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On Certain Length Functions Associated to a System of Parameters in Local Rings

Nguyen Tu Cuong, Nguyen Thai Hoa, and Nguyen Thi Hong Loan

Institute of Mathematics
P.O. Box 631, Bo Ho, Hanoi, Vietnam

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Abstract. We define in this paper two length functions $q_{M,\underline{x}}(\underline{n})$ and $J_{M,\underline{x}}(\underline{n})$ in d-variables $\underline{n} = (n_1, \ldots, n_d)$ associated to a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of an A-module M. Some properties of these functions are given, and thereby some relationships between them and the structure of M are clarified. We can also calculate these functions for generalized Cohen–Macaulay modules.

1. Introduction

This paper is concerned with the following submodule:

$$Q_M(\underline{x}) = \bigcup_{t>0}((x_1^{t+1}, \ldots, x_d^{t+1})M : x_1^t, ..., x_d^t),$$

where M is a finitely generated module over a Noetherian local ring $(A, \mathfrak{m})$ and $\underline{x} = (x_1 \cdots x_d)$ is a system of parameters (s.o.p. for short) on M. If M is a Cohen–Macaulay module, it is known by Hartshorne [10] that $Q_M(\underline{x}) = (x_1, \ldots, x_d)M$. The submodule $Q_M(\underline{x})$ is also used for studying the monomial conjecture with respect to $\underline{x}$ as follows: We say that the system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M satisfies the monomial conjecture if $x_1^t \cdots x_d^t M \notin (x_1^{t+1}, \ldots, x_d^{t+1})M$ for all $t > 0$ (see [11]). Clearly, $\underline{x}$ satisfies the monomial conjecture if and only if $Q_M(\underline{x}) \neq M$, i.e., $\ell(M/Q_M(\underline{x})) \neq 0$. This suggests the studying the lengths

$$q_{M,\underline{x}}(\underline{n}) = \ell\big(M/Q_M(\underline{x}(\underline{n}))\big)$$
$$J_{M,\underline{x}}(\underline{n}) = n_1 \cdots n_d e(\underline{x}; M) - q_{M;\underline{x}}(\underline{n})$$

as functions in $\underline{n} = (n_1, \ldots, n_d)$, where $\underline{x}(\underline{n}) = (x_1^{n_1}, \ldots, x_d^{n_d})$. The aim of this paper is to study the behavior of these functions and some relations between them and the structure of the module M.

This paper is divided into the following: In Sec. 2, we give some basic properties of the function $q_{M,\underline{x}}(\underline{n})$. It should be mentioned that, by [4], this function is just the length of generalized fractions defined in [16]. Thus, all results in this section are known in the case $M = A$ (see [13]), but our proofs, even in the general case, are more elementary.

In Sec. 3, we will show that the determinantal map

$$\delta : M/Q_M(\underline{x}) \to M/Q_M(\underline{y})$$

is an injective homomorphism, where $\underline{x}$, $\underline{y}$ are two systems of parameters on M such that $(\underline{y})A \subseteq (\underline{x})A$. This result, whose proof is quite simple, is a useful tool in the paper. As a first application, we give an uniform bound for the monomial conjecture for modules (Theorem 3.3). We consider in Sec. 4 the non-negative function

$$J_{M,\underline{x}}(\underline{n}) = n_1 \cdots n_d e(\underline{x}; M) - q_{M;\underline{x}}(\underline{n}).$$

Using the determinantal map in Sec. 3, we prove that the function $J_{M,\underline{x}}(\underline{n})$ is bounded above by a polynomial of degree $\leq \dim M - 2$. Moreover, the least degree of all polynomials in $\underline{n}$ bounding above the function $J_{M,\underline{x}}(\underline{n})$ is independent of the choice of the system of parameters $\underline{x}$ (Theorem 4.4). The last section is devoted to calculate the function $J_{M,\underline{x}}(\underline{n})$ when M is a generalized Cohen–Macaulay module (Theorem 5.1).

2. Function $q_{M,\underline{x}}(\underline{n})$

Throughout this paper, let $(A, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated A-module with $\dim M = d$. Let $\underline{x} = (x_1, \ldots, x_d)$ be an s.o.p. of M and $\underline{n} = (n_1, \ldots, n_d)$ a d-tuple of positive integers. We set

$$\underline{x}(\underline{n}) = (x_1^{n_1}, \ldots, x_d^{n_d}),$$
$$Q_M(\underline{x}) = \bigcup_{t>0} \big((x_1^{t+1}, \ldots, x_d^{t+1})M : x_1^t \cdots x_d^t\big),$$

and

$$Q_M(\underline{x}, \underline{n}) = Q_M(\underline{x}(\underline{n})).$$

Consider now the length

$$q_{M,\underline{x}}(\underline{n}) = \ell_A\big(M/Q_M(\underline{x}, \underline{n})\big)$$

as a function in $\underline{n}$. For simplicity, we write $q_{M,\underline{x}}(\underline{n}) = q_M(\underline{x})$ when $n_1 = \cdots = n_d = 1$.

The following simple lemma is helpful in the sequel.

Lemma 2.1. *With the above notions, the following statements are true:*

(i) *Let $\overline{M} = M/N$, where N is either an artinian submodule of M or $N = (0 :_M x_1)$. Then $\underline{x}$ is an s.o.p. of $\overline{M}$ and*

$$q_{M,\underline{x}}(\underline{n}) = q_{\overline{M},\underline{x}}(\underline{n}).$$

(ii) *Let $\widehat{M}$ be the $\mathfrak{m}$-adic completion of M. Then*

$$q_{M,\underline{x}}(\underline{n}) = q_{\widehat{M},\underline{x}}(\underline{n}).$$

Proof. (i) It is easy to check that

$$Q_{\overline{M}}(\underline{x}) = Q_M(\underline{x})/N,$$

and this implies (i).

(ii) follows immediately by the fact that $\widehat{A}$ is a faithfully flat A-module. ∎

Lemma 2.2. *With the above notations, there exists an epimorphism*

$$\psi : M'/Q_{M'}(\underline{x}') \to M/Q_M(\underline{x})$$

defined by $\psi(u' + Q_{M'}(\underline{x}')) = u + Q_M(\underline{x})$ for any $u \in M$, where $M' = M/x_1M$, u' is the image of u in M' and $\underline{x}' = (x_2, \dots, x_d)$.

Proof. It is trivial that ψ is surjective. One can easily check that ψ is really a homomorphism. ∎

Lemma 2.3. *With the notations as above,*

$$q_M(\underline{x}) \le e(\underline{x}; M).$$

Proof. We do it by induction on d. If $d = 1$, by Lemma 2.1(i), we may assume that depth $M > 0$. Therefore, M is a Cohen–Macaulay module. Hence, $q_M(\underline{x}) = e(\underline{x}; M)$ since $Q_M(x_1) = x_1M$.

Now, let $d > 1$. By Lemma 2.1(i), we may assume that depth $M > 0$ and x_1 is a non-zerodivisor of M. Set $M' = M/x_1M$. By the inductive hypothesis and Lemma 2.2, the epimorphism

$$\psi : M'/Q_{M'}(\underline{x}') \to M/Q_M(\underline{x})$$

implies that

$$q_M(\underline{x}) \le q_{M'}(\underline{x}') \le e(\underline{x}'; M') = e(\underline{x}; M)$$

as required. ∎

The following criterion of Cohen–Macaulayness, which is an easy consequence of Lemma 2.3, it not directly linked to the main object of this paper, but it came up while working on it.

Corollary 2.4. *The following statements are equivalent:*

(i) *M is Cohen–Macaulay;*
(ii) *$Q_M(\underline{x}) = \underline{x}M$, for any s.o.p. $\underline{x}$ of M;*
(iii) *There exists an s.o.p. $\underline{x}$ of M such that*

$$Q_M(\underline{x}) = \underline{x}M.$$

Proof. (i) $\Rightarrow$ (ii) follows by [10].

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) Assume that there exists an s.o.p. $\underline{x}$ of M such that $Q_M(\underline{x}) = \underline{x}M$. Then $\ell_A\big(M/Q_M(\underline{x})\big) = \ell_A\big(M/\underline{x}M\big) \geq e(\underline{x}; M)$. By Lemma 2.3, we have $\ell_A\big(M/\underline{x}M\big) \leq e(\underline{x}; M)$. Therefore, $\ell_A\big(M/\underline{x}M\big) = e(\underline{x}; M)$. So M is a Cohen–Macaulay module. ∎

3. Determinantal Map

Let $\underline{x} = (x_1, \ldots, x_n)$ be a sequence of elements in $\mathfrak{m}$. Let $\underline{y} = (y_1, \ldots, y_n)$ be another sequence of n elements such that $(y_1, \ldots, y_n)A \subseteq (x_1, \ldots, x_n)A$. Then there exists a matrix $B = (b_{ij})$, $b_{ij} \in A$, $1 \leq i, j \leq n$ such that

$$y_i = \sum_{j=1}^{n} b_{ij} x_j.$$

Put $\delta = \det B$. It easily follows from Crammer's rule that $\delta(\underline{x})A \subseteq (\underline{y})A$. Therefore, we obtain a canonical map

$$\delta : M/(\underline{x})M \to M/(\underline{y})M.$$

By [15, 5.1.15], we also have $\delta Q_M(\underline{x}) \subseteq Q_M(\underline{y})$. Therefore, we obtain a homomorphism

$$\delta : M/Q_M(\underline{x}) \to M/Q_M(\underline{y}),$$

which is independent of the choice of the matrix B by [15, 5.1.14]. The map δ is called the determinantal map. The following lemma is the key lemma of this section and also often used in next sections.

Lemma 3.1. *Let $\underline{x} = (x_1, \ldots, x_d)$ and $\underline{y} = (y_1, \ldots, y_d)$ be two systems of parameters of M such that $(\underline{y})A \subseteq (\underline{x})A$. Then the determinantal map*

$$\delta : M/Q_M(\underline{x}) \to M/Q_M(\underline{y})$$

is injective.

Proof. Since $\delta(\mathrm{Ann}_A(M)) = 0$, without any loss of generality, we may assume that $\mathrm{Ann}_A(M) = 0$. Then the ideal $(\underline{y})A$ is $\mathfrak{m}$-primary; so there exists a positive integer k such that $(\underline{x}(k))A \subseteq (\underline{y})A$. Therefore, we have a commutative diagram

$$\begin{tikzcd} M/Q_M(\underline{x}) \arrow[rr, "\delta"] \arrow[dr, "\delta_2"'] & & M/Q_M(\underline{y}) \arrow[dl, "\delta_1"] \\ & M/Q_M(\underline{x}(k)) & \end{tikzcd}$$

where $\delta, \delta_1, \delta_2$ are determinantal maps. It is easy to see that δ_2 is the map defined by multiplication by $x_1^{k-1} \cdots x_d^{k-1}$. Therefore, δ_2 is injective. Thus, δ is injective as required. ∎

An immediate consequence of Lemma 3.1 is the following corollary which will be often used in this paper.

Corollary 3.2. *Let $\underline{x} = (x_1, \ldots, x_d)$ and $\underline{y} = (y_1, \ldots, y_d)$ be two systems of parameters of M such that $(\underline{y})A \subseteq (\underline{x})A$. Then*

$$q_M(\underline{x}) \leq q_M(\underline{y}).$$

In Conjecture 1 of [11], Hochster conjectured for the case $M = A$ that, for every system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of A, one has

$$x_1^t \cdots x_d^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})A$$

for all $t > 0$. Hochster proved in [11] that this monomial conjecture is true for high powers of systems of parameters. He also gave an example which shows that the monomial conjecture is not true for modules. However, for a given system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M, we say that $\underline{x}$ satisfies the *condition of monomial conjecture* (MC) if

$$x_1^t \cdots x_d^t M \nsubseteq (x_1^{t+1}, \ldots, x_d^{t+1})M$$

for all $t > 0$. Then it is easy to see that a system of parameters $\underline{x}$ satisfies the condition (MC) if and only if $Q_M(\underline{x}) \neq M$. By the counterexample of Hochster mentioned above, we cannot show, in general, that every system of parameters of M satisfies the condition (MC). But we can give a uniform bound for high powers of all systems of parameters of M satisfying condition (MC) as follows:

Theorem 3.3. *There exists a constant N such that, for every system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M, $\underline{x}(\underline{n}) = (x_1^{n_1}, \ldots, x_d^{n_d})$ satisfies the condition (MC) for all $n_1, \ldots, n_d > N$.*

Proof. First, we show that, for every system of parameters $\underline{x} = (x_1, \ldots, x_d)$, there always exists an n_0 (possibly depending on $\underline{x}$) such that $\underline{x}(\underline{n}) = (x_1^{n_1}, \ldots, x_d^{n_d})$ satisfies the condition (MC) for all $n_1, \ldots, n_d \geq n_0$. In fact, it is well known that

$$\varinjlim M/(x_1^n, \ldots, x_d^n)M \cong H_{\mathfrak{m}}^d(M) \neq 0,$$

where the map

$$M/(x_1^n, \ldots, x_d^n)M \overset{f_{n,n+k}}{\rightarrow} M/\left(x_1^{n+k}, \ldots, x_d^{n+k}\right)M$$

is induced by multiplication by $(x_1 \cdots x_d)^k$. Thus, there exists an n_0 such that, for all $n \geq n_0$ and all $k > 0$, the map $f_{n,n+k}$ is non-zero. Therefore, we have

$$(x_1^{n+k}, \ldots, x_d^{n+k})M : (x_1 \cdots x_d)^k \neq M.$$

From this, we can easily verify that $Q_{M,\underline{x}}(\underline{n}) \neq M$ for all $n_1, \ldots, n_d \geq n_0$. Therefore, the system $(x_1^{n_1}, \ldots, x_d^{n_d})$ satisfies the condition (MC). Thus, we can now assume that there exists a system of parameters $\underline{y} = (y_1, \ldots, y_d)$ of M such that $\underline{y}$ satisfies the condition (MC). Moreover, without loss of generality, we may assume that $\mathrm{Ann}_A(M) = 0$. Thus, there exists a $r > 0$ such that $\mathfrak{m}^r \subseteq (\underline{y})A$. Therefore,

$$(x_1^{n_1}, \ldots, x_d^{n_d})A \subseteq (\underline{y})A$$

for any system of parameters $\underline{x} = (x_1, \dots, x_d)$ and for all $n_1, \dots, n_d \geq rd$. Set $N = rd$. Then

$$\ell_A(M/Q_M(\underline{y})) \leq \ell_A(M/Q_{M,\underline{x}}(\underline{n}))$$

for all $n_1, \dots, n_d \geq N$ by Corollary 3.2. Since $\underline{y}$ satisfies the condition (MC), $\ell_A(M/Q_M(\underline{y})) > 0$. It follows that $\ell_A(M/Q_{M,\underline{x}}(\underline{n})) > 0$, and so $\underline{x}(\underline{n})$ satisfies the condition (MC) for all $n_1, \dots, n_d \geq N$. ■

Remark. It should be noted that Strooker in [15, 11.2] has also proved an analogous result to Theorem 3.3 for the case $M = A$. But this proof used deep results on the annihilators of local cohomology. Our proof of Theorem 3.3, even for modules, is more elementary. On the other hand, since the monomial conjecture is not true for modules, so we cannot show that the integer N in Theorem 3.3 is equal to 1. So it is may be worth finding upper bounds for this integer N.

4. Function $J_{M,\underline{x}}(\underline{n})$

Keep all notations in the previous section. Let $\underline{x} = (x_1, \dots, x_d)$ be an s.o.p. of M. The difference

$$J_{M,\underline{x}}(\underline{n}) = n_1 \cdots n_d e(\underline{x}; M) - q_{M,\underline{x}}(\underline{n}),$$

which is non-negative by Lemma 2.3, can be considered as a function in $\underline{n}$. First, we need the following two auxiliary lemmas:

Lemma 4.1. *The following statements are true:*

(i) $J_{M,\underline{x}}(\underline{n}) = J_{\widehat{M},\underline{x}}(\underline{n}) = J_{M/H^0_{\mathfrak{m}}(M)}(\underline{n})$, *where $\widehat{M}$ is the $\mathfrak{m}$-adic completion of M.*

(ii) $J_{M,\underline{x}}(\underline{n}) \leq n_1 \cdots n_d J_M(\underline{x})$.

Proof. (i) follows from Lemma 2.1.

(ii) We denote by $\underline{\alpha}$ the d-tupel of integers $(\alpha, 1, \dots, 1)$. Then the map

$$\Phi : M/Q_{M,\underline{x}}(\underline{\alpha + 1}) \to M/Q_{M,\underline{x}}(\underline{\alpha})$$

defined by $\Phi(u + Q_{M,\underline{x}}(\underline{\alpha + 1})) = u + Q_{M,\underline{x}}(\underline{\alpha})$, $u \in M$, is an epimorphism. One can easily check that the map

$$\psi : M/Q_M(\underline{x}) \to \ker \Phi$$

defined by $\psi(u + Q_M(\underline{x})) = x_1^\alpha u + Q_{M,\underline{x}}(\underline{\alpha + 1})$, $u \in M$, is a monomorphism. It follows by induction on α that

$$\begin{aligned}\ell_A(M/Q_{M,\underline{x}}(\underline{\alpha + 1})) &= \ell_A(M/Q_{M,\underline{x}}(\underline{\alpha})) + \ell_A(\ker \phi)\\ &\geq \ell_A(M/Q_{M,\underline{x}}(\underline{\alpha})) + \ell_A(M/Q_M(\underline{x}))\\ &\geq (\alpha + 1)\ell_A(M/Q_M(\underline{x})).\end{aligned}$$

Since our proof is independent of the order of sequence $\underline{x}$, we finally obtain

$$q_{M,\underline{x}}(\underline{n}) \geq n_1 \cdots n_d q_M(\underline{x}).$$

Thus,

$$J_{M,\underline{x}}(\underline{n}) \leq n_1 \cdots n_d J_M(\underline{x}).$$

■

We set, for simplicity, $J_M(\underline{x}) = J_{M,\underline{x}}(\underline{n})$ when $n_1 = \cdots = n_d = 1$.

Lemma 4.2. *Let $\underline{x} = (x_1, \dots, x_{d-1}, x_d)$ and $\underline{y} = (x_1, \dots, x_{d-1}, y_d)$ be two systems of parameters M such that $(\underline{y})A \subseteq (\underline{x})A$. Then*

$$J_M(\underline{x}) \leq J_M(\underline{y}).$$

Proof. We use induction on d. It is trivial for the case $d = 1$. Suppose $d > 1$. By Lemma 4.1(i), we can assume that depth $M > 0$ and x_1 is a non-zerodivisor of M. Let $\overline{M} = M/x_1M$ and $\overline{A} = A/x_1A$. We put $\underline{x}' = (\bar{x}_2, \dots, \bar{x}_{d-1}, \bar{x}_d)$ and $\underline{y}' = (\bar{x}_2, \dots, \bar{x}_{d-1}, \bar{y}_d)$. Since $(\underline{y}')\overline{A} \subseteq (\underline{x}')\overline{A}$, we obtain, by Lemma 2.2, a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\varphi & \longrightarrow & \overline{M}/Q_{\overline{M}}(\underline{x}') & \xrightarrow{\varphi} & M/Q_M(\underline{x}) & \longrightarrow & 0 \\ & & \downarrow\delta' & & \downarrow\bar{\delta} & & \downarrow\delta & & \\ 0 & \to & \ker\psi & \to & \overline{M}/Q_{\overline{M}}(\underline{y}') & \xrightarrow{\psi} & M/Q_M(\underline{y}) & \to & 0, \end{array}$$

where the rows are exact sequences and $\delta, \bar{\delta}$ are determinantal maps. Therefore, by Lemma 3.1, the induced homomorphism δ' is injective. It follows from the inductive hypothesis that

$$\begin{aligned} J_M(\underline{x}) &= J_{\overline{M}}(\underline{x}') + \ell_A(\ker\varphi) \\ &\leq J_{\overline{M}}(\underline{y}') + \ell_A(\ker\psi) \\ &= J_M(\underline{y}). \end{aligned}$$

The lemma is proved. ■

Corollary 4.3. *The function $J_{M,\underline{x}}(\underline{n})$ is ascending, i.e., for $\underline{n} = (n_1, \dots, n_d)$, $\underline{m} = (m_1, \dots, m_d)$ with $n_i \geq m_i$, $i = 1, \dots, d$,*

$$J_{M,\underline{x}}(\underline{n}) \geq J_{M,\underline{x}}(\underline{m}).$$

Proof. Straightforward. ■

By Lemma 4.1(ii), we see that the function $J_{M,\underline{x}}(\underline{n})$ is bounded above by polynomial $n_1, \dots, n_d J_M(\underline{x})$. More generally, we can show the following theorem:

Theorem 4.4. *The least degree of all polynomials in $\underline{n}$ bounding above the function $J_{M,\underline{x}}(\underline{n})$ is independent of the choice of the s.o.p. $\underline{x}$.*

Proof. By Lemma 4.1(i), we can assume that $\mathrm{Ann}_A(M) = 0$ and depth $M > 0$. Let $\underline{y} = (y_1, \dots, y_d)$ be an s.o.p. of M. From [15, 8.2.5], there exists an s.o.p. $\underline{z} = (z_1, \dots, z_d)$ of M and positive integers $r_1, \dots, r_d$ such that

$$\begin{aligned} (x_1^{r_1}, \dots, r_d^{r_d})A &\subseteq (z_1, x_2^{r_2}, \dots, r_d^{r_d})A \subseteq \cdots \subseteq (z_1, \dots, z_d)A \subseteq \cdots \\ &\subseteq (z_1, \dots, z_{d-1}y_d)A \subseteq \cdots \subseteq (z_1, y_2, \dots, y_d) \subseteq (y_1, \dots, y_d)A. \end{aligned}$$

Note that, if $(x_1, \dots, x_{d-1}, y_d)A \subseteq (x_1, \dots, x_{d-1}, x_d)A$, then

$$(x_1^t, \dots, x_{d-1}^t, y_d^{dt})A \subseteq (x_1^t, \dots, x_{d-1}^t, x_d^t)A$$

for all $t \geq 1$. Therefore, applying Lemmas 4.1(ii) and 4.2 to the ascending sequence of ideals above $2d$-times, we obtain

$$J_{M,\underline{y}}(t) \leq \Big(\prod_{i=1}^{d} r_i\Big) d^{2d} J_{M,\underline{x}}(t)$$

for all $t \geq 1$. Similarly, there exists also a positive integer k such that

$$J_{M,\underline{x}}(t) \leq k J_{M,\underline{y}}(t)$$

for all $t \geq 1$. This shows that the least degree of all polynomials of one variable t bounding above $J_{M,\underline{x}}(t)$ is independent of the choice of $\underline{x}$. Now, the theorem easily follows from Corollary 4.3. ∎

Remark. It should be mentioned that, by using the theory of modules of generalized fractions of [19], Minh in [13] has defined a function

$$J_A(\underline{x}; \underline{n}) = n_1, \dots, n_d e(\underline{x}; A) - \ell_A\big(A(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))\big).$$

By [4, 2.3], we see that the length of generalized fractions $\ell_A\big(A(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))\big)$ is just equal to the function $q_{A,\underline{x}}(\underline{n})$. Therefore, the function $J_{A,\underline{x}}(\underline{n})$ is nothing else but the function $J_A(\underline{x}; \underline{n})$ defined by Minh as above. Thus, Theorem 4.4 is an extension to modules of the main result in Theorem 1.1 of [13]. However, the injective determinantal map in Lemma 3.1 enables us to obtain quite a simple and elementary proof for Theorem 4.4.

Following [6], we denote the new invariant defined in Theorem 4.4 by $pf(M)$. For convenience, we stipulate that the degree of the zero-polynomial is equal to $-\infty$. The following corollary is an immediate consequence of Lemma 4.1.

Corollary 4.6. *Let $\widehat{M}$ be the $\mathfrak{m}$-adic completion of M. Then*

$$pf(M) = pf\big(M/H^0_{\mathfrak{m}}(M)\big) = pf(\widehat{M}).$$

Proposition 4.7. *Suppose* $\dim M = d \geq 1$. *Then*

$$pf(M) \leq d - 2.$$

Proof. We prove by induction on d. In the proof of Lemma 2.3, we see that, if $d = 1$, $pf(M) = -\infty$ and so the proposition is true in this case. Assume that $d \geq 2$. By [4, 2.5], there exists an s.o.p. $\underline{x} = (x_1, \dots, x_d)$ of M such that the length $\ell_A\big(H^{d-1}_{\mathfrak{m}}(M)/x_1^{n_1} H^{d-1}_{\mathfrak{m}}(M)\big)$ is finite and independent of n_1, $e(\underline{x}', M_{n_1}) = e(\underline{x}; M)$ and the sequence

$$0 \to H^{d-1}_{\mathfrak{m}}(M)/x_1^{n_1} H^{d-1}_{\mathfrak{m}}(M) \to M_{n_1}/Q_{M_{n_1},\underline{x}'}(\underline{n}') \to M/Q_{M,\underline{x}}(\underline{n}) \to 0$$

is exact for all $n_1, \dots, n_d > 0$, where $M_{n_1} = M/x_1^{n_1}M$, $\underline{x}' = (x_2, \dots, x_d)$, $\underline{n}' = (n_2, \dots, n_d)$. Therefore,

$$J_{M,\underline{x}}(\underline{n}) = J_{M_{n_1},\underline{x}'}(\underline{n}') + \ell_A\big(H^{d-1}_{\mathfrak{m}}(M)/x_1^{n_1} H^{d-1}_{\mathfrak{m}}(M)\big).$$

If $d=2$, since $J_{M_{n_1},\underline{x}'}(\underline{n}')=0$, by (2.3), it follows that

$$J_{M,\underline{x}}(\underline{n}) = \ell_A\big(H^{d-1}_{\mathfrak{m}}(M)/x_1^{n_1}H^{d-1}_{\mathfrak{m}}(M)\big)$$

is independent of n_1. Thus, $pf(M)\leq 0$. If $d>2$, by (4.1)(ii), we have

$$J_{M,\underline{x}}(\underline{n}) \leq n_1 J_{M_{n_1},\underline{x}'}(\underline{n}').$$

Therefore,

$$J_{M,\underline{x}}(\underline{n}) \leq n_1\Big(J_{M_{n_1},\underline{x}'}(\underline{n}') + \ell_A\big(H^{d-1}_{\mathfrak{m}}(M)/x_1^{n_1}H^{d-1}_{\mathfrak{m}}(M)\big)\Big).$$

Hence, the proposition follows from the inductive hypothesis. ∎

We have shown in the previous section (Theorem 3.3), by using the non-vanishing of the highest local cohomology module $H^d_{\mathfrak{m}}(M)$, that the high powers of any system of parameters of M satisfy the condition (MC). Below, we will give a more elementary proof of this result by virtue of Proposition 4.7.

Corollary 4.8. *Let $\underline{x}=(x_1,\dots,x_d)$ be any system of parameters of M. Then there exists a positive integer k such that the system of parameters $\underline{x}(\underline{n}) = (x_1^{n_1},\dots,x_d^{n_d})$ satisfies the condition (MC) for all $n_1,\dots,n_d\geq k$.*

Proof. By Proposition 4.7, $J_{M,\underline{x}}(\underline{n})$ is bounded above by a polynomial of degree $d-2$, and note that $e(\underline{x}(\underline{n});M)$ is a polynomial in $\underline{n}$ of degree d. There exists a constant k such that

$$k^d e(\underline{x};M) > J_{M,\underline{x}}(k).$$

Therefore,

$$\ell_A\big(M/Q_M(\underline{x},\underline{n})\big) \geq \ell_A\big(M/Q_M(\underline{x},k)\big) > 0,$$

for all $n_1,\dots,n_d>k$. Thus, $\underline{x}(\underline{n})$ satisfies the condition (MC) for all $n_1,\dots,n_d>k$. ∎

5. Generalized Cohen–Macaulay Modules

The concept of generalized Cohen–Macaulay modules was first introduced in [7]. A module M is called a *generalized Cohen–Macaulay module* if and only if the ith local cohomology module $H^i_{\mathfrak{m}}(M)$ is finitely generated for all $i=0,\dots,d-1$. An important tool for studying generalized Cohen–Macaulay modules is the notion of standard systems of parameters defined in [20] as follows. A system of parameters $\underline{x}=(x_1,\dots,x_d)$ of M is called a *standard system of parameters* if

$$\ell_A\big(M/\underline{x}M\big) - e(\underline{x};M) = \ell_A\big(M/\underline{x}(2)M\big) - e(\underline{x}(2);M).$$

Then M is a generalized Cohen–Macaulay module if and only if M admits a standard system of parameters. Note that standard systems of parameters are also used to characterize Buchsbaum modules (see [17]). A module M is Buschsbaum if and only if every system of parameters of M is standard.

Let M be a generalized Cohen–Macaulay module and $\underline{x} = (x_1, \dots, x_d)$ a system of parameters. Sharp and Hamieh have shown in [16] that, for $n_1, \dots, n_d$ large enough,

$$J_{M,\underline{x}}(\underline{n}) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell_A(H^i_{\mathfrak{m}}(M)).$$

The main result of this section is relatively close to the above theorem of Sharp–Hamieh.

Theorem 5.1. *Let M be a generalized Cohen–Macaulay module and $\underline{x} = (x_1, \dots, x_d)$ a standard s.o.p. of M. Then*

$$J_{M,\underline{x}}(\underline{n}) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell_A\left(H^i_{\mathfrak{m}}(M)\right)$$

for all $n_1, \dots, n_d \geq 1$.

Proof. We prove by induction on d. For $d = 1$, we see in the proof of Lemma 2.3 that $J_{M,\underline{x}}(\underline{n}) = 0$, so in this case, the theorem is proved. Since $M/(0 :_M x_1)$ is generalized Cohen–Macaulay, without loss of generality, we can assume by Lemma 4.1(i) that x_1 is a non-zero divisor of M. Suppose now that $d > 1$. Put $M_{n_1} = M/x_1^{n_1}M$, $\underline{x}' = (x_2, \dots, x_d)$, and $\underline{n}' = (n_2, \dots, n_d)$. Then M_{n_1} is again a generalized Cohen–Macaulay module and $\underline{x}'$ is a standard system of parameters of M_{n_1}. Moreover, by [9], every standard system of parameters is an unconditioned strong d-sequence. Therefore, by [9, 2.3], we have

$$\begin{aligned} Q_M(\underline{x}, \underline{n}) &= \sum_{i=1}^{d} [(x_1^{n_1}, \dots, \widehat{x_i^{n_i}}, \dots, x_d^{n_d})M : x_i] + (x_1^{n_1}, \dots, x_d^{n_d})M \\ &= \sum_{i=1}^{d} [(x_1^{n_1}, \dots, \widehat{x_i^{n_i}}, \dots, x_d^{n_d})M : x_i] \end{aligned}$$

and

$$\begin{aligned} Q_{M_{n_1}}(\underline{x}', \underline{n}') &= \sum_{i=2}^{d} [(x_2^{n_2}, \dots, \widehat{x_i^{n_i}}, \dots, x_d^{n_d})M_{n_1} : x_i] \\ &\cong \sum_{i=2}^{d} [(x_1^{n_1}, \dots, \widehat{x_i^{n_i}}, \dots, x_d^{n_d})M : x_i]/x_1^{n_1}M. \end{aligned}$$

On the other hand, the exact sequence in (2.2)

$$0 \to \ker \psi \to M_{n_1}/Q_{M_{n_1}}(\underline{x}'; \underline{n}') \overset{\psi}{\to} M/Q_M(\underline{x}, \underline{n}) \to 0$$

implies

$$J_{M,\underline{x}}(\underline{n}) = J_{M_{n_1}\underline{x}'}(\underline{n}') + \ell_A(\ker \psi).$$

But, by [9, 2.3, 2.4], we can show that

$$
\begin{aligned}
\ker \psi &\cong \sum_{i=1}^{d}[(x_1^{n_1},\dots,\widehat{x_i^{n_i}},\dots,x_d^{n_d})M : x_i]/\\
&\quad \sum_{i=2}^{d}[(x_1^{n_1},x_2^{n_2}\dots,\widehat{x_i^{n_i}},\dots,x_d^{n_d})M : x_i]\\
&\cong [(x_2^{n_2},\dots,x_d^{n_d})M : x_1]/[((x_2^{n_2},\dots,x_d^{n_d})M : x_1)\\
&\quad \cap \sum_{i=2}^{d}[(x_1^{n_1},x_2^{n_2},\dots,\widehat{x_i^{n_i}},\dots,x_d^{n_d})M : x_i]]\\
&\cong [(x_2^{n_2},\dots,x_d^{n_d})M : x_1]/\sum_{i=2}^{d}[(x_2^{n_2}\dots,\widehat{x_i^{n_i}},\dots,x_d^{n_d})M : x_i]\\
&= [(x_2^{n_2},\dots,x_d^{n_d})M : x_1]/[(x_2^{2n_2},\dots,x_d^{2n_d})M : x_2^{n_2}\cdots x_d^{n_d}].
\end{aligned}
$$

Thus, by [14, 3.3],

$$\ker \psi \cong H_{\mathfrak{m}}^{d-1}(M).$$

Therefore, for all $n_1,\dots,n_d > 0$,

$$J_{M,\underline{x}}(\underline{n}) = J_{M_{n_1},\underline{x}'}(\underline{n}') + \ell_A\big(H_{\mathfrak{m}}^{d-1}(M)\big).$$

Note that, for all $i = 0,\dots,d-1$, $\ell_A(H_{\mathfrak{m}}^i(M)) < \infty$ and $x_1 H_{\mathfrak{m}}^i(M) = 0$, since $\underline{x}$ is a standard system of parameters. Thus, by simple calculation, we can check that

$$\sum_{i=1}^{d-1}\binom{d-1}{i-1}\ell_A\left(H_{\mathfrak{m}}^i(M)\right) = \sum_{i=2}^{d-2}\binom{d-2}{i-1}\ell_A\big(H_{\mathfrak{m}}^i(M_{n_1})\big)+\ell_A\big(H_{\mathfrak{m}}^{d-1}(M)\big).$$

Finally, by the inductive hypothesis, we thus obtain

$$
\begin{aligned}
J_{M,\underline{x}}(\underline{n}) &= J_{M_{n_1},\underline{x}'}(\underline{n}') + \ell_A\big(H_{\mathfrak{m}}^{d-1}(M)\big)\\
&= \sum_{i=2}^{d-2}\binom{d-2}{i-1}\ell_A\big(H_{\mathfrak{m}}^i(M_{n_1})\big) + \ell_A\big(H_{\mathfrak{m}}^{d-1}(M)\big)\\
&= \sum_{i=1}^{d-1}\binom{d-1}{i-1}\ell_A\big(H_{\mathfrak{m}}^i(M)\big),
\end{aligned}
$$

for all $n_1,\dots,n_d > 0$ as required. ■

Since every system of parameters of a Buchsbaum module is standard, Theorem 5.1 leads immediately to the following consequence.

Corollary 5.2. *Suppose M is a Buchsbaum module. Then*

$$J_{M,\underline{x}}(\underline{n}) = \sum_{i=1}^{d-1}\binom{d-1}{i-1}\ell_A\left(H_{\mathfrak{m}}^i(M)\right)$$

for any system of parameters $\underline{x}$ and for all $n_1,\dots,n_d \geq 1$.

Corollary 5.3. *Suppose M is a Buchsbaum module. Then*

$$e(\underline{x}; M) > \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell_A(H^i_{\mathfrak{m}}(M))$$

for every s.o.p. $\underline{x}$.

Proof. By Corollaary 5.2, we have

$$e(\underline{x}; M) = \ell_A(M/Q_M(\underline{x})) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell_A(H^i_{\mathfrak{m}}(M))$$

for all s.o.p. $\underline{x}$. Since every s.o.p. of the Buchschaum module M is standard, then, by [9, 2.3],

$$\ell_A(M/Q_M(\underline{x})) > 0,$$

and the corollary is proved. ∎

Corollary 5.4. *Suppose M is a Buchsbaum module. Then*

$$\ell_A(M/\underline{x}M) > \sum_{i=0}^{d-1} \binom{d}{i} \ell_A(H^i_{\mathfrak{m}}(M))$$

for every s.o.p. $\underline{x}$.

Proof. Since M is a Buchsbaum module,

$$\ell_A(M/\underline{x}M) = e(\underline{x}; M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(H^i_{\mathfrak{m}}(M)).$$

Therefore, we have, by Corollary 5.3,

$$\begin{aligned}\ell_A(M/\underline{x}M) &> \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell_A(H^i_{\mathfrak{m}}(M)) + \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(H^i_{\mathfrak{m}}(M)) \\ &= \sum_{i=0}^{d-1} \binom{d}{i} \ell_A(H^i_{\mathfrak{m}}(M))\end{aligned}$$

as required. ∎

It is easy to see that the invariant $\eta_A(M)$ defined in [18] by

$$\eta_A(M) = \sup\left\{ \frac{\ell_A(M/\underline{x}M)}{e(\underline{x}; M)} \,\middle|\, \underline{x} \text{ is an s.o.p. of } M \right\}$$

is finite if M is a generalized Cohen–Macaulay module. We give an upper bound of $\eta_A(M)$ for a Buchsbaum module as follows:

Corollary 5.5. *Suppose M is a Buchsbaum module with* depth $M = k > 0$ *and* $k < \dim M = d$. *Then*

$$\eta_A(M) < 1 + d - k.$$

Proof. Put

$$I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A\left(H^i_{\mathfrak{m}}(M)\right)$$

and

$$J(M) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell_A\left(H^i_{\mathfrak{m}}(M)\right).$$

Since M is a Buchsbaum module, by Corollary 5.3, we have

$$\frac{\ell_A(M/\underline{x}M)}{e(\underline{x}; M)} = 1 + \frac{I(M)}{e(\underline{x}; M)} < 1 + \frac{I(M)}{J(M)}$$

for all s.o.p. $\underline{x}$. It follows from

$$I(M) < (d-k)J(M)$$

that

$$\eta_A(M) < 1 + d - k$$

as required. ■

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