{14}$$

Proof. Let x be an lbs. We construct a nested sequence of boxes $[z^0, b] \supset [z^1, b] \supset \cdots \supset [x, b]$ such that $z^k \in G$ and z^k satisfies (14). Since $a \in G$, if $a \in H$, then a is the only lbs, hence, $x = a$ and the sequence $z^k = a\ \forall k$ satisfies the desired conditions. Now, let $a \notin H$ and suppose we have already defined $z^0, z^1, \ldots, z^h$ satisfying (14) and $z^k \in G$ for $k = 0, 1, \ldots, h$. If $z^h \in H$, then, since $z^h \in G$ and $z^h \leq x$, we must have $z^h = x$ (by the definition of an lbs), so that $z^k = x\ (\forall k \geq h+1)$ satisfies (14). Otherwise, $x^h = \rho(z^h) > z^h$, and by Proposition 17, the reverse polyblock $Q_{h+1} = [z^h, b] \setminus [z^h, x^h)$ still contains x. Let W_{h+1} be the set of proper vertices of Q_{h+1} that belong to G. Since G is normal, if $y \notin G$, then $[y, b] \cap G = \emptyset$, hence, $W_{h+1} = \emptyset$ would imply that $Q_{h+1} \cap G = \emptyset$, conflicting with $x \in Q_{h+1}$. Therefore, $W_{h+1} \neq \emptyset$ and there exists $z^{h+1} \in W_{h+1}$ such that $x \in [z^{h+1}, b]$. From Proposition 17, we know that $z^{h+1} = z^h + (x^h_i - z^h_i)e^i$ for some $i = i_h$. Since $z^{h+1} \in G$, the sequence $z^0, z^1, \ldots, z^{h+1}$ satisfies (14). Thus, a sequence $\{z^k\}$ satisfying (14) has been constructed.

It remains to show that such constructed sequences $(\{z^k\}, \{x^k\})$ tend to a common limit which is exactly x. First, by Lemma 1 (with the order $\leq$ replacing $\geq$), the sequence $z^0 = a \leq z^1 \leq z^2 \leq \cdots \leq b$ has a limit $\tilde{x}$. Now, by (14), $z^{k+1}_{i_k} = x^k_{i_k}$ while $z^{k+1} - z^k \to 0\ (k \to +\infty)$. Therefore,

$$x^k_{i_k} - z^k_{i_k} = z^{k+1}_{i_k} - z^k_{i_k} \leq \|z^{k+1} - z^k\| \to 0 \quad (k \to +\infty).$$

But by construction, $x^k - z^k = \rho(z^k) - z^k = \mu_k v$, so $x^k_{i_k} - z^k_{i_k} = \mu_k v_{i_k}$, hence, $\mu_k = (x^k_{i_k} - z^k_{i_k})/v_{i_k}$. Since $v_{i_k} \geq \min_{i=1,\ldots,n} v_i > 0$, it follows that $\mu_k \to 0$, and consequently, $x^k - z^k \to 0$, i.e., $\tilde{x} = \lim z^k = \lim x^k$. Since $z^k \in G\ \forall k$, it follows that $\tilde{x} \in G$. Also, $x^k = \rho(z^k) \in \partial^+ H^b \subset \partial^- H, \forall k$, hence, $\tilde{x} \in \partial^- H$, and so $\tilde{x} \in G \cap (\partial^- H)$. Finally, the fact $z^0 \leq z^1 \leq z^2 \leq \cdots$ implies that $\tilde{x} \leq x$, and since x is an lbs, it follows that $\tilde{x} = x$. ∎

We have thus proved the following characterization of the basic solutions of a monotonic system:

(i) Every upper basic solution x of a monotonic system (4)–(5) is characterized by a sequence $\{i_0, i_1, \ldots, i_k, \ldots\}$, where $i_k \in \{1, 2, \ldots, n\}$, such that x is the limit of the sequence $z^0 \geq z^1 \geq z^2 \geq \cdots$ defined by (12).

(ii) Every lower basic solution x of a monotonic system (4)–(5) is characterized by a sequence $\{i_0, i_1, \ldots, i_k, \ldots\}$ where $i_k \in \{1, 2, \ldots, n\}$, such that x is the limit of the sequence $z^0 \leq z^1 \leq z^2 \leq \cdots$ defined by (14).

Let us agree to call the sequence $\{i_0, i_1, \ldots, i_k, \ldots\}$ that determines an upper (or lower) basic solution x its *characteristic sequence* and i_k its kth *characteristic number*. For any $z \in R^n_+$ and $i \in \{1, 2, \ldots, n\}$, define

$$z_{[i]} = z - (z_i - \pi_i(z)e^i, \quad z^{[i]} = z + (\rho_i(z) - z_i)e^i, \tag{15}$$

where $\pi_i(z)$, $\rho_i(z)$ are the ith coordinate of $\pi(z)$ and $\rho(z)$, respectively. Also, write $z_{[i_0 i_1]}$ for $(z_{[i_0]})_{[i_1]}$ and analogously, $z^{[i_0 i_1]}$ for $(z^{[i_0]})^{[i_1]}$. Then for any upper basic solution x with characteristic sequence $\{i_0, i_1, \ldots\}$, we have

$$b_{[i_0 i_1 \cdots i_k]} \in H, \quad \forall k, \quad x = \lim_{k \to +\infty} b_{[i_0 i_1 \cdots i_k]}, \tag{16}$$

while for a lower basic solution x,

$$a^{[i_0 i_1 \cdots i_k]} \in G, \quad \forall k, \quad x = \lim_{k \to +\infty} a^{[i_0 i_1 \cdots i_k]}. \tag{17}$$

From the proofs of Propositions 19 and 20, it is easily seen that $[0, x] = \cap_{k=1}^{+\infty}[0, b_{[i_0 i_1 \cdots i_k]}]$ for an upper basic solution and $[x, b] = \cap_{k=1}^{+\infty}[a^{[i_0 i_1 \cdots i_k]}, b]$ for a lower basic solution.

Remark 2. Proposition 19 remains valid when we replace $\pi(z)$ by an arbitrary mapping $\pi : H \cap R^n_{++} \to \partial^+ G$ such that

$$\pi(z) = z - \lambda_z v, \quad \text{where } \lambda_z > 0, \ v_i \geq \eta > 0.$$

For example, under assumption (8), one can take $\pi(z) = z - \lambda_z z$, with $\lambda = \sup\{\alpha \mid (1-\alpha)z \in G\}$.

Also, Proposition 20 remains valid when we replace $\rho(z)$ by an arbitrary mapping $\rho : G \cap R^n_+ \to \partial^+ H^b$ such that

$$\rho(z) = z + \mu_z v, \quad \text{where } \mu_z > 0, \ v = b - z \in R^n_{++}.$$

For example, under the assumption $z < b \ \forall z \in G$, one can take $\rho(z) = z + \mu_z(b - z)$ with $\mu_z = \sup\{\alpha \mid z + \alpha(b - z) \in H^b\}$.

5. Optimization Under Monotonic Constraints

Given a monotonic system (6) and an increasing function $f(x)$, consider the following problems which are encountered in many important applications:

$$\text{(A)} \ \max\{f(x) \mid x \in G \cap H\}, \tag{18}$$

$$\text{(B)} \ \min\{f(x) \mid x \in G \cap H\}, \tag{19}$$

where $G := \{x \in R^n_+ \mid g(x) \leq 1\}$ and $H = \{x \in R^n_+ \mid h(x) \geq 1\}$, with $g(x)$, $h(x)$ being increasing functions on $[0, b] \subset R^n_+$, such that (7) is satisfied.

The next proposition, together with Propositions 19 and 20, provide a theoretical basis for a solution approach to these problems.

Proposition 21. *An increasing function $f(x)$ achieves its maximum over the set $G \cap H$ at an upper basic solution of the system (4)–(5), and its minimum at a lower basic solution.*

Proof. Let $x \in G \cap H$ be a feasible solution of Problem (A) and $\bar{x} = \pi(x)$. Then $x \le \bar{x}$, and since H is reverse normal, $\bar{x}$ still belongs to H, hence, $\bar{x}$ is a feasible solution which is at least as good as x. Clearly, $\bar{x}$ is an upper basic solution because $\bar{x} \le x' \in C \cap H$ implies $x' = \bar{x}$. Consequently, for any optimal solution of (A), there exists an optimal solution which is an upper basic solution. Analogously, the same holds for Problem (B). ■

Thus, a global maximizer of $f(x)$ must be sought among the upper basic solutions of the system (4)–(5), while a global minimizer must be sought among the lower basic solutions.

5.1. Maximization Problem

It was shown in the preceding section that, under assumption (7) where $0 < a < b$, every ubs of (6) is the limit of a sequence $b_{[i_0 i_1 \cdots i_k]}$, $k = 0, 1, \ldots$. Therefore, solving Problem (A) amounts to finding a suitable sequence $\{i_0, i_1, \ldots, i_k, \ldots\}$.

Let us introduce some definitions. Denote by Q the set of all vectors of the form $b_{[i_0 i_1 \cdots i_k]}$, for $k = 0, 1, \ldots$. Given a vector $z = b_{[i_0 i_1 \cdots i_k]}$, we say that a ubs x is covered by z if $x \in [0, b_{[i_0 i_1 \cdots i_k]}]$ (i.e., if its first $k+1$ characteristic numbers are exactly $i_0, i_1, \ldots, i_k$). Any vector $z \in Q$ determines a set of ubs's, namely the set $E(z)$ of all ubs's covered by z. By Proposition 15, $E(z) = \cup\{E(z_{[i]}) |\ i \notin I(\pi(z))\}$, so replacing a $z \in Q$ by $\{z_{[i]} |\ i \notin I(\pi(z))\}$ amounts to partitioning $E(z)$ into subsets $E(z_{[i]})$, $i \notin I(\pi(z))$. A vector $z \in T \subset Q$ is said to be an improper member of T if $z \le z'$ (hence, $E(z) \subset E(z')$) for some $z' \in T \setminus \{z\}$.

Now, we can outline the branch and bound procedure for maximizing $f(x)$ over $G \cap H$.

Start from $T_0 = \{b\}$, i.e., from the set $E(b)$ of all ubs's. Since $b \in H_a := \{x \in H |\ x \ge a\}$, if $b \in G$, then it is obviously an optimal solution. Otherwise, proceed to iteration $k = 1$. At iteration $k \ge 1$, we already have a set $T_k \subset Q$ which defines a collection of sets $\{E(z) |\ z \in T_k\} \subset E(b)$ such that $\cup_{z \in T_k} E(z)$ contains at least one optimal solution, if there is one. In the collection T_k, we can delete the improper members, the members $z \in T_k \setminus H_a$ (because $E(z) = \emptyset$ when $z \notin H_a$ in view of the reverse normality of H_a) and also delete all $z \in T_k$ such that $f(z) \le f(\bar{x})$, where $\bar{x}$ is the best feasible solution known up to this stage (indeed, no ubs covered by such z can be better than $\bar{x}$). Let Z_k be the set of remaining members of T_k. If $Z_k = \emptyset$, then $\bar{x}$ is an optimal solution (if no $\bar{x}$ exists, the problem is infeasible). If $Z_k \neq \emptyset$, select z^k with maximal $f(z^k)$, i.e., $z^k \in \mathrm{argmax}\{f(z) |\ z \in Z_k\}$. Since $z^k \in H_a$, if $z^k \in G$, then z^k is an optimal solution. Otherwise, compute $x^k = \pi(z^k)$ and replace z^k by the set $\{z^k_{[i]} |\ i \notin I(x^k)\}$ (i.e., further partition $E(z^k)$ into $E(z^k_{[i]})$, $i \notin I(x^k)$). Let T_{k+1} be the resulting set. Go to iteration $k+1$ with T_{k+1} in place of T_k.

It turns out that, whenever infinite, this branch and bound procedure generates a sequence $b_{[i_0]}, b_{[i_0 i_1]}, \ldots$, converging to an optimal solution.

We can thus state the following algorithm for solving Problem (A).

Algorithm 1. (For Problem (A), under assumption (7) with $a > 0$.) Select a vector $v \in R^n_{++}$ for the mapping $\pi : R^n_{++} \setminus G \to \partial^- G^\flat$ (see (11) and also Remark 1). Select a tolerance $\varepsilon > 0$.

Initialization. If $a \notin G$, terminate (the problem is infeasible because $G \cap H = \emptyset$). Otherwise, let $T_0 = \{b\}$. Let $\bar{x}$ be the best feasible solution available, $CBV = f(\bar{x})$. If no feasible solution is available, set $CBV = -\infty$. Set $k = 0$.

Step 1. In T_k delete all improper members, all $z \in T_k \setminus H_a$, and delete all z such that $f(z) \leq CBV + \varepsilon$. Let Z_k be the set of remaining members of T_k.

Step 2. If $Z_k = \emptyset$, then terminate: if $CBV > -\infty$, the current best feasible solution $\bar{x}$ is accepted as an ε-optimal solution of (A); if $CBV = -\infty$, the problem is infeasible.

Step 3. If $Z_k \neq \emptyset$, select $z^k \in \operatorname{argmax}\{f(z) \mid z \in Z_k\}$. If $z^k \in G$, then terminate (z^k is an optimal solution). Otherwise, compute $x^k = \pi(z^k)$. Update the current best value CBV and the current best feasible solution $\bar{x}$.

Step 4. Let $T_{k+1} = (Z_k \setminus \{z^k\}) \cup \{z^k - (z_i^k - x_i^k)e^i \mid i \notin I(x^k)\}$.

Step 5. Set $k \leftarrow k + 1$ and return to Step 1.

Proposition 22. *Assume $f(x)$ is upper semicontinuous on H. If Algorithm 1 is infinite, it generates at least one infinite sequence $b_{[i_0]}, b_{[i_0 i_1]}, \dots, b_{[i_0 i_1 \dots i_k]}, \dots$ converging to an optimal solution.*

Proof. Let us agree that z' is a *successor* of z if $z' \in \{z_{[1]}, \dots, z_{[n]}\}$; a *descendant* of z if $z' = z_{[\xi]}$ for some $\xi = (\xi_0, \xi_1, \dots, \xi_k)$, where k is a non-negative integer and $\xi_i \in \{1, \dots, n\}$, $i = 0, 1, \dots, k$. If the Algorithm is infinite, at least one successor of b, say $y^0 = b_{[i_0]}$, has infinitely many descendants. Then at least one successor of y^0, say $y^1 = y^0_{[i_1]} = b_{[i_0 i_1]}$, has infinitely many descendants, and so on. Continuing, we find an infinite sequence $y^0 = b_{[i_0]}, y^1 = b_{[i_0 i_1]}, \dots, y^k = b_{[i_0 \dots i_k]}, \dots$ such that $y^k \in H$, $\forall k$. By Proposition 19, $b_{[i_0 i_1 \cdots i_k]} \to \bar{z} \in G \cap H$. From the selection of z^k in Step 3, we have $f(b_{[i_0 i_1 \cdots i_k]}) \geq f(z), \forall z \in G \cap H$. Hence, by upper semicontinuity of $f(x)$ on H, $f(\bar{z}) \geq f(x)$, $\forall x \in G \cap H$, as was to be proved. ■

Remark 3. To alleviate storage problems which may arise in connection with the growth of T_k as the Algorithm proceeds, Step 5 of the Algorithm can be modified as follows. Let L be the maximum size allowed for $|T_k|$.

Step 5. If $|T_{k+1}| \leq L$, then set $k \leftarrow k + 1$ and return to Step 1. Otherwise, go to Step 6.

Step 6. Redefine $T_{k+1} = \{b - (b_i - x_i^k)e^i,\ i = 1, \dots, n\}$, set $k \leftarrow k + 1$ and return to Step 1.

With this modification, each time Step 6 occurs, the Algorithm is *restarted* from the last x^k. Restarting is a device for overcoming memory space limitations at the expense of more computational time in order to solve large scale problems.

Example 1. Consider the problem

$$\max\{\varphi(u(x)) \mid x \in D\}, \tag{20}$$

where $D \subset R^n_+$ is a non-empty compact convex set, $\varphi : R^m_+ \to R$ is an increasing function, $u(x) = (u_1(x), \dots, u_m(x))$, $u_i : D \to R_+$ being non-negative-valued continuous functions on D. By Proposition 1, this problem can be written as $\max\{\varphi(y) | y \in u(D)\} = \max\{\varphi(y) | y \in N[u(D)]\}$, i.e.,

$$\max\{\varphi(y) | y \in G\},$$

where $G := N[u(D)] = \{y \in R^m_+ | x \in D, \; y \le u(x)\}$. This is of course a problem (A), with $H = R^m_+$ and G being closed by continuity of $u(x)$. Furthermore, without loss of generality, we can assume

$$\max_{x \in D} u_i(x) > 0, \quad \forall i = 1, \dots, m. \tag{21}$$

It is then easily checked that there is a $y > 0$ satisfying $y \in G$, i.e., $\text{int}G \neq \emptyset$. Also, if every $u_i(x)$, $i = 1, \dots, m$ is concave or convex, then, for every $z \in R^n_+ \setminus \{0\}$, the point $\pi(z)$ as defined by (11) can be computed easily.

Example 2. Consider the problem

$$\max\{\langle c, x\rangle \mid x \in D, \; \varphi(u(x)) \le 1\}, \tag{22}$$

where D, φ and $u(x)$ are as previously. Observe that the set

$$H = \{y \in R^m_+ | \, u(x) \le y \text{ for some } x \in D\}$$

is closed and reverse normal, since $H = u(D) + R^m_+ = rN[u(D)]$. Define

$$\theta(y) = \begin{cases} \sup\{\langle c, x\rangle \mid x \in D, \; u(x) \le y\}, & \text{if } y \in H \\ -M, & \text{otherwise,} \end{cases} \tag{23}$$

where $M > 0$ is an arbitrary number such that $-M < \min\{\langle c, x\rangle \mid x \in D\}$. Since D is non-empty compact, clearly $-\infty < \theta(y) < +\infty, \forall y \in R^m_+$.

Proposition 23. *The function $\theta(y)$ is increasing and upper semicontinuous on R^m_+. If $u_1(x), \dots, u_m(x)$ are convex, then $\theta(y)$ is concave on the convex set $H = u(D) + R^m_+$.*

Proof. If $y \le y'$ and $y \notin H$, then $\theta(y) = -M$ while $\theta(y') \ge -M = f(y')$. But if $y \le y'$ and $y \in H$, then $\emptyset \neq \{x \in D | \, u(x) \le y\} \subset \{x \in D | \, u(x) \le y'\}$, hence, $\theta(y) \le \theta(y')$. Therefore, $\theta(y)$ is increasing. We now show the upper semicontinuity of $\theta(y)$. Since H is closed and $\theta(y) = -M \; \forall y \notin H$, it suffices to show the upper semicontinuity of $\theta(y)$ on H. Let $y^k \to y^0$ (where $y^k \in H$), and for each k, let x^k be such that $x^k \in D$, $u(x^k) \le y^k$, $\langle c, x^k\rangle = \theta(y^k)$. Since D is compact and $u(x)$ is continuous, we can assume $x^k \to x^0 \in D, u(x^0) \le y^0$. Then $\theta(y^0) \ge \langle c, x^0\rangle = \lim_k \langle c, x^k\rangle = \lim_k \theta(y^k)$, as desired. Finally, if every function $u_1, \dots, u_m$ is convex and $\theta(y^1) = \langle c, x^1\rangle$, $\theta(y^2) = \langle c, x^2\rangle$, where $x^i \in D$, $u(x^i) \le y^i$, $i = 1, 2$, then, for any $\alpha \in (0, 1)$, we have $x := \alpha x^1 + (1-\alpha)x^2 \in D$ and $u(x) \le \alpha u(x^1) + (1 - \alpha u(x^2)) \le y^1 + (1-\alpha)y^2 = y$. Hence, $\theta(\alpha y^1 + (1-\alpha)y^2) \ge \langle c, \alpha x^1 + (1-\alpha)x^2\rangle = \alpha\theta(y^1) + (1 - \alpha\theta(y^2))$, proving the concavity of $\theta(y)$ on $H = u(D) + R^m_+$. ∎

Proposition 24. *Problem (22) is equivalent to*

$$\max\{\theta(y) \mid \varphi(y) \le 1,\ y \in H\} \tag{24}$$

in the sense that if $\bar{x}$ solves (24), then $\bar{y} = u(\bar{x})$ solves (24), and conversely, if $\bar{y}$ solves (24) and $\theta(\bar{y}) = \langle c, \bar{x}\rangle$ for an optimal solution $\bar{x}$ of (23) (where $y = \bar{y}$), then $\bar{x}$ solves (22).

Proof. Let $\bar{x}$ solve (22) and $\bar{y} = u(\bar{x})$. Then $\varphi(\bar{y}) \le 1,\ \bar{y} \in H$. But for every $y \in R^m_+$ such that $\varphi(y) \le 1, y \in H$, we have $\theta(y) = \langle c, x\rangle$ for some $x \in D$, such that $u(x) \le y$ and hence, $\varphi(u(x)) \le 1$. Therefore, $\theta(y) \le \langle c, \bar{x}\rangle$, proving that $\bar{y}$ solves (24). Conversely, let $\bar{y}$ solve (24) and $\theta(\bar{y}) = \langle c, \bar{x}\rangle$ for an optimal solution $\bar{x}$ of (23). Then for every $x \in D$ such that $\varphi(u(x)) \le 1$, we have for $y = u(x) : \varphi(y) \le 1, y \in H$. Hence, on the one hand, $\theta(y) \le \theta(\bar{y}) = \langle c, \bar{x}\rangle$, on the other, $\langle c, x\rangle \le \theta(y)$, so $\langle c, x\rangle \le \langle c, \bar{x}\rangle$, i.e., $\bar{x}$ solves (24). ■

Again (24) is a Problem (A) in R^m, with $G = \{y \in R^m_+ \mid \varphi(y) \le 1\}$. Note that if $u_i(x), i = 1\ldots, m$, are convex, then $\theta(y)$ is the optimal value in a convex program.

Problems (20) and (22) with $\varphi(y) = \Pi^m_{i=1} y_i$ have been studied in [20] and [28], where some essential ideas of monotonic optimization have been first put forward. Computational experiments reported in these papers on two earlier versions of Algorithm 1 for instances of problems (20) and (22) with $n \le 15$ convincingly demonstrate the efficiency of the monotonic approach. Not only is this approach applicable to many problems so far known to be notoriously difficult, it even outperforms existing methods in several cases of interest.

5.2. Minimization Problem

In much the same way, we can derive the following algorithm for the minimization under monotonic constrains.

Algorithm 2. (For Problem (B), under assumption (7).) Select a vector $v \in R^n_{++}$ to define the mapping $\rho : H \to \partial^+ H^b$ (see (13) and also Remark 2). Select a tolerance $\varepsilon > 0$.

Initialization. Let $T_0 = Z_0 = \{a\}$. Let $\bar{x}$ be the best feasible solution available (the current best feasible solution), $CBV = f(\bar{x})$. If no feasible solution is available, set $CBV = +\infty$. Set $k = 0$.

Step 1. In T_k, delete all improper elements, all $z \in Z_k \setminus G$, and delete all z such that $f(z) \ge CBV - \varepsilon$. Let Z_k be the set of remaining elements of T_k.

Step 2. If $Z_k = \emptyset$, then terminate: if $CBV = +\infty$, the problem is infeasible; if $CBV < +\infty$, $\bar{x}$ is an ε-optimal solution.

Step 3. Select $z^k \in \text{argmin}\{f(x) \mid x \in Z_k\}$. If $z^k \in H$, then z^k is an optimal solution. Otherwise, compute $x^k = \rho(z^k)$. Update CBV and $\bar{x}$.

Step 4. Define $T_{k+1} = (Z_k \setminus \{z^k\}) \cup \{z^k + (x^k_i - z^k_i)e^i,\ i = 1, \ldots, n\}$.

Step 5. Set $k \leftarrow k+1$ and return to Step 1.

Proposition 25. *Assume that $f(x)$ is lower semicontinuous on G. If Algorithm 2 is infinite, it generates a sequence $a^{[i_0]}, a^{[i_0 i_1]}, \ldots, a^{[i_0 i_1 \cdots i_k]}$ converging to an optimal solution.*

Proof. Analogous to the proof of Proposition 22. ■

Remark 4. As with Algorithm 1, to alleviate storage problems in connection with the growth of T_k as the algorithm proceeds, Step 5 of Algorithm 2 can be modified as follows. Let L be the maximum size allowed for $|T_k|$.

Step 5. If $|T_{k+1}| \leq L$, then set $k \leftarrow k+1$ and return to Step 1. Otherwise, go to Step 6.

Step 6. Redefine $T_{k+1} = \{x_i^k e^i,\ i = 1, \ldots, n\}$, set $k \leftarrow k+1$ and return to Step 1.

With this modification, each time Step 6 occurs, Algorithm 2 is *restarted* from the last x^k. This restarting device enables us to overcome memory space limitations in solving large scale problems.

Example 3. Consider the problem

$$\min\{\varphi(u(x)) \mid x \in D\}, \tag{25}$$

where $D,\ \varphi,\ u(x)$ are as previously. This problem can be written as

$$\min\{\varphi(y) \mid y \in u(D)\} = \min\{\varphi(y) \mid y \in rN[u(D)]\},$$

or, equivalently, as

$$\min\{\varphi(y) \mid y \in H\}$$

with $H := rN[u(D)] = \{y \in [0, b] \mid x \in D,\ y \geq u(x)\}$, so this is a Problem (B) where $G = [0, b]$. The reverse normal set H is closed by continuity. As in Example 1, without loss of generality, we can assume that $\max_{x \in D} u_i(x) > 0\ \forall i = 1, \ldots, m$, i.e., that the normal set $[0, b] \setminus H$ has an interior point. Also, if $u_i(x),\ i = 1, \ldots, m$ are convex, then H is a reverse convex set, so for any $z \in H$, it is easy to compute the point $\rho(z)$ where the halfline from z in the direction of $e = (1, \ldots, 1) \in R_+^m$ meets $\partial^- H$.

Example 4. Consider the problem

$$\min\{\langle c, x\rangle \mid x \in D,\ \varphi(u(x)) \geq 1\} \tag{26}$$

with $D,\ \varphi,\ h$ as previously. Observe that the set

$$G = \{y \in R_+^m \mid y \leq u(x) \text{ for some } x \in D\}$$

is closed and normal, since $G = R_+^m \cap (u(D) - R_+^m) = N[u(D)]$. Define

$$\theta(y) = \begin{cases} \min\{\langle c, x\rangle \mid x \in D,\ y \leq u(x)\} & \text{if } y \in G, \\ M, & \text{otherwise,} \end{cases} \tag{27}$$

where $M > 0$ is an arbitrary number satisfying $M > \max\{\langle c, x\rangle \mid x \in D\}$. Since D is non-empty compact, clearly $-\infty < \theta(y) < +\infty\ \forall y \in R_+^m$ and it can easily be verified that the function $\theta(y)$ is lower semicontinuous and increasing (proof analogous to that of Proposition 23). Also, $\theta(y) < M \Leftrightarrow y \in G$.

Proposition 26. *Problem (26) is equivalent to*

$$\min\{\theta(y) \mid \varphi(y) \geq 1,\ \ y \in H\} \tag{28}$$

in the sense that if $\bar{x}$ solves (26), then $\bar{y} = u(\bar{x})$ solves (28) and conversely, if $\bar{y}$ solves (28) and $\theta(\bar{y}) = \langle c, \bar{x}\rangle$ for an optimal solution $\bar{x}$ of (27) (where $y = \bar{y}$), then $\bar{x}$ solves (26).

Proof. Analogous to the proof of Proposition 24. ∎

Thus, (26) appears to be a Problem (B) in R^m, with $H = \{y \in R^m_+ \mid \varphi(y) \geq 1\}$. If $u_i(x)$, $i = 1 \ldots, m$, are concave, then $\theta(y)$, for $y \in G$, is the optimal value of a convex program.

6. Optimization of Differences of Increasing Functions

Just as convex maximization methods can be extended to optimization of differences of convex functions, the above approach to monotonic optimization can be extended to optimization of differences of increasing functions. For the sake of convenience, we call *d.i. function* on $[a, b] \subset R^n_+$ any function which can be represented as a difference of two increasing functions on $[a, b]$. The set of all d.i. functions on $[a, b]$ forms a linear space, denoted by $DI[0, b]$, which is the linear space generated by increasing functions on $[a, b]$. The following proposition shows that $DI[a, b]$ includes a very large class of functions.

Proposition 27.

(i) *$DI\,[a, b]$ is a lattice with respect to the operations*

$$(f_1 \vee f_2)(x) = \max\{f_1(x), f_2(x)\},\ \ (f_1 \wedge f_2)(x) = \min\{f_1(x), f_2(x)\}.$$

(ii) *$DI\,[a, b]$ is dense in the space $C[a, b]$ of continuous functions on $[a, b]$ endowed with the usual supnorm.*

Proof. (i) Let $f_i = g_i - h_i$, where g_i, h_i are increasing on $[a, b]$. Noting that $f_1 = (g_1 + h_2) - (h_1 + h_2)$, $f_2 = (g_2 + h_1) - (h_1 + h_2)$ and setting $h = h_1 + h_2$, $p = g_1 + h_2$, $q = g_2 + h_1$ one has $f_1 \vee f_2 = \max\{p - h, q - h\} = \max\{p, q\} - h$, while $f_1 \wedge f_2 = \min\{p - h, q - h\} = \min\{p, q\} - h$. Since $\max\{p, q\}$ and $\min\{p, q\}$ are increasing, it follows that $f_1 \vee f_2$ and $f_1 \wedge f_2$ are d.i. on $[a, b]$.

(ii) A polynomial in $x \in R^n$ with positive coefficients is obviously an increasing function on R^n_+. Since an arbitrary polynomial $P(x)$ is a difference of two polynomials with positive coefficients: $P(x) = P_+(x) - P_-(x)$ where $P_+(P_-$, resp.) is the sum of all terms of P with positive (negative, resp.) coefficients, every polynomial is a d.i. function on any box $[a, b] \subset R^n_+$. But by the Weierstrass theorem, the set of polynomials on $[a, b]$ is dense in $C[a, b]$. Therefore, $DI[a, b]$ is dense in $C[a, b]$. ∎

Consider now the general *d.i. optimization problem*:

$$\text{(DIOP)}\qquad \begin{aligned} \min\ \ & f_1(x) - f_2(x), \\ \text{s.t.}\ \ & g_i(x) - h_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & x \in [0, b] \subset R^n_+, \end{aligned}$$

where f_1, f_2, g_i, h_i are increasing on $[a, b]$.

Proposition 28. *Any d.i. optimization problem can be reduced to minimizing an increasing function under monotonic constraints.*

Proof. We show that any (DIOP) can be transformed into an equivalent Problem (B). The transformation is performed in two steps.

Step 1. Reduce the problem to minimizing an increasing function under d.i. constraints. Let γ be any positive number such that $\gamma > f_2(b)$, i.e., $\gamma - f_2(x) > 0\ \forall x \in [0, b]$. We can rewrite (DIOP) as

$$\begin{aligned} \min \quad & f_1(x) + t \\ \text{s.t.} \quad & g_i(x) - h_i(x) \leq 0 \quad i = 1, \ldots, m, \\ & t + f_2(x) \geq \gamma, \\ & 0 \leq t \leq \gamma - f_2(0), \quad x \in [0, b]. \end{aligned}$$

Here, the function $(x, t) \mapsto f_1(x) + t$ is increasing and the constraints are d.i. on $[0, b] \times [0, \gamma - f_2(0)] \subset R^n_+ \times R_+$.

Step 2. Transform the resulting system of d.i. constraints into a monotonic system. By changing the notations, we can assume that this system of d.i. constraints has the form

$$g_i(x) - h_i(x) \leq 0 \quad i = 1, \ldots, p, \tag{29}$$

or, equivalently,

$$\bigvee_{i=1}^{p} [g_i - h_i](x) \leq 0.$$

Noting that $\vee_{i=1}^p [g_i - h_i](x) = g(x) - h(x)$ where $g = \vee_{i=1}^p [g_i + \Sigma_{j \neq i} h_j]$, $h = \Sigma_{i=1}^p h_i$ are increasing, we can rewrite (29) as

$$g(x) - h(x) \leq 0.$$

In turn, this inequality is equivalent to the monotonic system:

$$g(x) + u \leq \eta, \quad h(x) + u \geq \eta, \quad 0 \leq u \leq \eta - g(0),$$

where η is any positive number such that $\eta > g(b)$ (hence, for every $x \in [0, b] : g(x) \leq \eta$, i.e., $g(x) + u \leq \eta,\ u \geq 0$).

To sum up, Step 1 reduces the problem to minimizing an increasing function of (x, t) under a system of d.i. constraints in (x, t), then Step 2 converts the latter into a monotonic system in (x, t, u). The resulting problem, equivalent to the original (DIOP), is a Problem (B) in the variables (x, t, u). ■

Thus, at the expense of introducing at most two additional variables, any optimization problem involving differences of increasing functions can be reduced to minimizing or maximizing an increasing function under monotonic constraints. We close this paper with some applications.

6.1. Polynomial Programming

Denote by $\mathbf{P}(x)$ the set of polynomials in $x \in R^n$ with positive coefficients. As was already noticed, by grouping separately the terms with positive and the terms with negative coefficients, any polynomial $f(x)$ can be written as $f(x) = f_1(x) - f_2(x)$ with $f_1, f_2 \in \mathbf{P}(x)$. Therefore, any polynomial program can be written as a d.i. optimization problem (DIOP), where f_1, f_2 as well as g_i, h_i $(i = 1, \dots, m)$ all belong to $\mathbf{P}(x)$. By then applying further transformations described in Step 1 above and changing the notations, we can rewrite a polynomial program in the form

$$\min \quad f(x), \tag{30}$$

$$\text{s.t.} \quad g_i(x) - h_i(x) \leq 0 \quad i = 1, \dots, m, \tag{31}$$

$$x \in [0, b], \tag{32}$$

where $f, g_i, h_i \in \mathbf{P}(x)$, $i = 1, \dots, m$. Finally, by applying transformations described in Step 2 and changing the notations again, we obtain the following monotonic optimization problem:

$$\min \quad f(x), \tag{33}$$

$$\text{s.t.} \quad \max\{g_1, \dots, g_m\} + u \leq 1, \tag{34}$$

$$h(x) + u \geq 1, \tag{35}$$

$$(x, u) \in [0, b] \times [0, b_{n+1}], \tag{36}$$

where $b_{n+1} > g(b) - g(0)$ and $f, h, g_1, \dots, g_m \in \mathbf{P}(x)$. The latter problem is a Problem (B) (see (19)) with

$$G = \{(x, u) \mid \max\{g_1(x), \dots, g_m(x)\} + u \leq 1\}, \quad H = \{(x, u) \mid h(x) + u \geq 1\}.$$

The operator $\rho : G \cap R_+^{n+1} \to \partial H^\flat$ in Algorithm 2 for this problem is defined as follows:

$$z = (x, u) \mapsto \rho(z) = \max\{t \mid h(x + tb) + u + tb_{n+1} \geq 1\}.$$

This is an equation in t, of the form

$$\varphi(t) := h(x + tb) + u + tb_{n+1} = 1, \ 0 < t < 1,$$

where $\varphi(t)$ is a monotone increasing polynomial in t. Since the derivative $\varphi'(t)$ is readily available and is itself a polynomial in t with positive coefficients, i.e., an increasing function, this equation is very easy to solve. Therefore, Algorithm 2 reduces to solving a connected sequence of polynomial equations of one variable.

In the special case of *non-convex quadratic programming* problems, the computation of $\rho(z)$ is even simpler because it reduces to solving a mere quadratic equation of one variable.

6.2. A Problem of Smale

A challenging problem of global optimization which emerged from the complexity theory and is related to the arrangements of Fekete points on a sphere (see, e.g., [21]), consists of determining N points on a sphere such that the product of their mutual distances is maximized, i.e.,

$$\max \prod_{1\le i<j\le N} \|x^i - x^j\|, \quad \text{s.t. } \|x^i\| = 1 \ \ i = 1, \ldots, N.$$

By rewriting this problem as

$$\max \prod_{1\le i<j\le N} y_{ij}, \text{s.t. } y_{ij} \le \|x^i - x^j\|, \ \ 1 \le i < j \le N, \ \ \|x^i\| = 1, \ \ i = 1, \ldots, N,$$

we see that it has the form of a monotonic optimization problem, namely

$$\max \left\{ \prod_{1\le i<j\le N} y_{ij} \mid y = (y_{ij}) \in G \right\} \quad \text{with} \tag{37}$$

$$G = \{y = (y_{ij}) \mid y_{ij} \le \|x^i - x^j\| \ 1 \le i < j \le N, \ \|x^i\| = 1 \ \ i = 1, \ldots, N\}.$$

Here, the objective function is obviously increasing for $y = (y_{ij}) \ge 0$, while G is a normal set because $0 \le y' \le y$ and $y \in G$ imply $y' \in G$. Let $\alpha > 0$ be the product of mutual distances of any N chosen distinct points on the unit sphere. Since the distance between any two points on the unit sphere is at most 2, for any $y \in G$ and any (i, j) satisfying $1 \le i < j \le N$, we have $\alpha \le [N(N-1)/2 - 1]2y_{ij}$,

$$y_{ij} \ge \eta := \frac{\alpha}{N(N-1)-2}.$$

Therefore, if we define $H = \{y = (y_{ij} | \ y_{ij} \ge \eta\}$, then the problem (37) is the same as

$$\max \left\{ \prod_{1\le i<j\le N} y_{ij} \mid y = (y_{ij}) \in G \cap H \right\},$$

which is exactly a Problem (A). For solving this problem by Algorithm 1, the computational burden comes from the determination of $\pi(z)$ as defined from (11) for each given $z = (z_{ij}) \notin G$. In fact, computing $\pi(z)$ for the above set G amounts to solving the distance geometry problem

$$\min\{\lambda \mid \lambda z_{ij} \le \|x^i - x^j\| \ \ 1 \le i < j \le N, \ \ \|x^i\| = 1 \ \ i = 1, \ldots, N\}. \tag{38}$$

(Given positive numbers $\delta_{ij} = \lambda z_{ij}$, find N points $x^1, \ldots, x^N$ on the unit sphere, such that the distance between any two points x^i, x^j equals at least δ_{ij}.) This is still a difficult problem, which, however, can be solved, in principle, by currently available methods of non-convex quadratic programming (see, e.g., [25]), or also by the above-developed method of monotonic optimization (then each problem (38) reduces to a sequence of quadratic equations of one real variable).

7. Conclusion

We have presented a theory of normal sets and polyblocks and have shown how it provides a general mathematical framework for the study of monotonic systems of inequalities and monotonic optimization problems, including optimization problems involving d.i. functions. We have illustrated the applicability of this approach by examples of problems from generalized multiplicative programming, non-convex quadratic optimization, and more generally, polynomial programming. These difficult problems of non-convex global optimization have attracted considerable interest in recent years. In a companion paper [26], devoted especially to monotonic optimization, we will discuss these and other applications in greaater detail.

References

1. S. Azarm, Local monotonicity in optimal design, Ph.D. Thesis, University of Michigan, Ann Arbor, 1984.
2. S. Azarm and P. Papalambros, An automated procedure for local monotonicity analysis, *ASME Journal of Mechanisms, Transmissions, and Automation in Design* **106** (1984) 82–89.
3. R. J. Duffin, E. L. Peterson, and C. Zener, *Geometric Programming*, Wiley, New York, 1966.
4. C. R. Hammond and G. E. Johnson, A general approach to constrained optimal design based on symbolic mathematics, in: *Advances in Design Automation — Design Methods, Computer Graphics and Expert Systems*, S. S. Rao (ed.), ASME, Vol. 1, 1987, pp. 31–40.
5. P. Hansen, B. Jaumard, and S. H. Lu, Some further results on monotonicity in globally optimal design, *Journal of Mechanics, Transmissions and Automation in Design* **111** (1989) 345–352.
6. R. Horst and H. Tuy, *Global Optimization (Deterministic Approaches)*, 3rd ed., Springer-Verlag, Berlin-New York, 1996.
7. H. Konno and T. Kuno, Generalized multiplicative and fractional programming, *Annals of Operations Research* **25** (1990) 147–162.
8. H. Konno, Y. Yajima, and T. Matsui, Parametric simplex algorithms for solving a special class of non-convex minimization problems, *Journal of Global Optimization* **1** (1991) 65–81.
9. H. Konno, P. T. Thach, and H. Tuy, *Optimization on Low Rank non-convex Structures*, Kluwer Academic Publishers, Dordrecht - Boston - London, 1997.
10. H. L. Li and P. Papalambros, A production system for use of global optimization knowledge, *ASME Journal of Mechanisms, Transmissions, and Automation in Design* **107** (1985) 277–284.
11. V. L. Makarov and A. M. Rubinov, *Mathematical Theory of Economic Dynamic and Equilibria*, Springer-Verlag, Berlin-New York, 1977.
12. T. S. Motzkin and E. G. Strauss, Maxima for graphs and a new proof of a theorem of Turan, *Canad. J. Math.* **17** (1965) 533–540.
13. H. Nikaido, *Economic Theory and Convex Structures*, Academic Press, New York, 1969.
14. P. Papalambros and H. L. Li, Notes on the operational utility of monotonicity in optimization, *ASME Journal of Mechanisms, Transmissions, and Automation in Design* **105** (1993) 174–180.
15. P. Papalambros and D. J. Wilde, *Principles of Optimal Design — Modeling and Computation*, Cambridge University Press, New York, 1986.
16. P. Pardalos and G. L. Xue (eds.), *Journal of Global Optimization*, Special issue on molecular and protein conformations, **11** (1997).
17. U. Passy, Global solutions of mathematical programs with intrinsically concave functions, in: *Advances in Geometric Programming*, M. Avriel (ed.), Plenum Press, New York, 1980.
18. K. M. Ragsdell and D. T. Philipps, Optimal design of a class of welded structures using geometric programming, *ASME Journal of Engineering for Industry* **98** (1976) 1021–1025.
19. A. Rubinov and B. M. Glover, Duality for increasing positively homogenuous functions and normal sets, *Recherche Operationnelle (Operations Research)* **32** (1998) 105–123.
20. A. Rubinov, H. Tuy, and H. Mays, Algorithm for a monotonic global optimization problem, SIMS, University of Ballarat, Australia, 1998 (preprint).

21. E. Saff and A. Kuijlaars, Distributing many points on a sphere, *Mathematical Intelligencer* **10** (1997) 5–11.
22. H. Tuy, The complementary convex structure in global optimization, *Journal of Global Optimization* **2** (1992) 21–40.
23. H. Tuy, D.C. Optimization: Theory, methods and algorithms, in: *Handbook on Global Optimization*, R. Horst and P.M. Pardalos (eds.), Kluwer Academic Publishers, Dordrecht-Boston-London, 1995, pp.149–216.
24. H. Tuy, *Convex Analysis and Global Optimization*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1998.
25. H. Tuy, Normal branch and bound algorithms for general non-convex quadratic programming, in: *Combinatorial and Global Optimization*, P.M. Pardalos, A. Migdalas, and R. Burkard (eds.), World Scientific Publishing Co., 1999 (to appear).
26. H. Tuy, Monotonic optimization: Problems and solution approaches, Institute of Mathematics, Hanoi, 1999 (preprint).
27. H. Tuy, The MCCNFP with a fixed number of non-linear arc costs: Complexity and approximation, in: *Approximation and Complexity in Numerical Optimization: Continuous and Discrete Problems*, P.M. Pardalos (ed.), Kluwer Academic Publishers, 1999 (to appear).
28. H. Tuy and Le Tu Luc, A new approach to optimization under monotonic constraint, Institute of Mathematics, Hanoi, 1998 (preprint).
29. D. J. Wilde, A maximum activity principle for eliminating over-constrained optimization cases, *ASME Journal of Mechanisms, Transmissions, and Automation in Design* **108** (1986) 312–314.