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Quasi-Continuous Modules Relative to Module Classes*

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Abstract. We investigate modules M with the property that, for each submodule N_1 in some given class $\mathcal{X}$ of modules and submodule N_2 with $N_1 \cap N_2 = 0$, there exist submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$ and $N_i \subseteq M_i$ $(i = 1, 2)$.

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unital right modules. Let R be a ring and M an R-module. Let $E(M)$ denote the injective hull of any module M. The module M is called *quasi-continuous* if $\theta(M) \subseteq M$ for every idempotent endomorphism θ of $E(M)$. Quasi-continuous modules form an important class of modules which have been extensively studied in recent years (see, for example, [3–12]). In particular, in [3, 2.10] or [6, Theorem 2.8], we find the following result:

Proposition 1.1. *The following statements are equivalent for a module M:*

(i) *M is quasi-continuous;*
(ii) *for all submodules N_1, N_2 with $N_1 \cap N_2 = 0$ there exist submodules M_1, M_2 such that $M = M_1 \oplus M_2$ and $N_i \subseteq M_i$ $(i = 1, 2)$;*
(iii) (a) for any submodule N of M, there exists a direct summand K of M such that N is essential in K; and
(b) for all direct summands K, L of M with $K \cap L = 0$, the submodule $K \oplus L$ is also a direct summand of M.

In this paper we investigate modules M which satisfy condition (ii) in Proposition 1.1, where N_1 is chosen to belong to a given class of R-modules. One motivation is the following simple observation. Let R be a ring which is not right Noetherian. Then there exist an infinite index set I and injective R-modules M_i $(i \in I)$ such that the module $M = \oplus_{i \in I} M_i$ is not quasi-continuous (see [6, Proposition 2.10]). Let N_1, N_2

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be submodules of M such that N_1 is finitely generated and $N_1 \cap N_2 = 0$. There exists a finite subset J of I such that $N_1 \subseteq \oplus_{j \in J} M_j$. Since $\oplus_{j \in J} M_j$ is an injective R-module, it follows that there exists an injective submodule U of $\oplus_{j \in J} M_j$ such that N_1 is essential in U. Because $N_1 \cap N_2 = 0$, we have $U \cap N_2 = 0$, and it is rather easy to prove that there exists a submodule U' of M such that $M = U \oplus U'$ and $N_2 \subseteq U'$. Thus, M satisfies condition (ii) of Proposition 1.1 in case N_1 is finitely generated.

A second motivation is that, in [1], we studied modules M which satisfy condition (iii)(a) of Proposition 1.1 in case N belongs to a given class of R-modules. It turns out that, in this case, there are two generalizations of condition (iii)(a) for M, and it is interesting to see how the restricted version of condition (ii) behaves in relation to the restricted version of condition (iii)(a). The relationship between these concepts is established in Theorem 2.10 below.

We investigate modules, which are quasi-continuous relative to a class $\mathcal{X}$ of R-modules, in two different ways, corresponding to conditions (ii) and (iii) in Proposition 1.1, and give some of their general properties. We also consider what happens when the class of modules in question is a specific class, for example, the class of Goldie torsion modules.

Relative quasi-continuous modules have been considered by other authors. For example, Page [11] considers quasi-continuous modules relative to a torsion theory τ. Oshiro [9] also considers relative quasi-continuous modules but his approach differs from that of Page. In fact, although Oshiro's stand point is rather different from ours, his definition in terms of condition (iii) in Proposition 1.1 is essentially the same as ours.

2. Modules with Property $Q(\mathcal{X})$

Consider any ring R and R-module M. A submodule K of M is *closed* (in M) if K has no proper essential extension in M. By Zorn's Lemma, it can easily be shown that every submodule N of M is essential in a closed submodule K of M, and in this case, we call K a *closure* of N (in M). Moreover, for any submodule N of M, another easy Zorn's Lemma argument shows that the collection S of submodules L of M such that $N \cap L = 0$ contains a maximal member. Any maximal member of S is called a *complement* of N (in M). A submodule K of M is called a *complement* if there exists a submodule N of M such that K is a complement of N in M. It can easily be verified that a submodule K of M is closed if and only if K is a complement. The module M is called an *extending* module if every closed submodule is a direct summand of M, i.e., M satisfies condition (iii)(a) of Proposition 1.1.

Now, recall the following result:

Lemma 2.1. *See, e.g., [4, Lemma 5]. Given R-modules U, V, the module U is V-injective if and only if, for any submodule X of the R-module $W = U \oplus V$ with $X \cap U = 0$, there exists a submodule U' of W such that $W = U \oplus U'$ and $X \subseteq U'$.*

By a *class* $\mathcal{X}$ of R-modules, we mean a collection of R-modules which contains a zero module and is closed under isomorphisms. Any module in a class $\mathcal{X}$ will be called an *$\mathcal{X}$-module*. By an *$\mathcal{X}$-submodule* N of an R-module M, we mean a submodule N of M such that N is an *$\mathcal{X}$-module*.

Given a class $\mathcal{X}$ of R-modules, we say that an R-module M *satisfies property* $Q(\mathcal{X})$ ("Q" for quasi-continuous) if, for each $\mathcal{X}$-submodule N and submodule L of M with $N \cap L = 0$, there exist submodules N', L' such that $M = N' \oplus L'$, $N \subseteq N'$ and $L \subseteq L'$. Two extremes are given in the next result.

Proposition 2.2. *Let R be any ring and let $\mathcal{M}$ and $\mathcal{I}$ denote the classes of all R-modules and of all injective R-modules, respectively.*

(i) *An R-module M is quasi-continuous if and only if M satisfies $Q(\mathcal{M})$. In this case M satisfies $Q(\mathcal{X})$ for any class $\mathcal{X}$ of R-modules.*
(ii) *Every R-module satisfies $Q(\mathcal{I})$.*

Proof. (i) By [3, 2.10].

(ii) Let M be any R-module. Let N be an injective submodule and L a submodule of M such that $N \cap L = 0$. Then $M = N \oplus N'$ for some submodule N' of M. Because N is N'-injective, Lemma 2.1 applies to give a submodule L' of M such that $M = N \oplus L'$ and $L \subseteq L'$. Thus, M satisfies $Q(\mathcal{I})$. ■

Now, we make three elementary introductory observations. The first is the following.

Lemma 2.3. *Let $\mathcal{X}$ be any class of R-modules and M an R-module which satisfies $Q(\mathcal{X})$. Then any direct summand of M satisfies $Q(\mathcal{X})$.*

Proof. Suppose M_1 and M_2 are submodules of M such that $M = M_1 \oplus M_2$. Let N be an $\mathcal{X}$-submodule and L a submodule of M_1 such that $N \cap L = 0$. Consider the submodules N and $L \oplus M_2$ of M. By hypothesis, there exist submodules N' and L' of M such that $M = N' \oplus L'$, $N \subseteq N'$, and $L \oplus M_2 \subseteq L'$. Hence, $L' = L' \cap (M_1 \oplus M_2) = M_2 \oplus (L' \cap M_1)$, $M = N' \oplus L' = N' \oplus (L' \cap M_1) \oplus M_2$, and $M_1 = (L' \cap M_1) \oplus [(N' + M_2) \cap M_1]$. Note that $N \subseteq N' \cap M_1 \subseteq (N' + M_2) \cap M_1$ and $L \subseteq L' \cap M_1$. Thus, M_1 satisfies $Q(\mathcal{X})$. ■

Our second elementary observation is the following:

Lemma 2.4. *Let $\mathcal{X}$ be any class of R-modules, U an $\mathcal{X}$-module, and M any R-module such that the R-module $U \oplus M$ satisfies $Q(\mathcal{X})$. Then U is M-injective.*

Proof. Let L be any submodule of the module $X = U \oplus M$ such that $U \cap L = 0$. There exist submodules N' and L' of X such that $X = N' \oplus L'$, $U \subseteq N'$ and $L \subseteq L'$. Clearly, $N' = U \oplus (N' \cap M)$ and $X = U \oplus U'$, where $U' = (N' \cap M) \oplus L'$. Note that $L \subseteq U'$. By Lemma 2.1, U is M-injective. ■

Our third observation is as follows:

Lemma 2.5. *Let $\mathcal{X}$ be any class of R-modules and M an R-module which satisfies $Q(\mathcal{X})$. Let N be any $\mathcal{X}$-submodule of M and L any complement of N in M. Then $M = N' \oplus L$ for some closure N' of N in M.*

Proof. Since $N \cap L = 0$, it follows that $M = N' \oplus L'$ for some submodules N', L' such that $N \subseteq N'$ and $L \subseteq L'$. But $L' \cap N = 0$ gives $L = L'$. Moreover, $N \oplus L$, essential in M (see, for example, [3, 1.10]), gives $N = (N \oplus L) \cap N'$, essential in N'. Clearly, N' is closed in M, so that N' is a closure of N in M.

Given any class $\mathcal{X}$ of R-modules, we denote by $\mathcal{X}^e$ the class of R-modules with an essential $\mathcal{X}$-submodule. We shall say that the class $\mathcal{X}$ is *essentially closed* if $\mathcal{X} = \mathcal{X}^e$. For example, the class $\mathcal{I}$ of injective modules and the class $\mathcal{U}$ of modules with finite uniform dimension are both essentially closed. Note that $\mathcal{X}^e$ is an essentially closed class for any class $\mathcal{X}$. For an essentially closed class $\mathcal{X}$, we have the following immediate corollary of Lemma 2.5:

Corollary 2.6. *Let $\mathcal{X}$ be any essentially closed class of R-modules and M an R-module which satisfies $Q(\mathcal{X})$. Let N be an $\mathcal{X}$-submodule of M. Then $M = N' \oplus L'$ for any closure N' of N and complement L' of N in M.*

Recall that a module M is extending if every closed submodule (equivalently, complement) is a direct summand of M. Let $\mathcal{X}$ be a class of R-modules. In [1], a module M is defined to be *type* 1 $\mathcal{X}$*-extending* if, for every $\mathcal{X}$-submodule N of M, every complement of N in M is a direct summand of M. On the other hand, a module M is *type* 2 $\mathcal{X}$*-extending* if, for every $\mathcal{X}$-submodule N of M, every closure of N in M is a direct summand of M. Now, we define the module M to be $\mathcal{X}$*-quasi-continuous* if M satisfies the following two properties:

$(C1)_{\mathcal{X}}$ For any $\mathcal{X}$-submodule N of M, there exists a direct summand K of M such that N is essential in K, and

$(C3)_{\mathcal{X}}$ For any $\mathcal{X}$-submodule K which is a direct summand of M and direct summand L of M such that $K \cap L = 0$, the submodule $K \oplus L$ is also a direct summand of M.

The concept of $\mathcal{X}$-quasi-continuous modules is due to Oshiro [9] although his approach is somewhat different. In fact, if $\mathcal{X}$ is an essentially closed class of R-modules and $\mathcal{B}$ is the collection of $\mathcal{X}$-submodules of an $\mathcal{X}$-quasi-continuous R-module M, then M is $\mathcal{B}$-quasi-continuous in the sense of [9]. Conversely, if $\mathcal{B}$ is a collection of submodules of an R-module M such that M is $\mathcal{B}$-quasi-continuous in Oshiro's sense, then M is $\mathcal{X}$-quasi-continuous, where $\mathcal{X}$ is the essentially closed class of R-modules which are either zero or isomorphic to a member of $\mathcal{B}$.

Lemma 2.7. *Given a class $\mathcal{X}$ of R-modules, an R-module M satisfies* $(C3)_{\mathcal{X}}$ *if and only if, for all summands P, Q of M such that $P \in \mathcal{X}$ and $P \cap Q = 0$, there exists a submodule P' of M such that $M = P \oplus P'$ and $Q \subseteq P'$.*

Proof. Necessity. Let P and Q be direct summands of M such that $P \in \mathcal{X}$ with $Q \cap P = 0$. Then, by hypothesis, $Q \oplus P$ is a direct summand of M. Hence, $M = P \oplus Q \oplus Q''$ for some submodule Q'' of M. Thus, $P' = Q \oplus Q''$ has the requisite properties.

Conversely, let K and L be direct summands of M such that $K \in \mathcal{X}$ and $K \cap L = 0$. There exists a submodule K' of M such that $M = K \oplus K'$ and $L \subseteq K'$. But $M = L \oplus L'$ for some submodule L'. Hence, $K' = L \oplus (K' \cap L')$. Thus, $M = K \oplus L \oplus (K' \cap L')$. Then M satisfies $(C3)_{\mathcal{X}}$. ■

Lemma 2.8. *Let $\mathcal{X}$ be a class of R-modules and M an R-module which satisfies $Q(\mathcal{X})$. Then M is type* 1 *$\mathcal{X}$-extending and M satisfies* $(C1)_{\mathcal{X}}$.

Proof. Let K be any $\mathcal{X}$-submodule of M and L a complement of K in M. By hypothesis, there exist submodules M_1, M_2 of M with $M = M_1 \oplus M_2$, $K \subseteq M_1$, and $L \subseteq M_2$.

Since $K \cap M_2 = 0$, it follows that $L = M_2$. Thus, M is type 1 $\mathcal{X}$-extending. Moreover, $K \oplus L$ is an essential submodule of M, and

$$(K \oplus L) \cap M_1 = K + (L \cap M_1) \subseteq K + (M_2 \cap M_1) = K \subseteq M_1,$$

so that K is essential in M_1. Thus, M satisfies $(\mathrm{C1})_{\mathcal{X}}$. ∎

Corollary 2.9. *Let $\mathcal{X}$ be a class of R-modules and M an R-module which satisfies $Q(\mathcal{X})$. Then M is $\mathcal{X}$-quasi-continuous.*

Proof. By Lemmas 2.7 and 2.8. ∎

Theorem 2.10. *Let R be any ring and $\mathcal{X}$ any essentially closed class of R-modules. The following statements are equivalent for an R-module M.*

(i) *M satisfies $Q(\mathcal{X})$;*
(ii) *M is $\mathcal{X}$-quasi-continuous and type 1 $\mathcal{X}$-extending;*
(iii) *M is type 1 and type 2 $\mathcal{X}$-extending and M satisfies $(\mathrm{C3})_{\mathcal{X}}$.*

Proof. (i) $\Rightarrow$ (ii). By Lemma 2.8 and Corollary 2.9.

(ii) $\Rightarrow$ (iii). Let N be any $\mathcal{X}$-submodule of M and let K be any closure of N. By hypothesis, $K \in \mathcal{X}$, and because M satisfies $(\mathrm{C1})_{\mathcal{X}}$, K is a direct summand of M. It follows that M is type 2 $\mathcal{X}$-extending and (iii) follows immediately.

(iii) $\Rightarrow$ (i). Let A, B be submodules of M such that $A \in \mathcal{X}$ and $A \cap B = 0$. Let K be a complement of A in M with $B \subseteq K$. Since M is type 1 $\mathcal{X}$-extending, it follows that K is a direct summand of M. Because M is type 2 $\mathcal{X}$-extending, there exists a direct summand L of M such that A is essential in L. Because $\mathcal{X}$ is essentially closed, $L \in \mathcal{X}$. Also, we have $L \cap K = 0$. Since M has $(\mathrm{C3})_{\mathcal{X}}$, $L \oplus K$ is a direct summand of M. Then $M = L \oplus K \oplus P$ for some submodule P and $A \subseteq L$, $B \subseteq K \oplus P$. Thus, M has $Q(\mathcal{X})$. ∎

3. Classes of Modules

Let R be any ring. The basic question we wish to consider in this section is, if $\mathcal{X}$ and $\mathcal{Y}$ are classes of R-modules which are related in some way and M is an R-module which satisfies $Q(\mathcal{X})$, does M also satisfy $Q(\mathcal{Y})$? The first result is clear.

Lemma 3.1. *Let $\mathcal{X} \subseteq \mathcal{Y}$ be classes of R-modules. Then every R-module which satisfies $Q(\mathcal{Y})$ also satisfies $Q(\mathcal{X})$.*

Lemma 3.2. *Let $\mathcal{X}$ be any class of R-modules. Then a non-singular R-module M satisfies $Q(\mathcal{X})$ if and only if M satisfies $Q(\mathcal{X}^e)$.*

Proof. Because $\mathcal{X} \subseteq \mathcal{X}^e$, the sufficiency follows by Lemma 3.1. Conversely, suppose M satisfies $Q(\mathcal{X})$, let N be an $\mathcal{X}^e$-submodule of M and L a submodule of M with $N \cap L = 0$. There exists an $\mathcal{X}$-submodule K of M such that K is essential in N. Clearly, $K \cap L = 0$ and hence, $M = M_1 \oplus M_2$ for some submodules M_1, M_2 such that $K \subseteq M_1$

and $L \subseteq M_2$. Since N/K is singular, it follows that $N/(N \cap M_1) \cong (N + M_1)/M_1$ is singular. However, $M/M_1 \cong M_2$ which is non-singular. Thus, $N = N \cap M_1 \subseteq M_1$. Hence, M satisfies $Q(\mathcal{X}^e)$. ■

Given a positive integer n and classes $\mathcal{X}_i$ $(1 \le i \le n)$ of R-modules, $\mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n$ denote the class of R-modules of the form $M_1 \oplus \cdots \oplus M_n$, where M_i is an $\mathcal{X}_i$-module for all $1 \le i \le n$.

Proposition 3.3. *Let n be a positive integer and let $\mathcal{X}_i$ $(1 \le i \le n)$ be classes of R-modules. Then an R-module M satisfies $Q(\mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n)$ if and only if M satisfies $Q(\mathcal{X}_i)$ for all $1 \le i \le n$.*

Proof. The necessity follows by Lemma 3.1. ■

Conversely, suppose M satisfies $Q(\mathcal{X}_i)$ for all $1 \le i \le n$. Let N be any $(\mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n)$-submodule of M and L a submodule of M such that $N \cap L = 0$. Then $N = N_1 \oplus \cdots \oplus N_n$ for some $\mathcal{X}_i$-submodule N_i $(1 \le i \le n)$ of M. Now, $N_1 \cap (N_2 \oplus \cdots \oplus N_n \oplus L) = 0$ so that $M = M_1 \oplus M_2$ for some submodules M_1, M_2 such that $N_1 \subseteq M_1$ and $N_2 \oplus \cdots \oplus N_n \oplus L \subseteq M_2$. By Lemma 2.3, M_2 satisfies $Q(\mathcal{X}_i)$ for all $2 \le i \le n$. By induction on n, there exist submodules M_3, M_4 of M_2 such that $M_2 = M_3 \oplus M_4$, $N_2 \oplus \cdots \oplus N_n \subseteq M_3$ and $L \subseteq M_4$. Hence, $M = (M_1 \oplus M_3) \oplus M_4$, $N_1 \oplus \cdots \oplus N_n \subseteq M_1 \oplus M_3$ and $L \subseteq M_4$. It follows that M satisfies $Q(\mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n)$. ■

For any class $\mathcal{X}$ of R-modules, let $\mathcal{X}^{\oplus}$ denote the class of all R-modules which are finite direct sums of $\mathcal{X}$-modules. Proposition 3.3 has the following immediate corollary:

Corollary 3.4. *Let $\mathcal{X}$ be any class of R-modules. Then an R-module M satisfies $Q(\mathcal{X})$ if and only if M satisfies $Q(\mathcal{X}^{\oplus})$.*

Before we proceed, we mention a basic fact about closed submodules. Its proof can be found in [3, 1.10].

Lemma 3.5. *Let K be a closed submodule of a module M and L a closed submodule of K. Then L is a closed submodule of M.*

The next result is an analog of Proposition 3.3. Note that, if a module M satisfies $Q(\mathcal{X}_i^e)$ $(1 \le i \le n)$, then M satisfies $Q(\mathcal{X}_1^e \oplus \cdots \oplus \mathcal{X}_n^e)$ by Proposition 3.3. In fact we can say more.

Theorem 3.6. *Let n be a positive integer, let $\mathcal{X}_i$ $(1 \le i \le n)$ be classes of R-modules, and let $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n$. Then an R-module M satisfies $Q(\mathcal{X}^e)$ if and only if M satisfies $Q(\mathcal{X}_i^e)$ for all $1 \le i \le n$.*

Proof. Since $\mathcal{X}_i \subseteq \mathcal{X}$ and hence $\mathcal{X}_i^e \subseteq \mathcal{X}^e$ for all $1 \le i \le n$, the necessity follows by Lemma 3.1.

Conversely, suppose M satisfies $Q(\mathcal{X}_i^e)$ for all $1 \le i \le n$. Let N be an $\mathcal{X}^e$-submodule and L a submodule of M such that $N \cap L = 0$. There exists a closed submodule N' of M such that N is essential in N'. Note that $N' \cap L = 0$. There exist $\mathcal{X}_i$-submodules N_i $(1 \le i \le n)$ of N such that $N_1 \oplus \cdots \oplus N_n$ is essential in N. There exists a closed submodule N_1' of N' such that N_1 is essential in N_1'. By Lemma 3.5, N_1' is closed in M. By Zorn's Lemma, there exists a complement L' of N_1 (or N_1') in M such that $N_2 \oplus \cdots \oplus N_n \oplus L \subseteq L'$. By Corollary 2.6, $M = N_1' \oplus L'$, because M satisfies $Q(\mathcal{X}_1^e)$.

Now, $N' = N_1' \oplus (N' \cap L')$ and $(N_1 \oplus \cdots \oplus N_n) \cap (N' \cap L') = N_2 \oplus \cdots \oplus N_n$ is essential in $N' \cap L'$. Thus, $N' \cap L' \in \mathcal{Y}^e$, where $\mathcal{Y} = \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_n$. But L' satisfies $Q(\mathcal{X}_i^e)$ for all $2 \leq i \leq n$, by Lemma 2.3. By induction on n, L' satisfies $Q(\mathcal{Y}^e)$. There exist submodules P, Q of L' such that $L' = P \oplus Q$, $N' \cap L' \subseteq P$, and $L \subseteq Q$. Finally, note that $M = N_1' \oplus P \oplus Q$, $N \subseteq N' \subseteq N_1' \oplus P$, and $L \subseteq Q$. It follows that M satisfies $Q(\mathcal{X}^e)$. ■

We now apply some of the results in this section to specific classes of modules. Let R be any ring. Let $\mathcal{U}$ denote the class of R-modules with finite uniform dimension, $\mathcal{U}_1$ the class of R-modules which are uniform or zero, $\mathcal{G}$ the class of finitely generated R-modules, and $\mathcal{G}_1$ the class of cyclic R-modules.

First, note that the classes $\mathcal{U}$ and $\mathcal{U}_1$ are both essentially closed and any $\mathcal{U}$-module is an essential extension of a finite direct sum of $\mathcal{U}_1$-modules. Thus, Theorem 3.6 immediately gives:

Corollary 3.7. *An R-module M satisfies $Q(\mathcal{U})$ if and only if M satisfies $Q(\mathcal{U}_1)$.*

In view of Corollary 3.7 it is natural to ask whether any module with $Q(\mathcal{G}_1)$ also satisfies $Q(\mathcal{G})$. We do not know the answer to this question.

Note that $\mathcal{U} \subseteq \mathcal{G}^e$. By Lemmas 3.1 and 3.2, any non-singular R-module, which satisfies $Q(\mathcal{G})$, also satisfies $Q(\mathcal{U})$. The converse is false in general. Let R be a domain which is not right Öre. Then the right R-module R is non-singular and satisfies $Q(\mathcal{U})$ vacuously since it has no uniform submodules, but R_R does not satisfy $Q(\mathcal{G})$. For commutative domains, the conditions $Q(\mathcal{U})$ and $Q(\mathcal{G})$ are equivalent for torsion-free modules, as the next result shows.

Proposition 3.8. *Let R be a commutative domain. Then the following statements are equivalent for a torsion-free R-module M.*

(i) *M satisfies $Q(\mathcal{G})$;*

(ii) *M satisfies $Q(\mathcal{U})$;*

(iii) *M is quasi-continuous.*

Proof. (i) ⇒ (ii). By Lemmas 3.1 and 3.2.

(ii) ⇒ (iii). By Lemma 2.8, M is type 1 $\mathcal{U}$-extending. Now, M is extending by Theorem 4.8 in [1]. It follows that $M = M_1 \oplus M_2$ for some injective submodule M_1 and $\mathcal{U}$-submodule M_2 by Theorem 5 in [5]. By Lemma 2.3, M_2 satisfies $Q(\mathcal{U})$ and hence, M_2 is quasi-continuous. Next, $M_1 = \oplus_{i \in I} M_{1i}$, where M_{1i} is indecomposable injective for all $i \in I$. By Lemma 2.3, for each $i \in I$, $M_{1i} \oplus M_2$ satisfies $Q(\mathcal{U})$ and hence, $M_{1i} \oplus M_2$ is quasi-continuous and M_2 is M_{1i}-injective (see, for example, [6, Proposition 2.10]). Thus, M_2 is M_1-injective by Proposition 1.5 in [6] and M is quasi-continuous by Corollary 2.14 in [6].

(iii) ⇒ (i). By Proposition 2.2. ■

Let R be a right Noetherian ring. Then $\mathcal{G} \subseteq \mathcal{U}$. Hence, any right R-module with $Q(\mathcal{U})$ also satisfies $Q(\mathcal{G})$. In fact, any R-module with $Q(\mathcal{U})$ is quasi-continuous as we show next.

Proposition 3.9. *Let R be a right Noetherian ring. Then a right R-module M is quasi-continuous if and only if M satisfies $Q(\mathcal{U})$.*

Proof. The necessity is clear by Proposition 2.2. Conversely, suppose M satisfies $Q(\mathcal{U})$. By Lemma 2.8, M is type 1 $\mathcal{U}$-extending. Now, $M = \oplus_{i\in I} M_i$ for some uniform submodules M_i $(i \in I)$ of M by Lemma 4.5 in [1]. Fix $i \in I$. For each $j \in I\backslash\{i\}$, the $\mathcal{U}$-module $M_i \oplus M_j$ satisfies $Q(\mathcal{U})$ (Lemma 2.3) and hence is quasi-continuous. Thus, M_i is M_j-injective for each $j \in I\backslash\{i\}$ by Proposition 2.10 in [6], and M_i is $(\oplus_{j\in I\backslash\{i\}} M_j)$-injective by Proposition 1.5 in [6]. By Theorem 2.13 in [6], M is quasi-continuous. ■

We do not know, for a right Noetherian ring R, whether all right R-modules with $Q(\mathcal{G})$ are quasi-continuous. Next, we confirm a remark made in the introduction.

Proposition 3.10 *Let M_i $(i \in I)$ be injective R-modules. Then the R-module $M = \oplus_{i\in I} M_i$ satisfies $Q(\mathcal{G})$.*

Proof. Let N be any finitely generated submodule of M and let L be any submodule of M such that $N \cap L = 0$. There exists a finite subset J of I such that $N \subseteq \oplus_{j\in J} M_j$. Because $\oplus_{j\in J} M_j$ is injective, there exists an injective submodule M' of M such that N is essential in M'. Now, $M = M' \oplus M''$ for some submodule M'' of M and $M' \cap L = 0$. Because M' is M''-injective, there exists a submodule L' of M such that $M = M' \oplus L'$ and $L \subseteq L'$ (Lemma 2.1). It follows that M satisfies $Q(\mathcal{G})$. ■

If M_i $(i \in I)$ are injective R-modules for some ring R, then the R-module $\oplus_{i\in I} M_i$ does not satisfy $Q(\mathcal{U})$ in general. To demonstrate this fact, we first prove the following result.

Proposition 3.11. *Let R be any ring and let M_i $(i \in I)$ be any collection of indecomposable injective R-modules. Then $M = \oplus_{i\in I} M_i$ satisfies $Q(\mathcal{U})$ if and only if M is quasi-injective.*

Proof. Suppose M satisfies $Q(\mathcal{U})$. Since U is essentially closed, it follows that M is type 2 $\mathcal{U}$-extending by Theorem 2.10. Thus, M is quasi-injective by Corollary 3.6 in [2]. The converse is clear. ■

Example. Let R be a ring which has finite right uniform dimension but is not right Noetherian. Then there exist injective right R-modules M_n $(n \in \mathbf{N})$ such that $\oplus_{n\in\mathbf{N}} M_n$ does not satisfy $Q(\mathcal{U})$.

Proof. Because R is not right Noetherian, by Theorem 4.1 in [13], there exist simple R-modules S_n $(n \in \mathbf{N})$ such that $\oplus_{n\in\mathbf{N}} E(S_n)$ is not injective. Since R_R has finite uniform dimension, $E(R_R) = E_1 \oplus \cdots \oplus E_k$ for some positive integer k and indecomposable injective R-modules E_i $(1 \leq i \leq k)$. Let $E = E_1 \oplus \cdots \oplus E_k \oplus (\oplus_{n\in\mathbf{N}} E(S_n))$. Then E is not quasi-injective because $\oplus_{n\in\mathbf{N}} E(S_n)$ is not $(E_1 \oplus \cdots \oplus E_k)$-injective. By Proposition 3.11, E does not satisfy $Q(\mathcal{U})$. ■

4. Direct Sums

Let R be a ring and M_i $(1 \leq i \leq n)$ a finite collection of R-modules. We say that the modules M_i $(1 \leq i \leq n)$ are *relatively injective* if M_i is M_j-injective for all $1 \leq i \neq j \leq n$. It is well known that the module $M = M_1 \oplus \cdots \oplus M_n$ is quasi-continuous if and only if the modules M_i $(1 \leq i \leq n)$ are quasi-continuous and relatively injective (see, for example, [6, Corollary 2.14]). We now generalize this fact by proving:

Theorem 4.1. *Let $\mathcal{X}$ be an essentially closed class of R-modules such that $\mathcal{X}$ is closed under submodules. Let M_i $(1 \leq i \leq n)$ be a finite collection of relatively injective R-modules. Then the R-module $M = M_1 \oplus \cdots \oplus M_n$ satisfies $Q(\mathcal{X})$ if and only if M_i satisfies $Q(\mathcal{X})$ for all $1 \leq i \leq n$.*

Proof. Necessity. Follows by Lemma 2.3.

Conversely, suppose M_i satisfies $Q(\mathcal{X})$ for all $1 \leq i \leq n$. By induction on n, to prove that M satisfies $Q(\mathcal{X})$, we can suppose without loss of generality that $n = 2$. Let N be an $\mathcal{X}$-submodule and L a submodule of $M = M_1 \oplus M_2$ such that $N \cap L = 0$. Let N' be a closure of N in M. Because N is essential in N', we have $N' \in \mathcal{X}$ and $N' \cap L = 0$. Thus, without loss of generality, we can suppose $N = N'$, i.e., N is closed in M.

Suppose next that $N \cap M_1 = 0$. Because M_1 is M_2-injective, Lemma 2.1 allows us to assume without loss of generality that $N \subseteq M_2$. Then Corollary 2.6 gives $M_2 = N \oplus H$ for any complement H of N in M_2. By Lemma 2.4, N is H-injective. But M_2 being M_1-injective implies N is M_1-injective and hence, N is $(H \oplus M_1)$-injective (see, for example, [6, Proposition 1.5]). But $M = N \oplus (H \oplus M_1)$ and $N \cap L = 0$ so that, applying Lemma 2.1 again, there exists a direct summand M' of M such that $M = N \oplus M'$ and $L \subseteq M'$.

In general, $N \cap M_2$ is an $\mathcal{X}$-submodule of M, because $\mathcal{X}$ is closed under submodules, and there exists a closed submodule K of N such that $N \cap M_2$ is essential in K. By Lemma 3.5, K is a closed submodule of M. Moreover, K is an $\mathcal{X}$-submodule of M, $K \cap M_1 = 0$, and $K \cap L = 0$. By the above argument, $M = K \oplus K'$ for some submodule K' such that $L \subseteq K'$. Note that $N = K \oplus (N \cap K')$, so that $N \cap K'$ is a closed submodule of M by Lemma 3.5. Moreover, $(N \cap K') \cap M_2 \subseteq K \cap K' = 0$. By the above argument, $M = (N \cap K') \oplus K''$ for some submodule K'' such that $L \subseteq K''$. Hence,

$$M = K \oplus K' = K \oplus (N \cap K') \oplus (K' \cap K'') = N \oplus (K' \cap K''),$$

and $L \subseteq K' \cap K''$. It follows that M satisfies $Q(\mathcal{X})$. ∎

For any ring R, the class $\mathcal{U}$ of R-modules with finite uniform dimension is essentially closed and is also closed under submodules. Thus, Theorem 4.1 has the following immediate corollary:

Corollary 4.2. *Let M_i $(1 \leq i \leq n)$ be a finite collection of relatively injective R-modules. Then the R-module $M = M_1 \oplus \cdots \oplus M_n$ satisfies $Q(\mathcal{U})$ if and only if M_i satisfies $Q(\mathcal{U})$ for all $1 \leq i \leq n$.*

Examples of classes of modules which are both essentially closed and closed under submodules include the class $\mathcal{T}$ of Goldie torsion modules and the class $\mathcal{F}$ of Goldie torsion-free (i.e., non-singular) modules. As an application of Theorem 4.1, we next characterize modules which satisfy $Q(\mathcal{T})$. For any module M, $Z_2(M)$ will denote the Goldie torsion submodule of M, i.e., $Z_2(M)/Z(M) = Z(M/Z(M))$, where $Z(N)$ denotes the singular submodule of any module N.

Theorem 4.3. *Let $\mathcal{T}$ denote the class of Goldie torsion R-modules. Then R-module M satisfies $Q(\mathcal{T})$ if and only if $M = Z_2(M) \oplus M'$ for some submodule M' of M such that $Z_2(M)$ is quasi-continuous and M'-injective.*

Proof. First, suppose M satisfies $Q(\mathcal{T})$. Because $\mathcal{T}$ is essentially closed, $Z_2(M)$ is a closed $\mathcal{T}$-submodule of M and hence, $M = Z_2(M) \oplus M'$ for some submodule M' of M by Lemma 3.5. By Lemma 2.4, $Z_2(M)$ is M'-injective and by Lemma 2.3 and Theorem 2.10, $Z_2(M)$ is quasi-continuous.

Conversely, suppose $M = Z_2(M) \oplus M'$, $Z_2(M)$ is quasi-continuous and $Z_2(M)$ is M'-injective. Clearly, $\mathrm{Hom}(Z_2(M), M') = 0$ and hence, M' is $Z_2(M)$-injective. Clearly, M' also satisfies $Q(\mathcal{T})$. Moreover, by Proposition 2.2, $Z_2(M)$ satisfies $Q(\mathcal{T})$. Finally, Theorem 4.1 gives that M satisfies $Q(\mathcal{T})$. ∎

Using Theorem 4.3, we can show that, for the class $\mathcal{T}$, not every $\mathcal{T}$-quasi-continuous module satisfies $Q(\mathcal{T})$. For example, let S be a simple **Z**-module and let M denote the **Z**-module $S \oplus \mathbf{Z}$. Because S is not **Z**-injective, Theorem 4.3 shows that M does not satisfy $Q(\mathcal{T})$. Since the only $\mathcal{T}$-submodules of M are 0 and S, it is easy to check that M satisfies $(C1)_{\mathcal{T}}$ and $(C3)_{\mathcal{T}}$, i.e., M is $\mathcal{T}$-quasi-continuous.

Theorem 4.1 for the class $\mathcal{T}$ is as follows:

Theorem 4.4. *Let M_i $(1 \le i \le n)$ be a finite collection of R-modules and let $M = M_1 \oplus \cdots \oplus M_n$. Then M satisfies $Q(\mathcal{T})$ if and only if M_i satisfies $Q(\mathcal{T})$ for all $1 \le i \le n$ and $Z_2(M_i)$ is M_j-injective for all $1 \le i \ne j \le n$.*

Proof. First, suppose M satisfies $Q(\mathcal{T})$. By Lemma 2.3, M_i satisfies $Q(\mathcal{T})$ and hence, by Theorem 4.3, $M_i = Z_2(M_i) \oplus M_i'$ for some submodule M_i', for all $1 \le i \le n$. Let $1 \le i \ne j \le n$. Then $M_i \oplus M_j = Z_2(M_i) \oplus M_i' \oplus M_j$ satisfies $Q(\mathcal{T})$ and hence, $Z_2(M_i) \oplus M_j$ satisfies $Q(\mathcal{T})$ by Lemma 2.3. By Lemma 2.4, $Z_2(M_i)$ is M_j-injective.

Conversely, suppose M_i satisfies $Q(\mathcal{T})$ for all $1 \le i \le n$ and that $Z_2(M_i)$ is M_j-injective for all $1 \le i \ne j \le n$. To prove that M satisfies $Q(\mathcal{T})$, we can suppose without loss of generality that $n = 2$. By Theorem 4.3, for $i = 1, 2$, M_i contains a submodule M_i' such that $M_i = Z_2(M_i) \oplus M_i'$. Then

$$M = M_1 \oplus M_2 = Z_2(M_1) \oplus Z_2(M_2) \oplus M_1' \oplus M_2' = Z_2(M) \oplus M',$$

where $M' = M_1' \oplus M_2'$. By hypothesis, the modules $Z_2(M_1)$ and $Z_2(M_2)$ are relatively injective and satisfy $Q(\mathcal{T})$. Hence, $Z_2(M)$ satisfies $Q(\mathcal{T})$, i.e., $Z_2(M)$ is quasi-continuous (see Proposition 1.1). Moreover, Theorem 4.3 gives that $Z_2(M_1)$ is M_1'-injective and hence, $Z_2(M_1)$ is M'-injective. Similarly, $Z_2(M_2)$ is M'-injective. Thus, $Z_2(M)$ is M'-injective. By Theorem 4.3, M satisfies $Q(\mathcal{T})$. ∎

There is an analog to Theorems 4.3 and 4.4 for the class $\mathcal{F}$ of non-singular R-modules.

Theorem 4.5. *Let $\mathcal{F}$ denote the class of non-singular R-modules. Then an R-module M satisfies $Q(\mathcal{F})$ if and only if $M = Z_2(M) \oplus M'$ for some quasi-continuous submodule M' of M such that $Z_2(M)$ is M'-injective.*

Proof. Suppose M satisfies $Q(\mathcal{F})$. Let M' be a complement of $Z_2(M)$ in M. Then M' is an $\mathcal{F}$-submodule of M and $Z_2(M)$ is a complement of M'. By Corollary 2.6, $M = Z_2(M) \oplus M'$. The rest of the proof is straightforward being analogous to the proof of Theorem 4.3. ■

Corollary 4.6. *Let M_i $(1 \leq i \leq n)$ be a finite collection of R-modules and let $M = M_1 \oplus \cdots \oplus M_n$. Then M satisfies $Q(\mathcal{F})$ if and only if $M_i = Z_2(M_i) \oplus M_i'$ for some quasi-continuous submodule M_i' such that $Z_2(M_i)$ is M_i'-injective for all $1 \leq i \leq n$ and M_i' is M_j-injective for all $1 \leq i \neq j \leq n$.*

Proof. Similar to the proof of Theorem 4.4. ■

Note in particular that Theorems 4.3 and 4.5 together give that a module M is quasi-continuous if and only if M satisfies $Q(\mathcal{T})$ and $Q(\mathcal{F})$ (see [6, Corollary 2.14]).

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