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On Capacity of Labeled Petri Net Languages

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Abstract. In this paper we consider a complexity characteristic of languages and some properties of classes of languages characterized by it. Using this characteristic, a necessary condition for labeled Petri net languages is established. It is a sharpening and a generalizing of necessary conditions for free-labeled Petri net languages in [1].

1. Introduction

The Petri net has been developed as a mathematical model of parallel and distributed processing systems. In the past few years, the theory of Petri nets, in particular, the languages acceptable by them (Petri net languages) were investigated extensively by many authors (see, for example [1–3, 8, 10–12]).

The Petri net languages have become popular in describing the sequential behavior of the above systems. Up to now, although we have had some examples of non-Petri net languages, we do not have a criterion for recognizing whether a given language is Petri net language or not. In [1] a necessary condition for the class of free-labeled Petri net languages (FP-languages) was given. In this paper we are concerned with the class of labeled Petri net languages (LP-languages). Thus, the present paper could be seen as a continuation of work [1].

The notions and definitions of labeled Petri net (LP-net) and of LP-languages are recalled in Sec. 2. In Sec. 3, a complexity characteristic of languages and its properties are considered. Implementing it, a necessary condition for the class of LP-languages is given in Sec. 4. In Sec. 5, some properties of classes of languages characterized by it are examined. Finally, we close the paper with some open problems.

2. Notations and Definitions

2.1. For a finite alphabet Σ, we denote Σ^* (resp. Σ^r, $\Sigma^{\leq r}$) the set of all words (resp. the set of all words of length r, of length at most r) on the alphabet Σ, and Λ denotes the empty word. For any word $\omega \in \Sigma^*$, $l(\omega)$ denotes the length of ω. Every subset $L \subseteq \Sigma^*$ is called a language over the alphabet Σ. Let N be the set of all non-negative integers and $N^+ = N\backslash\{0\}$.

Definition 1. *A labeled Petri net $\mathcal{N}$ is given by a list:*

$$\mathcal{N} = (P, T, I, O, \sigma, \mu_0, M_f),$$

where

$P = \{p_1, ..., p_n\}$ is a finite set of places;
$T = \{t_1, ..., t_m\}$ is a finite set of transitions, $P \cap T = \emptyset$;
$I : P \times T \to N$ is an input function;
$O : T \times P \to N$ is an output function;
$\sigma : T \to \Sigma$ is a labeled function, where Σ is a finite *output alphabet*;
$\mu_0 : P \to N$ is an initial marking;
$M_f = \{\mu_{f_1}, ..., \mu_{f_k}\}$ is a finite set of final markings.

We can extend the labeled function σ for a sequence as follows:

$$\text{if } t = t_1 t_2 \dots t_n, \quad \text{then } \sigma(t) = \sigma(t_1)\sigma(t_2) \cdots \sigma(t_n).$$

Definition 2. *A marking μ (global configuration) of a Petri net $\mathcal{N}$ is a function from the set of places P to N:*

$$\mu : P \to N.$$

The marking μ can also be defined as an n-vector $\mu = (\mu_1, ..., \mu_n)$ with $\mu_i = \mu(p_i)$ and $|P| = n$.

Definition 3. *A transition $t \in T$ is said to be firable at the marking μ if and only if*

$$\forall p \in P : \mu(p) \geq I(p, t).$$

Let t be firable at μ, and if t fires, then the Petri net $\mathcal{N}$ shall change its state from marking μ to a new marking μ' which is defined as follows:

$$\forall p \in P : \mu'(p) = \mu(p) - I(p, t) + O(t, p).$$

We set $\delta(\mu, t) = \mu'$ and the function δ is said to be *function of changing state of the net.*

A *firing sequence* can be defined as a sequence of transitions such that the firing of each of its prefix will be led to a marking at which the next transition will be firable. By $\mathcal{F}_{\mathcal{N}}$, we denote the set of all firing sequences of the net $\mathcal{N}$.

We now extend the function δ for a firing sequence by induction as follows:

Let $t \in T^*$, $t_j \in T$, μ be a marking at which tt_j is a firing sequence. Then

$$\begin{cases} \delta(\mu, \Lambda) &= \mu, \\ \delta(\mu, tt_j) &= \delta(\delta(\mu, t), t_j). \end{cases}$$

Definition 4. *The language acceptable by labeled Petri net* $\mathcal{N}$ is the set:

$$L(\mathcal{N}) = \{x \in \Sigma^* | \exists t \in T^* : (x = \sigma(t)) \wedge (t \in \mathcal{F}_{\mathcal{N}}) \wedge (\delta(\mu_0, t) \in M_f)\}.$$

The set of all labeled Petri net languages is denoted by $\mathcal{L}$.

2.2. The class of LP-languages can be classified by various restrictions made on the labeled function σ. For example, if σ is an isomorphism, then it may be omitted completely by choosing $\Sigma = T$ and $\mathcal{N}$ is said to be *a free-labeled Petri net* (FP-net). The language acceptable by a FP-net is called an *FP-language.*

The set of all FP-languages is denoted by $\mathcal{L}^f$. The class $\mathcal{L}^f$ has been examined in [1,12].

It is obvious that $\mathcal{L}^f \subseteq \mathcal{L}$. We now show that $\mathcal{L}^f \subset \mathcal{L}$.

Let $\Sigma = \{a\}$, and $L_k = \{a^k\}$, $\quad k = const., k \in N$.

We consider $L = \Sigma^* \backslash L_k$. It is clear that L is regular. It is well known that every regular language is an LP-language (see [9]), so $L \in \mathcal{L}$. But in [12], Starke has proved that over one-letter alphabet, $L \in \mathcal{L}^f$ if and only if L is finite or $L = \Sigma^*$, therefore, $L \notin \mathcal{L}^f$. We obtain $\mathcal{L}^f \subset \mathcal{L}$. ■

3. A Complexity Characteristic of Languages

3.1. Let $L \subseteq \Sigma^*$. We define two equivalence relations $E_{\leq r}(\mathrm{mod}\, L)$ in $\Sigma^{\leq r}$ (and $E_r(\mathrm{mod}\, L)$ in Σ^r) as follows:

$\forall x_1, x_2 \in \Sigma^{\leq r}$ (and $\forall x_1, x_2 \in \Sigma^r$):

$$x_1 E_{\leq r} x_2(\mathrm{mod}\, L) \Leftrightarrow \forall \omega \in \Sigma^* : x_1\omega \in L \leftrightarrow x_2\omega \in L.$$
$$(x_1 E_r x_2(\mathrm{mod}\, L) \Leftrightarrow \forall \omega \in \Sigma^* : x_1\omega \in L \leftrightarrow x_2\omega \in L).$$

It is easy to show that the relations $E_{\leq r}$ (mod L), E_r (mod L) are reflexive, symmetric, and transitive. Therefore, they are equivalence relations.

We define

$$G_L(r) = \mathrm{Rank}\, E_{\leq r}\ (\mathrm{mod}\, L),$$
$$H_L(r) = \mathrm{Rank}\, E_r\ (\mathrm{mod}\, L).$$

Remark that the functions $G_L(r)$, $H_L(r)$, in general, are not given by algorithms. Nevertheless, we here call them to be functions. They are considered to be complexity characteristics of the language L over $\Sigma^{\leq r}$ and over Σ^r. In the sequel, we shall use them for formulating a necessary condition for LP-languages.

First, we give some of their simple properties:

(1) $E_{\leq r}$ (mod L), E_r (mod L) are right-invariant equivalence relations.

(2) $\forall r \in N$:

$$1 \leq H_L(r) \leq G_L(r) \leq \mathrm{Exp}(r),$$

where $\mathrm{Exp}(r)$ denotes some exponential function of r.

(3) $G_L(r)$ is a non-decreasing function.

(4)

$$G_{\overline{L}}(r) = G_L(r).$$

$$H_{\overline{L}}(r) = H_L(r).$$

(5)

$$G_{L_1 \cup L_2}(r) \leq G_{L_1}(r) \cdot G_{L_2}(r).$$

$$G_{L_1 \cap L_2}(r) \leq G_{L_1}(r) \cdot G_{L_2}(r).$$

$$H_{L_1 \cup L_2}(r) \leq H_{L_1}(r) \cdot H_{L_2}(r).$$

$$H_{L_1 \cap L_2}(r) \leq H_{L_1}(r) \cdot H_{L_2}(r).$$

3.2. Now, we estimate the functions $G_L(r)$, $H_L(r)$ for some languages L:

Example 1. Let $L_1 = \Sigma^*$. We have

$$(\forall r \in N^+)(\forall x_1, x_2 \in \Sigma^{\leq r})(\forall \omega \in \Sigma^*) : x_1\omega \in L_1 \quad \text{and} \quad x_2\omega \in L_1.$$

It follows that $x_1 E_{\leq r} x_2 \pmod{L_1}$. Therefore, $G_{L_1}(r) = H_{L_1}(r) = 1$.

Denote $\overline{L_1} = \Sigma^* \backslash L_1 = \emptyset$. We obtain $G_{\overline{L_1}}(r) = H_{\overline{L_1}}(r) = 1$.

Example 2. Let $\Sigma = \{a, b\}$ and

$$L_2 = \left\{a^m b^n \mid m, n \in N^+\right\}.$$

Denote:

$$W_1 = \left\{a^m \mid 1 \leq m \leq r\right\};$$

$$W_2 = \left\{a^m b^k \mid m + k \leq r; \quad k \geq 1\right\};$$

$$W_3 = \left\{\omega \in \Sigma^{\leq r} \mid \omega \notin W_1 \cup W_2\right\}.$$

We have $\Sigma^{\leq r} = W_1 \cup W_2 \cup W_3$ and $W_1 \cap W_2 \cap W_3 = \emptyset$.

It is easy to prove that all words in every W_i, $i = 1, 2, 3$, are equivalent by the relation $E_{\leq r} \pmod{L_2}$. Therefore, $G_{L_2}(r) = 3$.

Example 3. Let $\Sigma = \{a, b\}$ and

$$L_3 = \left\{a^n b^n \mid n \in N^+\right\}.$$

Denote $W = \{a, a^2, \ldots, a^r\}$. We have $W \subset \Sigma^{\leq r}$, $|W| = r$, and $a^i \overline{E}_{\leq r} a^j \pmod{L_3}$ with $i \neq j$. Therefore, $G_{L_3}(r) \geq |W| = r$.

Example 4. Let $|\Sigma| = k \geq 2$ and

$$L_4 = \left\{x x^R \mid x \in \Sigma^*\right\},$$

where x^R is the inverse image of x.

It is easy to show that, if $x_1, x_2 \in \Sigma^r$, $x_1 \neq x_2$, then $x_1 \overline{E}_r x_2 \pmod{L_4}$. We obtain $H_{L_4}(r) = |\Sigma^r| = k^r$.

Example 5. Let $|\Sigma| = k \geq 2, c \notin \Sigma$ and

$$L_5 = \left\{ xcx \mid x \in \Sigma^+ \right\}.$$

It can verify that, if $x_1, x_2 \in \Sigma^{\leq r}, x_1 \neq x_2$, then $x_1 \overline{E}_{\leq r} x_2 \pmod{L_5}$. Therefore, $G_{L_5}(r) = |\Sigma^{\leq r}| = k(k^r - 1)/(k-1)$.

4. A Necessary Condition for LP-Languages

4.1. In [1], we have formulated some necessary conditions for FP-languages. In this part, by using the functions $G_L(r)$, $H_L(r)$, we will establish a necessary condition for LP-languages. The obtained result is a sharpening and a generalizing of one result in [1].

Theorem 1. *Let L be accepted by an LP-net with m transitions and n places. Denote* $k = \min\{m, n\}$. *There exists a polynomial P_k of degree k such that, for any integer* $r \geq 1$,

$$H_L(r) \leq P_k(r),$$
$$G_L(r) \leq P_k(r).$$

Proof. First, we prove the theorem for the case where $\mathcal{N}$ is an FP-net, and then generalize the result for the case where $\mathcal{N}$ is an LP-net.

Let $\mathcal{N} = (P, T, I, O, \mu_0, M_f)$ be an FP-net with $|T| = m$, $|P| = n$. We denote by $S_{\leq r}$ the set of all reachable markings of $\mathcal{N}$ by firing at most r transitions. We now prove that if $L = L(\mathcal{N})$, then

$$\forall r \in N^+ : G_L(r) \leq |S_{\leq r}|.$$

To prove this, we assume the contrary, i.e., $G_L(r) > |S_{\leq r}|$. Therefore, there are $x_1, x_2 \in T^{\leq r}$ such that $x_1 \overline{E}_{\leq r} x_2 \pmod{L}$, and $\delta(\mu_0, x_1) = \delta(\mu_0, x_2)$. It follows that

$$\forall \omega \in T^* : \delta(\delta(\mu_0, x_1), \omega) = \delta(\delta(\mu_0, x_2), \omega).$$

It means

$$\forall \omega \in T^* : x_1\omega \in L \leftrightarrow x_2\omega \in L.$$

We obtain $x_1 E_{\leq r} x_2 \pmod{L}$. This conflicts with hypothesis $x_1 \overline{E}_{\leq r} x_2 (\mathrm{mod} L)$. Therefore, $G_L(r) \leq |S_{\leq r}|$.

We now estimate $|S \leq r|$. There are two ways to do this.

First, we prove $|S_{\leq r}| \leq P_n(r)$ with $|P| = n$.

Denote

$$\mu_0 = (a_1, .., a_n); \quad a = \max a_i, \quad 1 \leq i \leq n.$$
$$l = \max |O(t_j, p_i) - I(p_i, t_j)|, \quad 1 \leq i \leq n; \quad 1 \leq j \leq m.$$

Let $t = t_{j_1} t_{j_2} \cdots t_{j_p}$, $p \leq r$, be any firing sequence of $\mathcal{N}$. The equation of state change by firing t can be determined as follows: $\delta(\mu_0, t_{j_1}) = \mu'$ with $\forall p_i \in P$:

$$\mu'(p_i) = \mu_0(p_i) + (O(t_{j_1}, p_i) - I(p_i, t_{j_1})),$$
$$\mu'(p_i) \leq a + l.$$

$\delta(\mu_0, t_{j_1} \cdots t_{j_p}) = \mu^{(p)}$ with $\forall p_i \in P$:

$$\mu^{(p)}(p_i) = \mu^{(p-1)}(p_i) + (O(t_{j_p}, p_i) - I(p_i, t_{j_p})),$$
$$\mu^{(p)}(p_i) \leq a + p \cdot l \leq a + l \cdot r.$$

Therefore, $\forall r \in N^+$:

$$|S_{\leq r}| \leq (a + lr)^n = P_n(r).$$

Second, we show $|S_{\leq r}| \leq P_m(r)$ with $|T| = m$. We define the matrices I^-, O^+, D as follows:

$$I^-[j,i] = (I(p_i, t_j))_{m\times n},$$
$$O^+[j,i] = (O(t_j, p_i))_{m\times n},$$
$$D = O^+ - I^-,$$

and set

$$e[j] = (0, \ldots, 0, \underbrace{1}_{j\text{th}}, 0, \ldots, 0)_{1\times m}.$$

Let $t = t_{j_1}t_{j_2}\cdots t_{j_p}$, $p \leq r$, be any firing sequence of $\mathcal{N}$. Firing t, the equation of state change is also determined by another way as follows:

$$\delta(\mu_0, t_{j_1}) = \mu' = \mu_0 + e[j_1]D.$$
$$\delta(\mu_0, t_{j_1} \cdots t_{j_p}) = \mu^{(p)} = \mu^{(p-1)} + e[j_p]D.$$

We obtain

$$\delta(\mu_0, t_{j_1} \cdots t_{j_p}) = \mu_0 + e[j_1]D + \cdots + e[j_p]D.$$

We set $e[j]D = \nu_j$, $j = 1, \ldots, m$, and f_j is the number of occurrences of transition t_j in t. We can now express the equation of state change in the following form:

$$\begin{cases} \mu^{(p)} &= \mu_0 + \sum_{j=1}^m f_j \nu_j, \\ \sum_{j=1}^m f_j &\leq r. \end{cases}$$

It follows that $|S_{\leq r}|$ equals at most the number of non-negative integer solutions of inequality $\sum_{j=1}^m f_j \leq r$. In [1], we have proved that this number equals $C^r_{m+r} = (m+r)!/r!m! \leq (m+r)^m$. Therefore, $\forall r \in N^+$:

$$|S_{\leq r}| \leq (m + r)^m = P_m(r).$$

Combining both results of estimating $|S_{\leq r}|$, we obtain

$$G_L(r) \leq |S_{\leq r}| \leq P_k(r), \text{ with } k = \min\{m, n\}.$$

We now generalize the above result to LP-nets. The essential observation is that the methods of estimating $|S_{\leq r}|$ depend only on the length of firing sequence t, not on which components t_j occur in sequence t. At the same time, the labeled function σ is a non-erasing mapping, i.e., if $x = \sigma(t)$, then $l(x) = l(t)$. So all arguments of estimating $|S_{\leq r}|$ for FP-nets still hold for LP-nets.

Finally, from the property $\forall r \in N : H_L(r) \leq G_L(r)$, it follows that $H_L(r) \leq P_k(r)$. ∎

4.2. Using Theorem 1, we can show a series of rather simple languages not being acceptable by any LP-nets.

Example 6. Let $|\Sigma| = k \geq 2$ and $c \notin \Sigma$. We consider the following languages:

$$L_4 = \{xx^R \mid x \in \Sigma^*\},$$
$$L_5 = \{xcx \mid x \in \Sigma^*\},$$

where x^R is the inverse image of x.

We have proved in Examples 4 and 5 that $H_{L_4}(r) = k^r$ and $G_{L_5}(r) = k(k^r - 1)/(k-1)$. According to Theorem 1, we have the languages $L_4 \notin \mathcal{L}$ and $L_5 \notin \mathcal{L}$.

Example 7. Let $\Sigma = \{0, 1, a\}$ and

$$L_7 = \{\omega a^k \mid \omega \in \{0,1\}^*, k = B(\omega)\},$$

where $B(\omega)$ is the integer represented by ω as a binary number.

By an argument analogous to that used in Example 6, it is easy to show that $H_{L_7}(r) = 2^r$. Therefore, by Theorem 1, the language $L_7 \notin \mathcal{L}$.

Example 8. Let $|\Sigma| = k \geq 2$ and

$$L_8 = \{\tau_1\tau_2 \cdots \tau_n\tau_0 \mid \forall i : \tau_i \in \Sigma^*, \quad l(\tau_i) = l = \text{const.} : \quad \exists \tau_i = \tau_0\}.$$

For every subset $W = \{P_1, P_2, \ldots, P_q\} \subseteq \Sigma^l$, W is associated with a word $\alpha(W) = P_1 P_2 \cdots P_q \underbrace{P_q \cdots P_q}_{k^l - q \text{ times}} \in \Sigma^r$, with $r = l \cdot k^l$.

It is easy to verify that

$$\alpha(W)\omega \in L_8 \leftrightarrow \omega \in W.$$

Therefore,

$$H_{L_8}(r) \geq |2^{\Sigma^l}| \geq 2^{k^l} = C^r,$$

with $C = 2^{1/l}$. According to Theorem 1, the language $L_8 \notin \mathcal{L}$.

Example 9. Let $|\Sigma| = k \geq 2$ and $c \notin \Sigma$. We define

$$L_9 = \{\tau_1 c \tau_2 c \cdots c \tau_n c \tau_0 \mid \forall i : \tau_i \in \Sigma^*, l(\tau_i) \leq l = \text{const.}; \exists \tau_i = \tau_0\}.$$

Similarly as in Example 8, each subset $W = \{P_1, P_2, \ldots, P_q\} \subseteq \Sigma^{\leq l}$ is associated with a word:

$$\beta(W) = P_1 c P_2 c \cdots c P_q c \in \Sigma^{\leq r},$$

where $r = (|\Sigma^{\leq l}|)(l+1) = \big((k(k^l - 1))/(k-1)\big)(l+1)$, with $k = |\Sigma|$.

It is easy to see that

$$\beta(W)\omega \in L_9 \leftrightarrow \omega \in W.$$

Therefore,

$$G_{L_9}(r) \geq 2^{|\Sigma^{\leq l}|} = C^r,$$

with $C = 2^{1/(l+1)}$. According to Theorem 1, the language $L_9 \notin \mathcal{L}$.

5. Some Properties of Classes of Languages Characterized by $G_L(r)$

5.1. It is well known that the class of regular languages is the simplest in Chomsky's hierarchy. We now show that it is also the simplest class by the complexity characteristic $G_L(r)$.

Theorem 2. *A language L is regular if and only if there exists a constant C, such that, for any $r \in N^+$,*

$$G_L(r) \leq C.$$

Proof. (a) "Only if" part. Let $L \subseteq \Sigma^*$ and L be regular. We recall the Myhill–Nerode equivalence relation $E(\text{mod}\,L)$ defined as follows: $\forall x_1, x_2 \in \Sigma^*$:

$$x_1 E x_2\,(\text{mod}\,L) \Leftrightarrow \forall \omega \in \Sigma^* \;:\; x_1\omega \in L \leftrightarrow x_2\omega \in L.$$

Denote $I_L = \text{Rank}\,E\,(\text{mod}\,L)$.

Myhill–Nerode have proved that L is regular if and only if there exists a constant C, such that $I_L \leq C$ (see, e.g., [7]). If $x_1 E x_2 (\text{mod}\,L)$, then $x_1 E_{\leq r} x_2\,(\text{mod}\,L)$. Therefore, $G_L(r) \leq I_L$. We obtain $G_L(r) \leq C$ for all $r \in N^+$.

(b) "If" part. We assume that there is a constant C such that $\forall r \in N^+$, $G_L(r) \leq C$. Because $G_L(r)$ is non-decreasing and bounded, there exists $\lim G_L(r) = q$, $q = \text{const.}$ when $r \longrightarrow \infty$. Since the values of $G_L(r)$ are integer, so there is a constant r_0, such that $\forall r \geq r_0 : \; G_L(r) = q$.

To prove L is regular, we assume the contrary that L is not regular. By Myhill–Nerode's theorem, $I_L = +\infty$. Therefore, there is an infinite sequence $x_1, x_2, \ldots, x_k, \ldots$ $x_i \in \Sigma^*$, $x_i \neq x_j$ and $x_i \overline{E} x_j (\text{mod}\,L)$. From this sequence, we pick up the finite sequence $x_1, x_2, \ldots, x_q, x_{q+1}$ and set $k = \max\{l(x_1), \ldots, l(x_{q+1})\}$. We now choose $r = \max\{k, r_0\}$. We obtain $x_i \overline{E}_{\leq r} x_j$ (mod L) for $i \neq j$. It follows that $G_L(r) \geq q + 1$. Thus, there is r, $r \geq r_0$ but $G_L(r) \neq q$. This contradicts the property that $\forall r \geq r_0$, $G_L(r) = q$. It follows that L is regular. ■

Example 10. We consider the following languages:

$$L_2 = \{a^m b^n \mid m, n \geq 1\},$$
$$L_3 = \{a^n b^n \mid n \geq 1\}.$$

In Examples 2 and 3, we have shown that $G_{L_2}(r) = 3$ and $G_{L_3}(r) \geq r$. By Theorem 2, it follows that L_2 is regular, but L_3 is not regular.

5.2. In [9] Peterson has shown that all LP-languages are context-sensitive. We now have the following theorem:

Theorem 3.

(a) *There is a context-free language L (i.e., it is rather simple in Chomsky's hierarchy) but $G_L(r)$ is very large.*

(b) *Conversely, there is a context-sensitive but not context-free language L (i.e., it is rather complicated in Chomsky's hierarchy) but $G_L(r)$ is rather small.*

Thus, the function $G_L(r)$ gives a complexity of L, but, in general, it does not characterize a computational complexity of L.

Proof. (a) We examine

$$L_4 = \{xx^R | x \in \Sigma^* stdi\}, \quad |\Sigma| = k \geq 2.$$

It is easy to see that L_4 is a context-free language because it is acceptable by a push-down automaton. But in Example 4, we have shown that $G_{L_4}(r) \geq H_{L_4}(r) = k^r$; it means the degree of $G_{L_4}(r)$ gets maximum.

(b) We consider $L_{11} = \{a^n b^n c^n / n \geq 1\}$. One can prove that L_{11} is a context-sensitive but not context-free language (see, e.g., [7]). We now estimate $G_{L_{11}}(r)$. Let W denote the set of all words in $\Sigma^{\leq r}$ having one of the following forms:

(1) a^{k_1}, with $1 \leq k_1 \leq r$;
(2) $a^{k_1} b^{k_2}$, with $k_1 + k_2 \leq r$; $k_2 \leq k_1$;
(3) $a^{k_1} b^{k_2} c^{k_3}$, with $k_1 + k_2 + k_3 \leq r$, $k_1 = k_2, k_3 \leq k_2$;
(4) b.

It is easy to prove that every word $x \in \Sigma^{\leq r}$ is equivalent by the relation $E_{\leq r}$ (mod L_{11}) to some word in W. Therefore, $G_{L_{11}}(r) \leq |W|$. By estimating the number of elements in W, we obtain

$$G_{L_{11}}(r) \leq |W| \leq \frac{r^3}{3!} + \frac{r^2}{2!} + r + 1.$$

It means the degree of $G_{L_{11}}(r)$ is rather small.

5.3. We close the paper with some open problems.

(i) Is the necessary condition in Theorem 1 also sufficient? There are reasons to believe that the answer could be negative.
(ii) Remark that the necessary condition in Theorem 1 is not trivial only when the output alphabet Σ consists of $k \geq 2$ letters. So it is interesting to give another necessary condition with $k = 1$.

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