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Survey

Ninety Years of the Brouwer Fixed Point Theorem*

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Abstract. This historical article surveys the development of areas of mathematics directly related to the nearly ninety-year-old Brouwer fixed point theorem. We are mainly concerned with equivalent formulations and generalizations of the theorem. Also, we deal with the KKM theory and various equilibrium problems closely related to the Brouwer theorem.

1. Introduction

The Brouwer fixed point theorem is one of the most well known and important existence principles in mathematics. Since the theorem and its many equivalent formulations or extensions are powerful tools in showing the existence of solutions for many problems in pure and applied mathematics, many scholars have been studying its further extensions and applications. The purpose of this article is to survey the development of areas of mathematics directly related to the nearly ninety-year-old theorem. We are mainly concerned with equivalent formulations and generalizations of the theorem. Moreover, we deal with the Knaster–Kuratowski–Mazurkiewicz Theory (KKM theory for short) and various equilibrium problems closely related to the Brouwer theorem.

Generalizations of the Brouwer theorem have appeared in relation to the theory of topological vector spaces in mathematical analysis. The compactness, convexity, single-valuedness, continuity, self-mapness, and finite dimensionality related to the Brouwer theorem are all extended and, moreover, for the case of infinite dimension, it is known that the domain and range of the map may have different topologies. This is why the Brouwer theorem has so many generalizations. Current study of its generalizations concentrates on a more general class of compact or condensing multimaps defined on convex subsets of more general topological vector spaces.

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Along with these developments, a large number of equivalent formulations of the Brouwer fixed point theorem have been found. One of the earliest was a theorem of Knaster, Kuratowski, and Mazurkiewicz, which initiated the so-called KKM theory. At first, the basic theorems in the KKM theory were established for convex subsets of topological vector spaces, and later, for various generalized abstract convexities. These basic theorems have many applications to various equilibrium problems.

Other directions of the generalizations in topology are studies of spaces having the fixed point property, various degree or index theories, the Lefschetz fixed point theory, the Nielsen fixed point theory, and the fixed point theorems in the Atiyah–Singer index theory which generalizes the Lefschetz theory. However, we will not follow these lines of study.

In closing our introduction, we quote an excellent expression on the current status of the fixed point theory as follows:

"Fixed points and fixed point theorems have always been a major theoretical tool in fields as widely apart as differential equations, topology, economics, game theory, dynamics, optimal control, and functional analysis. Moreover, more or less recently, the usefulness of the concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major weapon in the arsenal of the applied mathematician."

M. Hazewinkel,
Editor's Preface to [120].

2. Works of Poincaré and Bohl

The Bolzano intermediate value theorem in 1817 was generalized by Poincaré [211, 212] in order to apply to the three body problem as follows:

Let $\xi_1, \xi_2, \dots, \xi_n$ be n continuous functions of n variables $x_1, x_2, \dots, x_n$; the variable x_i is subjected to vary between the limits $+a_i$ and $-a_i$. Suppose that for $x_i = a_i$, ξ_i is constantly positive, and for $x_i = -a_i$, ξ_i is constantly negative; I say that there will exist a system of values of x for which all the ξ's vanish.

For the proof, he referred to a theorem of Kronecker in a paper on functions of several variables. This paper was known to initiate the theory of the topological degree of maps (see Browder [34] and references therein). Later, Poincaré [213] published the argument on the continuation invariance of the index which is the basis for the proof of the above theorem.

Poincaré's theorem is nowadays called the Bolzano–Poincaré–Miranda theorem because it was proved by Miranda [168], who also showed that it was equivalent to the Brouwer fixed point theorem. It should be noted that Kaniel [127] misquoted Poincaré's theorem and a number of authors followed (see [194]).

The second forerunner of the Brouwer theorem was given by Bohl [24, p. 185] as follows:

Let a domain (G) $-a_i \le x_i \le a_i$ $(i = 1, 2, \dots, n)$ be given. In this domain, let $f_1, f_2, \dots, f_n$ be continuous functions of x which do not have a common zero. Then there is a point $(u_1, u_2, \dots, u_n)$ in the boundary of G such that

$$f_i(u_1, u_2, \ldots, u_n) = N \cdot u_i, \quad N < 0 \quad (i = 1, 2, \ldots, n).$$

The following theorem can be regarded as contained in this theorem:

There do not exist n continuous functions $F_1, F_2, \ldots, F_n$, defined on the domain (G) $-a_i \leq x_i \leq a_i$ $(i = 1, 2, \ldots, n)$, which have no common zero and which fulfill for the points of the boundary of (G)

$$F_i = x_i \quad (i = 1, 2, \ldots, n).$$

Hence, Bohl proved for the first time that the boundary of a cube is not a retract of the solid cube, which is equivalent to the Brouwer theorem.

For Bohl's work, Bing [21] wrote:

"The result is frequently called the Brouwer Fixed Point Theorem although the work of Brouwer [8] was probably preceded by that of Bohl [4]. . . . In proving the theorem, Bohl considered differentiable maps and used Green's Theorem to show that equivalent integrals did not match if the n-cell had a fixed point free map into itself."

The following is called the non-retract theorem:

Theorem 1. *For $n \geq 1$, $\mathbf{S}^{n-1}$ is not a retract of $\mathbf{B}^n$.*

Smart [253] wrote:

"Bohl [24] proved a result equivalent to the non-retraction theorem but apparently did not go on to obtain the Brouwer theorem."

On the other hand, Dugundji and Granas [52] claimed that the non-retract theorem was due to Borsuk and the following to Bohl:

Theorem 2. *Every continuous $F : \mathbf{B}^{n+1} \to \mathbf{R}^{n+1}$ has at least one of the following properties:*

(a) *F has a fixed point;*
(b) *there is an $x \in \mathbf{S}^n$ such that $x = \lambda F x$ for some $0 < \lambda < 1$.*

This follows from Bohl's first theorem: If $f = I - F$ is continuous and fails to have a fixed point, then Bohl's conclusion implies (b).

Note that the concept of retraction is due to Borsuk [27] and that the negation of condition (b) is the so-called Leray–Schauder boundary condition.

3. The Brouwer Fixed Point Theorem

In 1910, the Brouwer theorem appeared.

Theorem 3. [29] *A continuous map from an n-simplex to itself has a fixed point.*

It is clear that, in this theorem, the n-simplex can be replaced by the unit ball $\mathbf{B}^n$ or any compact convex subset of $\mathbf{R}^n$. This theorem appeared as Satz 4 in [29]. At the end of this paper, it is noted that "Amsterdam, July 1910" by Brouwer himself.

Some authors were confused with the theorem that appeared in [28]. According to Bing [21], "even before Brouwer's paper [29] appeared, reference had been made to the Brouwer Fixed Point Theorem." (See Hadamard's reference in [258, p. 472].) In fact, Hadamard gave a proof of the Brouwer theorem using the Kronecker indices in the appendix of Tannaery [258].

According to Freudenthal [74] (where [29] is listed as "1911D"), Hadamard knew the Brouwer theorem (without proof) from a letter of Brouwer (dated January 4, 1910).

Brouwer [29] gave a proof of his theorem using the simplicial approximation technique and notions of degree. According to Bing again, Brouwer himself proved the theorem by showing that homotopic maps of an $(n-1)$-sphere onto itself have the same degree (or rotation of vector fields), hence, there is no retraction of an n-cell onto its boundary. Hence, each map of an n-cell into itself is not fixed point free. (For further comments on Brouwer's works on fixed point, see [49], and on degree theory, see [245, 246].

Alexander [1] proved a theorem of Brouwer [28] using the index of a map and applied it to obtain the Brouwer fixed point theorem. Birkhoff and Kellogg [22] also gave a proof of the theorem of Brouwer by using classical methods in calculus and determinant theory. The same line of proof of the Brouwer theorem can be found in Dunford and Schwartz [53].

Knaster, Kuratowski, and Mazurkiewicz [133] gave a proof of the Brouwer theorem using combinatorial techniques. They used the Sperner lemma [254] and showed that the non-retract theorem holds.

Later, there appeared proofs using algebraic topology, various degree theories, or differential forms. Hirsch [105] gave a proof of the non-retract theorem using the method of geometric topology, and Milnor [166] gave an analytic proof. There were also many other proofs of the Brouwer theorem, and a simple proof using advanced calculus was given by Rogers [224] and others.

Recently, there have been very interesting proofs of the Brouwer theorem. Kulpa [142] deduced a generalization of the Brouwer theorem from the Fubini theorem and the Weierstrass approximation theorem, and applied it to give a simple proof of the fundamental theorem of algebra. More recently, Su [257] gave a completely elementary proof that the Borsuk–Ulam theorem implies the Brouwer theorem by a direct construction.

The Brouwer theorem itself gives no information about the location of fixed points. However, effective ways have been developed to calculate or approximate the fixed points. Such techniques are important in various applications including calculation of economic equilibria. The first such algorithm was the simplicial algorithm proposed by Scarf [228] and later developed in the so-called homotopy or continuation methods for calculating zeros of functions. (For details of this topic, see [71, 129, 273] and others.)

4. Sperner's Combinatorial Lemma: From 1928

In 1928, Sperner [254] gave the following combinatorial lemma and its applications:

Lemma 1. [254] *Let K be a simplicial subdivision of an n-simplex $v_0v_1\ldots v_n$. To each vertex of K, let an integer be assigned in such a way that, whenever a vertex u of K lies on a face $v_{i_0}v_{i_1}\cdots v_{i_k}$ $(0 \leq k \leq n,\ 0 \leq i_0 < i_1 < \cdots < i_k \leq n)$, the number assigned* *to u is one of the integers* $i_0, i_1, \ldots, i_k$. *Then the total number of those n-simplexes of K, whose vertices receive all* $n+1$ *integers* $0, 1, \ldots, n$, *is odd. In particular, there is at least one such n-simplex.*

Fifty years after the birth of this lemma, at a conference in Southampton, England, in 1979, Sperner himself listed early applications of his lemma as follows:

(1) invariance of dimension [254];
(2) invariance of region [254];
(3) theorem of verification (Rechtfertigungssatz) in Menger's theory of dimension [165];
(4) Brouwer's fixed point theorem [133];
(5) matrices with elements ≥ 0 [60], theorems on eigenvalues of such matrices by Perron, Frobenius, and others.

There appeared a number of generalizations of the lemma, which was applied to the following:

(6) antipodal theorems [58, 261]; which include the Lusternik–Schnirelmann theorem on a cover of the n-sphere $\mathbf{S}^n$ consisting of $n+1$ closed subsets and the Borsuk–Ulam theorem on a continuous map $f : \mathbf{S}^n \to \mathbf{R}^n$;
(7) derivation of the Sperner lemma from the Brouwer fixed point theorem [272];
(8) constructive proof of the Fundamental Theorem of Algebra [141];
(9) approximation algorithm to approximate Brouwer fixed point [3, 140, 228, etc].

In the later years, Sperner unified his own lemma and its extensions due to Tucker and Fan [36, 58]. (For the details, see Sperner's articles in [71].)

5. The KKM Theorem: From 1929

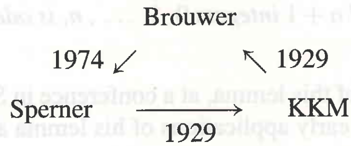
In 1928, Knaster, Kuratowski, and Mazurkiewicz [133] obtained the following so-called KKM theorem from the Sperner lemma [254]:

Theorem 4. [133] *Let* A_i $(0 \leq i \leq n)$ *be* $n+1$ *closed subsets of an n-simplex* $p_0 p_1 \cdots p_n$. *If the inclusion relation*

$$p_{i_0} p_{i_1} \cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0} p_{i_1} \cdots p_{i_k}$ $(0 \leq k \leq n,\ 0 \leq i_0 < i_1 < \cdots < i_k \leq n)$, *then* $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

A special case or dual form of the KKM theorem is already given in [254]. The KKM theorem follows from the Sperner lemma and is used to obtain one of the most direct proofs of the Brouwer theorem. Therefore, it was conjectured that these three theorems are mutually equivalent. This was clarified by Yoseloff [272]. In fact, these three theorems are regarded as a sort of mathematical trinity (see diagram below). All are extremely important and have many applications.



Moreover, many important results in nonlinear functional analysis and other fields are known to be equivalent to these three theorems. Only less than a dozen of those results are shown in textbooks such as Aubin [8], Aubin and Ekeland [9], and Zeidler [275] and in surveys such as Gwinner [92] and others. Further, usefulness of these three theorems can also be seen in [102, 115, 175, 273, etc.]

From the KKM theorem, we can deduce the concept of KKM maps as follows: Let E be a vector space and $D \subset E$. A multimap (set-valued function or map) $G : D \multimap E$ is called a *KKM map* if

$$\operatorname{co} N \subset G(N)$$

holds for each non-empty finite subset N of D.

Granas [88] gave some examples of KKM maps as follows:

(i) *Variational problems.* Let C be a convex subset of a vector space E and $\phi : C \to \mathbf{R}$ a convex function. Then $G : C \multimap C$ defined by

$$Gx = \{y \in C : \phi(y) \leq \phi(x)\} \quad \text{for } x \in C$$

is a KKM map.

(ii) *Best approximation.* Let C be a convex subset of a vector space E, p a seminorm on E, and $f : C \to E$ a function. Then $G : C \multimap C$ defined by

$$Gx = \{y \in C : p(fy - y) \leq p(fy - x)\} \quad \text{for } x \in C$$

is a KKM map.

(iii) *Variational inequalities.* Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space, C a convex subset of H, and $f : C \to H$ a function. Then $G : C \multimap C$ defined by

$$Gx = \{y \in C : \langle fy, y - x \rangle \leq 0\} \quad \text{for } x \in C$$

is a KKM map.

The study of properties of such KKM maps and their applications is appropriately called the KKM theory (see [187, 191]). In the framework of this theory, various fixed point theorems and many other consequences are obtained (see Sec. 7). As part of the development of this theory, there have been many results equivalent to the Brouwer theorem, especially in nonlinear functional analysis and mathematical economics. For the classical results, see Granas [88].

Relatively early equivalent forms of the Brouwer theorem are as follows:

- Poincaré's theorem (1883);
- Bohl's non-retract theorem (1904);
- Brouwer's fixed point theorem (1912);
- Sperner's combinatorial lemma (1928);
- Knaster–Kuratowski–Mazurkiewicz theorem (1929);

- Caccioppoli's fixed point theorem (1930);
- Schauder's fixed point theorem (1930);
- Tychonoff's fixed point theorem (1935);
- von Neumann's intersection lemma (1937);
- intermediate value theorem of Bolzano–Poincaré–Miranda (1941);
- Kakutani's fixed point theorem (1941);
- Bohnenblust–Karlin's fixed point theorem (1950);
- Hukuhara's fixed point theorem (1950);
- Fan–Glicksberg's fixed point theorem (1952);
- main theorem of mathematical economics on Walras equilibria of [46,78,172] (1955);
- Kuhn's cubic Sperner lemma (1960);
- Fan's KKM theorem (1961);
- Fan's geometric or section property of convex sets (1961);
- Fan's theorem on sets with convex sections (1966);
- Hartman–Stampacchia's variational inequality (1966);
- Browder's variational inequality (1967);
- Scarf's intersection theorem (1967);
- Fan–Browder's fixed point theorem (1968);
- Fan's best approximation theorems (1969);
- Fan's minimax inequality (1972);
- Himmelberg's fixed point theorem (1972);
- Shapley's generalization of the KKM theorem (1973);
- Tuy's generalization [262] of the Walras excess demand theorem (1976);
- Fan's matching theorems 1984.

Many generalizations of those theorems are also known to be equivalent to the Brouwer theorem. Recently, Horvath and Lassonde [263] obtained intersection theorems of the KKM-, Klee-, and Helly-type, which are all equivalent to the Brouwer theorem.

6. Early Extensions of the Brouwer Theorem: 1920s–1950s

The Brouwer theorem was extended to continuous selfmaps of compact convex subsets of

(1) C[0, 1] by Caccioppoli [36];

(2) normed spaces by Schauder [233, 234]; and

(3) locally convex topological vector spaces by Tychonoff [263].

All those results are included in Lefschetz-type fixed point theorems, which are in turn contained in the Leray–Schauder theory as extended by Browder and others (see [264]).

Note that Birkhoff–Kellogg [22], Schauder [233], and Tychonoff [263] applied their results to the existence of solutions of certain differential and integral equations.

There also appeared extensions for maps, which were not selfmaps of compact convex subsets, as follows:

Theorem 5. [133] *If $f : \mathbf{B}^n \to \mathbf{R}^n$ is a continuous map such that f maps $\mathbf{S}^{n-1} = Bd\,\mathbf{B}^n$ back into $\mathbf{B}^n$, then f has a fixed point.*

This was originally stated for a simplex instead of $\mathbf{B}^n$, and is the origin of the so-called Rothe boundary condition.

Theorem 6. [234] *If C is a closed convex subset of a Banach space, then every compact continuous map $f : C \to C$ has a fixed point.*

This is the second theorem of Schauder [234], and it is especially convenient in application. Note that this follows from (2) by using Mazur's result [162] that the convex closure of a compact set in a Banach space is compact. It is later recognized that the closedness of C and the completeness of the space are not necessary. The third Schauder theorem is:

Theorem 7. [234] *If C is a weakly compact convex subset of a separable Banach space, then every weakly continuous map $f : C \to C$ has a fixed point.*

This also follows from (2) by considering the weak topology, and was generalized by Krein and Šmulian [138] as follows:

Theorem 8. [138] *Let H be a closed convex subset of a Banach space. If $f : H \to H$ is weakly continuous such that $f(H)$ is separable and the weak closure of $f(H)$ is weakly compact, then f has a fixed point.*

For Caccioppoli's fixed point theorem [36] and for the role of the separability in the above two theorems, see [7].

The KKM fixed point theorem was extended by Rothe:

Theorem 9. [28] *Let V be a closed ball of a Banach space E and $f : V \to E$ a compact continuous map such that $f(\mathrm{Bd}\,V) \subset V$. Then f has a fixed point.*

Altman [4] showed that the Rothe condition $f(\mathrm{Bd}\,V) \subset V$ can be replaced by the following:

$$\|fx - x\|^2 \geq \|fx\|^2 - \|x\|^2 \quad \text{for all } x \in \mathrm{Bd}\,V.$$

Note that those conditions are all particular to the so-called Leray–Schauder condition.

Applications of theorems of Brouwer [29], Rothe [225], Schauder [234], and Tychonoff [263] appeared in many textbooks for the existence of solutions. We list some of them:

- nonlinear systems of equations;
- systems of inequalities;
- integral equations;
- ordinary differential equations satisfying Lipschitz condition;
- peano's theorem on the existence of solutions of ordinary differential equations;
- alternating current circuits (periodic solutions of systems of ordinary differential equations);
- elliptic partial differential equations;
- problems in mathematical physics.

One interesting application of the Brouwer theorem is due to Zeeman [274], who described a model of brain.

Lomonosov [153] gave a proof of the existence of invariant subspaces in operator theory, that is, for any completely continuous linear map f from a Banach space X into itself, there exists a closed subspace X_0 satisfying $f(X_0) \subset X_0$ and $\{0\} \subset_{\neq} X_0 \subset_{\neq} X$. Here, a completely continuous map is a continuous map sending bounded sets into compact sets. (For further information on this topic, see [40] and references therein.)

Maehara [156] deduced the Jordan curve theorem from the Brouwer theorem.

On the other hand, Kakutani [126] showed the existence of a fixed-point-free continuous selfmap (even for a homeomorphism) of the unit ball in an infinite-dimensional space. Therefore, the compactness in the above theorems on an finite-dimensional case cannot be replaced by bounded closedness or by weak compactness. Moreover, Dugundji [51] showed that a normed vector space is finite-dimensional if and only if every continuous selfmap of its unit ball has a fixed point.

Tychonoff's theorem was applied to obtain the following by Markov [157]:

Theorem 10. [157, 164] *Let K be a compact convex subset of a topological vector space E. Let $\mathcal{F}$ be a commuting family of continuous affine maps from K into itself. Then $\mathcal{F}$ has a common fixed point $p \in K$, that is, $fp = p$ for each $f \in \mathcal{F}$.*

Later, Kakutani [124] gave a direct proof and several applications.

The Markov–Kakutani theorem was generalized to larger classes of maps by Day [45] and others.

Earlier, Schauder asked in Problem 54 of [161] whether a continuous selfmap of a compact convex subset of any topological vector space has a fixed point. If the space is Hausdorff locally convex or admissible in the sense of Klee [132], then Schauder's conjecture holds. For some particular spaces, it also holds. However, the problem is not resolved yet in its full generality, even when the space is metrizable [132]. (For this problem, see [118, 199] and references therein.)

In the mid-1930s, the Leray–Schauder theory [149] appeared. It assigns a degree to certain maps and establishes properties of the degree which lead to fixed point and domain invariance theorems. This was first done for Banach spaces, and was later developed by Leray [148], Nagumo [169], Altman [5] and others for locally convex spaces. When the space is Banach, Granas [86] obtained a homotopy extension theorem, which yields many useful conclusions of the theory while avoiding the more complicated notions of the degree. Moreover, Klee [132] established the theory without local convexity.

On the other hand, Schaefer [231] showed that the problem of solvability of an equation $x = fx$, for a completely continuous map f on a locally convex space E, reduces to finding *a priori* bounds of all possible solutions for the family of equations $x = \lambda fx, \lambda \in (0, 1)$. This fact is called the Leray–Schauder alternative by Granas [70] and its various extensions and modifications have played a basic role in various applications to nonlinear problems (see also [192, 207]).

It is often said that the last theorem in Sec. 2 can be obtained in the framework of Leray and Schauder [149], which seems not to be directly related to the so-called Leray–Schauder boundary condition. This condition seems to have originated from [24, 232] (see Fishel, MR 50#8177) and has frequently appeared since the 1960s. It is assumed that it was first called the Leray–Schauder condition by Petryshyn [208]. (For the literature on the theory without using degree theory, see [197].)

Independent of the generalizations of the Brouwer theorem, Nikodym [176] and Mazur and Schauder [164] initiated the abstract approach to problems in calculus of variations. Their result can be stated as follows:

Theorem 11. [164] *Let E be a reflexive Banach space and C a closed convex set in E. Let ϕ be a lower semicontinuous convex and coercive (that is, $|\phi(x)| \to \infty$ as $\|x\| \to \infty$) real function on C. If ϕ is bounded from below, then at some $x_0 \in C$, the function ϕ attains its minimum.*

This is a very useful generalization of the classical Bolzano–Weierstrass theorem and was applied to a number of concrete problems in calculus of variations by Mazur and Schauder. However, these results were never published (see [88]). Later, this theorem was generalized to the variational inequality problems in the framework of the KKM theory (see [183, 184]).

In the 1950s, there were remarkable generalizations of the Schauder and Tychonoff theorems. The following was due to Hukuhara in 1950.

Theorem 12. [112] *Let X be a non-empty convex subset of a locally convex Hausdorff topological vector space E and $f : X \to X$ a compact continuous map. Then f has a fixed point.*

This is called the Schauder–Tychonoff fixed point theorem in [52, 53]. (See also [251]).

In 1955, Krasnoselskiĭ proved the following theorem which combines the contraction principle of Banach [10] and the Schauder theorem.

Theorem 13. [137] *Let E be a Banach space, T a bounded closed convex subset of E, and $A, B : T \to E$ operators such that*

(a) $A\phi + B\psi \in T$ for $\phi, \psi \in T$;

(b) A is completely continuous;

(c) B is a Banach contraction (that is, there exists $q < 1$ such that $\|B\phi_1 - B\phi_2\| \leq q\|\phi_1 - \phi_2\|$ for $\phi_1 \phi_2 \in T$).

Then there is a $\phi \in T$ such that $A\phi + B\phi = \phi$.

Note that, when A is the zero operator, this is (particular to) the Banach contraction principle; when B is zero, this is the second Schauder theorem.

This type of theorem is useful in establishing existence theorems for perturbed operator equations and other problems. Since then, there have been many generalizations and other applications (see, for example, [35, 37].)

In the same year, Darbo [44] introduced a new type of fixed point theorem for non-compact maps.

Recall that Kuratowski [143] defined the measure of non-compactness, $\alpha(A)$, of a bounded subset A of a metric space (X, d):

$$\alpha(A) = \inf\{\varepsilon > 0 : A \text{ can be covered by a finite number of sets of diameter less than or equal to } \varepsilon\}.$$

Let $T : X \to X$ be a continuous map. Darbo calls T an *α-contraction* for any given bounded set A in X, $T(A)$ is bounded in X and

$$\alpha[T(A)] \leq k\alpha(A),$$

where the constant k fulfills the inequality $0 \leq k < 1$.

Theorem 14. [44] *If G is a closed bounded convex subset of a Banach space X and $T : G \to G$ is an α-contraction, then T has a fixed point.*

Note that the class of α-contractions contains completely continuous maps and all Banach contractions as well [11]. (For generalizations of Darbo's theorem to Φ-condensing maps, see Sec. 11.)

7. Extensions to Multimaps and Applications: 1940s and 1950s

Independent of the above progress, in 1928, von Neumann [265] obtained the following minimax theorem, which is one of the fundamental theorems in game theory developed by him.

Theorem 15. [265] *Let $f(x, y)$ be a continuous real-valued function defined for $x \in K$ and $y \in L$, where K and L are arbitrary bounded closed convex sets in two Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$. If, for every $x_0 \in K$ and for every real number α, the set of all $y \in L$ such that $f(x_0, y) \leq \alpha$ is convex, and if, for every $y_0 \in L$ and for every real number β, the set of all $x \in K$ such that $f(x, y_0) \geq \beta$ is convex, then we have*

$$\max_{x\in K}\min_{y\in L} f(x, y) = \min_{y\in L}\max_{x\in K} f(x, y).$$

(For the history of earlier proofs of the theorem, see [43, 267].

The theorem was later extended by von Neumann to the following intersection lemma:

Lemma 2. [266] *Let K and L be two compact convex sets in the Euclidean spaces $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, and let us consider their Cartesian product $K \times L$ in $\mathbf{R}^{m+n}$. Let U and V be two closed subsets of $K \times L$ such that, for any $x_0 \in K$, the set U_{x_0}, of $y \in L$ such that $(x_0, y) \in U$, is non-empty, closed, and convex such that, for any $y_0 \in L$, the set V_{y_0}, of all $x \in K$ such that $(x, y_0) \in V$, is non-empty, closed and convex. Under these assumptions, U and V have a common point.*

Von Neumann proved this by using a notion of integral in Euclidean spaces and applied this to the problems of mathematical economics. We adopted the above formulations of Theorem and Lemma in [125].

According to Debreu [47],

"Ironically that Lemma, which, through Kakutani's Corollary, had a major influence in particular on economic theory and on the theory of games, was not required to obtain either one of the results that von Neumann wanted to establish. The Minimax theorem, as well as his theorem on optimal balanced growth paths, can be proved by elementary means."

Recall that a multimap $F : X \multimap Y$, where X and Y are topological spaces, is *upper semicontinuous* (u.s.c.) whenever, for any $x \in X$ and any neighborhood U of Fx, there exists a neighborhood V of x satisfying $F(V) \subset U$.

In order to give simple proofs of von Neumann's Lemma and the minimax theorem, Kakutani obtained the following generalization of the Brouwer theorem to multimaps:

Theorem 16. [125] *If $x \to \Phi(x)$ is an upper semicontinuous point-to-set mapping of an r-dimensional closed simplex S into the family of non-empty closed convex subset of S, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

Equivalently,

Corollary 1. [125] *Theorem 16 is also valid even if S is an arbitary bounded closed convex set in a Euclidean space.*

As Kakutani noted, Corollary 1 readily implies von Neumann's Lemma, and later Nikaido [174] noted that these two results are directly equivalent.

This was the begining of the fixed point theory of multimaps having a vital connection with the minimax theory in game theory and the equilibrium theory in economics.

According to Debreu [47] again:

"However, the formulation given by Kakutani is far more convenient to use, and his proof is distinctly more appealing.

One of the earliest, and most important, applications of the theorem of Kakutani was made by Nash [70] in his proof of the existence of an equilibrium for a finite game. It was followed by several hundred applications in the theory of games and in economic theory. In the latter, Kakutani's theorem has been for more than three decades the main tool for proving the existence of an economic equilibrium (a recent survey by Debreu [47] quotes some three hundred and fifty instances). Other areas of applications were mathematical programming, control theory and the theory of differential equations."

In the 1950s, Kakutani's theorem was extended to Banach spaces by Bohnenblust and Karlin [25] and to locally convex Hausdorff topological vector spaces by Fan [57] and Glicksberg [83]. These extensions were mainly used to extend von Neumann's works in the above. Moreover, they were known to be included in the extensions, due to Eilenberg and Montgomery [55] or Begle [12], of Lefschetz's theorem to u.s.c. maps of a compact *lc*-space into the family of its non-empty compact acyclic subsets.

The first remarkble generalization of von Neumann's minimax theorem was Nash's theorem [171] on equilibrium points of non-cooperative games. The following was formulated by Fan [64]:

Theorem 17. [171] *Let $X_1, X_2, \dots, X_n$ be n (≥ 2) non-empty, compact, convex sets, each in a real Hausdorff topological vector space. Let $f_1, f_2, \dots, f_n$ be n real-valued continuous functions defined on $\Pi_{i=1}^n X_i$. If, for each $i = 1, 2, \dots, n$ and for any given point $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \Pi_{j \neq i} X_j$, $f_i(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ is a quasi-concave function on X_i, then there exists a point $(\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n) \in \Pi_{i=1}^n X_i$ such that*

$$f_i(\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n) = \max_{y_i \in X_i} f_i(\widehat{x}_1, \dots, \widehat{x}_{i-1}, y_i, \widehat{x}_{i+1}, \dots, \widehat{x}_n) \quad (1 \leq i \leq n).$$

Further, von Neumann's minimax theorem was extended by Sion [252] to arbitrary topological vector spaces as follows:

Theorem 18. [252] *Let X, Y be a compact convex set in a topological vector space. Let f be a real-valued function defined on $X \times Y$. If,*

(1) *for each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous, quasi-convex function on Y, and*
(2) *for each fixed $y \in Y$, $f(x, y)$ is an upper semicontinuous, quasi-concave function on X,*

then we have

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y).$$

Here, f is *lower semicontinuous* whenever the set $\{y \in Y : f(x, y) > r\}$ is open, and *quasi-concave* whenever $\{x \in X : f(x, y) > r\}$ is convex for each $r \in \mathbf{R}$. Further, f is *upper semicontinuous* whenever $\{x \in X : f(x, y) < r\}$ is open, and *quasi-convex* whenever $\{y \in Y : f(x, y) < r\}$ is convex for each $r \in \mathbf{R}$.

Sion's proof was based on the KKM theorem and this seems to be the first application of the theorem after KKM [133].

As for the Brouwer theorem, in the mid-1960s, algorithms on constructive processes approximating effectively to the values of the Kakutani fixed points were developed. For the literature, see Secs. 3 and 4.

In closing this section, we quote two stories on the Brouwer and Kakutani theorems.

In [45], Brouwer denied the existence of a fixed point in his earlier theorem [28], and claimed that there can be only ε-fixed points for each $\varepsilon > 0$, because the Bolzano–Weierstrass theorem is invalid in the intuitionistic mathematics. Note that his theorem in [28] implies the Brouwer fixed point theorem as Alexander [1] showed. Here, we see Brouwer's fate of denying one of his great accomplishments of his younger days because of his own philosophy.

Comparing the Brouwer and Kakutani theorems, Franklin [73] quoted a private survey:

"... 96% of all mathematicians can state the Brouwer fixed point theorem, but only 5% can prove it. Among mathematical economists, 95% can state it, but only 2% can prove it (and these are all ex-topologists). ... while 96% of mathematicians can *state* the Brouwer fixed-point theorem, only 7% can state the Kakutani theorem."

8. Establishment of the KKM Theory: From the 1960s–1980s

In 1961, a milestone of the history of the KKM theory was established by Fan [61]. He extended the KKM theorem to infinite-dimensional spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

Lemma 3. [61] *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in $\cup_{i=1}^n F(x_i)$;*
(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\cap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the KKMF theorem. Fan also obtained the following geometric or section property of convex sets, which is equivalent to Lemma 3.

Lemma 4. [61] *Let X be a compact convex set in a topological vector space. Let A be a closed subset of $X \times X$ with the following properties:*

(i) $(x, x) \in A$ *for every* $x \in X$;
(ii) *For any fixed* $y \in X$, *the set* $\{x \in X : (x, y) \notin A\}$ *is convex (or empty).*

Then there exists a point $y_0 \in X$ *such that* $X \times \{y_0\} \subset A$.

Fan applied Lemma 4 to give a simple proof [61] of the Tychonoff theorem and to prove two results [62] generalizing the Pontrjagin–Iohvidov–Krein theorem on the existence of invariant subspaces of certain linear operators. Also, Fan [63] applied his KKMF lemma to obtain an intersection theorem (concerning sets with convex sections) which implies the Sion minimax theorem [252] and the Tychonoff theorem [263]. The main results of Fan [63] were extended by Ma [154], who obtained a generalization of the Nash theorem for the infinite case.

On the other hand, Debrunner and Flor [48] proved an extension theorem of monotone sets. This generalized earlier works of Minty [167] and Grünbaum [91] have interesting applications to nonlinear elliptic boundary value problems. Since then, the fixed point theory of multimaps have become closely related to the study of monotone operators (see [33, 34, 275]).

Moreover, "a theorem concerning sets with convex sections" was applied to prove the following results in [64]:

- an intersection theorem (which generalizes the von Neumann lemma [266]);
- an analytic formulation (which generalizes the equilibrium theorem of Nash [171] and the minimax theorem of Sion [252]);
- a theorem on systems of convex inequalities of Fan [59];
- extremum problems for matrices;
- a theorem of Hardy–Littlewood–Pólya concerning doubly stochastic matrices;
- a fixed point theorem generalizing the results of Iohvidov [119] and Tychonoff [263];
- extensions of monotone sets;
- invariant vector subspaces;
- an analog of Helly's intersection theorem for convex sets.

In the same year, Hartman and Stampacchia [100] introduced the following variational inequality:

Lemma 5. [100] *Let K be a compact convex subset in $\mathbf{R}^n$ and $f : K \to \mathbf{R}^n$ a continuous map. Then there exists $u_0 \in K$ such that*

$$(f(u_0), v - u_0) \geq 0 \quad for \quad v \in K,$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbf{R}^n$.

Using this result, the Hartman and Stampacchia [100] obtained existence and uniqueness theorems for (weak) uniformly Lipschitz continuous solutions of Dirichlet boundary value problems associated with certain nonlinear elliptic differential functional equations. Later, Lemma 5 is known to be equivalent to the Brouwer theorem.

Lemma 5 was extended by Browder [32] while trying to extend the theorems of Schauder and Tychonoff motivated by Halpern's work [97] on fixed point theorems for outward maps:

Theorem 19. [32] *Let E be a locally convex topological vector space, K a compact convex subset of E, and T a continuous mapping of K into E^*. Then there exists an element u_0 of K such that*

$$(T(u_0), u - u_0) \geq 0$$

for all u in K.

Here, E^* is the topological dual of E and $(,)$ denotes the pairing between elements of E^* and elements of E. This theorem was later extended and improved by Park [181] and many others by pointing out that the local convexity is superfluous.

On the other hand, Browder [33] restated Fan's geometric lemma [61] in the convenient form of a fixed point theorem by means of the Brouwer theorem and the partition of unity argument. Since then, the following is known as the Fan–Browder fixed point theorem:

Theorem 20. [33] *Let K be a non-empty compact convex subset of a topological vector space. Let T be a map of K into 2^K, where, for each $x \in K$, $T(x)$ is a non-empty convex subset of K. Suppose further that, for each y in K, $T^{-1}(y) = \{x \in K : y \in T(x)\}$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

Later, this is also known to be equivalent to the Brouwer theorem. Browder [33] applied his theorem to a systematic treatment of interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. This was also applied by Borgin and Keiding [26] and Yannelis and Prabhakar [271], to the existence of maximal elements in mathematical economics. For further development of generalizations and applications of the Fan–Browder theorem, we refer to [181, 191].

Motivated by Browder's works [32, 33] on fixed point theorems, Fan in 1969 deduced the following from his geometric lemma:

Theorem 21. [65] *Let X be a non-empty compact convex set in a normed vector space E. For any continuous map $f : X \to E$, there exists a point $y_0 \in X$ such that*

$$\|y_0 - f(y_0)\| = \min_{x \in X} \|x - f(y_0)\|.$$

(In particular, if $f(X) \subset X$, then y_0 is a fixed point of f.)

Fan [65] also obtained a generalization of this theorem to locally convex Hausdorff topological vector spaces. These are known as best approximation theorems and are applied to obtain generalizations of the Brouwer theorem and some non-separation theorems on upper demicontinuous (u.d.c.) multimaps.

Moreover, Fan established a minimax inequality from the KKMF theorem:

Theorem 22. [66] *Let X be a compact convex set in a Hausdorff topological vector space. Let f be a real function defined on $X \times X$ such that*

(a) *For each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous function of y on X.*
(b) *For each fixed $y \in X$, $f(x, y)$ is a quasi-concave function of x on X. Then the minimax inequality*

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x)$$

holds.

Fan gave applications of his inequality as follows:

- a variational inequality (extending [32, 64]);
- a geometric formulation of the inequality (equivalent to the Fan–Browder theorem);
- separation properties of upper demicontinuous multimaps, coincidence, and fixed point theorems;
- properties of sets with convex sections [64];
- a fundamental existence theorem in potential theory;

Furthermore, Fan [67, 68] introduced a KKM theorem with a coercivity (or compactness) condition for non-compact convex sets and, from this, extended many known results to non-compact cases. We list some main results as follows:

- generalizations of the KKM theorem for non-compact cases;
- geometric formulations;
- fixed point and coincidence theorems;
- generalized minimax inequality (extending Allen's variational inequality [2]);
- a matching theorem for open (closed) covers of convex sets;
- the 1978 model of the Sperner lemma;
- another matching theorem for closed covers of convex sets;
- a generalization of Shapley's KKM theorem [239];
- results on sets with convex sections;
- a new proof of the Brouwer theorem.

While closing a sequence of lectures delivered at the NATO-ASI, Montreal, 1983, Fan listed various fields in mathematics which have applications of KKM maps, as follows:

- potential theory;
- Pontrjagin spaces or Bochner spaces in inner product spaces;
- operator ideals;
- weak compactness of subsets of locally convex topological vector spaces;
- function algebras;
- harmonic analysis;
- variational inequalities;
- free boundary value problems;
- convex analysis;
- mathematical economics;
- game theory;
- mathematical statistics.

We may add the following fields to this list: nonlinear functional analysis, approximation theory, optimization theory, fixed point theory, and others.

In the 1980s, many recognized the following open-valued version of the KKM theorem:

Theorem 23. *Let D be the set of vertices of Δ_n and $G : D \multimap \Delta_n$ a KKM map (that is, $\mathrm{co}\, A \subset G(A)$ for each subset A of D) with open values. Then $\bigcap_{a \in D} G(a) \neq \emptyset$.*

This is a simple consequence of the KKM theorem in view of Theorem 4 in [241], which shows the existence of a closed-valued KKM map $F : D \multimap \Delta_n$ such that $Fx \subset Gx$ for all $x \in D$. It is later known to be equivalent to the KKM theorem itself. (For the history of generalizations and applications of the above open-valued version of the KKM theorem, see [190, 205]).

9. Intersection Theorems and Equilibrium Problems

Intersection theorems concern those conditions under which members of a certain subset of a cover of a given set have a non-empty intersection. Such intersection theorems on the standard simplex or other convex sets were given by the covering property of Sperner [254], the KKM theorem [133], the KKMF theorem due to Fan [61], Scarf's theorem [228], the KKMS theorem due to Shapley [239], Gale's theorem [79], Ichiishi's theorem [116], the intersection theorems of Horvath and Lassonde [111], and others. These theorems are applied to the existence of solutions of mathematical programming problems, to economic equilibrium theory, and to game theoretic problems.

The KKMS theorem is a very useful tool to show that the core of any balanced, non-transferable utility game is non-empty, a result first shown in [229] by means of a constructive method being related to the methods introduced in [228, 230]. In fact, Shapley [239] extended the KKM theorem on closed covers of a simplex to the case of more general closed covers of a simplex incorporating the notion of balancedness, and obtained a theorem now called the KKMS theorem. Shapley proved the theorem constructively using an analogous generalization of the Sperner lemma [254].

Let $N = \{1, \ldots, n\}$ and $\mathcal{N}$ the family of all non-empty subsets of N. Let $\{e^i : i \in N\}$ be the standard basis of $\mathbf{R}^n$, that is, e^i is an n-vector whose ith coordinate is 1 and 0 otherwise. Let Δ be the simplex $\mathrm{co}\{e^i : i \in N\}$ and, for an $S \in \mathcal{N}$, let Δ^S be the face of Δ spanned by $\{e^i : i \in S\}$, that is, $\Delta^S = \mathrm{co}\{e^i : i \in S\}$. A subfamily $\mathcal{B}$ of $\mathcal{N}$ is said to be *balanced* if there are non-negative weights λ^S, $S \in \mathcal{B}$, such that $\sum_{S\in\mathcal{B}} \lambda^S e^S = e^N$, where e^S denotes the n-vector whose ith coordinate is 1 if $i \in S$ and 0 otherwise. It is easily seen that $\mathcal{B}$ is balanced if and only if $m^N \in \mathrm{co}\{m^S : S \in \mathcal{B}\}$, where m^S denotes the center of gravity of the face Δ^S, that is, $m^S = \sum_{i\in S} e^i/|S|$.

Theorem 24. [239] *Let $\{C_S : S \in \mathcal{N}\}$ be a family of closed subsets of Δ such that, for each $T \in \mathcal{N}$,*

$$\Delta^T \subset \bigcup_{S\subset T} C_S.$$

Then there is a balanced family $\mathcal{B}$ such that

$$\bigcap_{S\in\mathcal{B}} C_S \neq \emptyset.$$

Since Scarf's core theorem was very important in mathematical economics and since Shapley's proof of the KKMS theorem was rather complicated, several authors explored the logical connection between Scarf's theorem and fixed point theory, either by proving the KKMS theorem from a standard fixed point theorem or by proceeding directly to Scarf's theorem via a different route. Kannai [128] showed that Scarf's theorem [228] is equivalent to the Brouwer theorem. Todd [260] applied the Kakutani theorem [125] to prove a special case of the KKMS theorem, which is sufficient to prove the core theorem. An easy non-constructive proof of the KKMS theorem due to Ichiishi [113] was based on a coincidence theorem of Fan [65]. Keiding and Thorlund–Peterson [130] proved the core theorem through the KKM theorem. Also, Ichiishi [141] initiated a cooperative extension of the non-cooperative game and, more systematically, in particular, his theorem [117] includes as special cases the Nash equilibrium theorem in non-cooperative game theory and Scarf's core theorem in cooperative game theory. Moreover, Ichiishi [116] obtained a dual version of the KKMS theorem using Fan's coincidence theorem, and then applied it to the core theorem.

Shapley and Vohra [240] gave proofs of both Scarf's core theorem and the KKMS theorem involving either Kakutani's fixed point theorem or Fan's coincidence theorem. Komiya [134] gave a proof of the KKMS theorem based on the Kakutani theorem, the separating hyperplane theorem, and the Berge maximum theorem. Krasa and Yannelis [135] gave a proof of the KKMS theorem by means of the Brouwer theorem, the separating hyperplane theorem, and a continuous selection theorem. Zhou [276] considered intersection theorems close to the Ichiishi theorem and the KKMS theorem. Finally, Herings [101] gave a very elementary and simple proof of the KKMS theorem using only the Brouwer theorem and some elementary calculus. This shows that the KKMS theorem and the Brouwer theorem should be regarded as "equivalent" since it is elementary to prove the Brouwer theorem using the KKMS theorem.

By an *equilibrium problem*, Blum and Oettli [23] understood the problem of finding

$$\hat{x} \in X \text{ such that } f(\hat{x}, y) \leq 0 \text{ for all } y \in X, \tag{EP}$$

where X is a given set and $f : X \times X \to \overline{\mathbf{R}}$ is a given function.

We consider more general problems as follows:

A *quasi-equilibrium problem* is to find

$$\hat{x} \in X \text{ such that } \hat{x} \in S(\hat{x}) \text{ and } f(\hat{x}, z) \leq 0 \text{ for all } z \in S(\hat{x}), \tag{QEP}$$

where X and f are as above and $S : X \multimap X$ is a given multimap.

A *generalized quasi-equilibrium problem* is to find

$$\hat{x} \in X \text{ and } \hat{y} \in T(\hat{x}) \text{ such that } \hat{x} \in S(\hat{x}) \text{ and } f(\hat{x}, \hat{y}, z) \leq 0 \text{ for all } z \in S(\hat{x}), \tag{GQEP}$$

where X and S are the same as above, Y is another given set, $T : X \multimap Y$ is another multimap, and $f : X \times Y \times X \to \overline{\mathbf{R}}$ is a given function.

These problems contain as special cases, for instance, optimization problems, problems of the Nash-type equilibrium, complementarity problems, fixed point problems, variational inequalities, minimax theorems, and many others. There are many variations or generalizations of these problems (see [152, 177] and references therein).

It should be emphasized that the main tools of various equilibrium problems are intersection theorems, fixed point theorems, and their equivalent formulations. Recently, each field of study of equilibrium problems becomes very productive with a large number of literature.

For example, the von Neumann minimax theorem and its extended versions have been applied to different branches of mathematics; and even in mathematical analysis, they have been applied to function algebras and extension theorems for nonlinear operators and inequalities. Moreover, many authors have contributed elementary proofs of various minimax theorems (say, set-theoretical or not using any equivalent form of the Brouwer theorem). Consequently, the literature on the minimax theory is by now extensive (see, for instance, [250]).

10. Convex-Valued Multimaps: 1960s–1990s

Since the 1960s, there have been many fixed point theorems generalizing the Brouwer or Kakutani theorems for single-valued or multi-valued maps defined on convex subsets of Hausdorff topological vector spaces.

For single-valued continuous maps, Fan [63] showed that Schauder's conjecture is valid for a topological vector space E on which its topological dual E^* separates points.

Halpern [97] considered new boundary conditions called outwardness and, later, inwardness, and obtained fixed point theorems for maps satisfying these conditions. For a topological vector space, E, a compact convex subset K of E, and a continuous map $f : K \to E$ satisfying certain inwardness or outwardness, generalizations of the Brouwer theorem were due to Fan [65], Halpern [97], Reich [218], Sehgal and Singh [237] and others whenever E is locally convex, and to Halpern and Bergman [99], Kaczynski [122], Roux and Singh [226], Sehgal, Singh, and Whitfield [238] whenever E^* separates points of E. In the sequel, t.v.s. means a Hausdorff topological vector space.

Kakutani's convex-valued u.s.c. multimaps are further extended as follows: For a subset X of a t.v.s. E, a map $F : X \multimap E$ is called

(i) *upper demicontinuous* (u.d.c.) if, for each $x \in X$ and open half-space H in E containing Fx, there exists an open neighborhood N of x in X such that $f(N) \subset H$ (see [65]).

(ii) *upper hemicontinuous* (u.h.c.) if, for each $h \in E^*$ and for any real α, the set $\{x \in X : \sup \operatorname{Re} h(Fx) < \alpha\}$ is open in X (see [41, 144, 180]).

(iii) *generalized u.h.c.* if, for each $p \in \{\operatorname{Re} h : h \in E^*\}$, the set $\{x \in X : \sup p(Fx) \geq p(x)\}$ is compactly closed in X (see [38, 82, 185, 188, 247, 248]).

For those maps with compact convex domains, the Kakutani theorem was extended by Browder [32], Fan [65, 66], Glebov [82], Halpern [98], Cellina [38], Reich [218, 221], Cornet [41], Lasry–Robert [144], and Simons [247, 248] for a locally convex t.v.s. E, and by Granas–Liu [90], Park [180, 185, 188] and others for a t.v.s. E on which E^* separates points.

In order to assure the existence of a fixed point of maps $f : X \to E$ or $F : X \multimap E$, we need the following:

(1) Certain continuity of the map like the generalized u.h.c.. The topology of the domain X is not necessarily the same as the relative topology of X in E.

(2) Certain compactness on X. If X is not compact, then certain compactness or coercivity condition suffices for the existence of fixed points.

(3) Certain boundary conditions. Until the mid-1960s, we only had a few such conditions, for example, that of Altman [4], Rothe [225], or the Leray–Schauder condition.

Halpern [97] first introduced the outward and, later, inward sets:

Let E be a t.v.s. and $X \subset E$. The inward and outward sets of X at $x \in E$, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X - x), \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

Let X be a non-empty convex subset of a vector space E. Following [65], the *algebraic boundary* $\delta_E(X)$ of X in E is the set of all $x \in X$ for which there exists $y \in E$ such that $x + ry \notin X$ for all $r > 0$. If E is a t.v.s., the *topological boundary* $\operatorname{Bd} X = \operatorname{Bd}_E X$ is the complement of $\operatorname{Int}_E X$ of X. It is known that $\delta_E(X) \subset \operatorname{Bd} X$ and, in general, $\delta_E(X) \neq \operatorname{Bd} X$.

A map $F : X \multimap E$ is said to be

(i) *inward* if $Fx \cap I_X(x) \neq \emptyset$ for each $x \in \operatorname{Bd} X$,
outward if $Fx \cap O_X(x) \neq \emptyset$ for each $x \in \operatorname{Bd} X$.

(ii) *weakly inward* if $Fx \cap \overline{I_X(x)} \neq \emptyset$ for each $x \in \operatorname{Bd} X$,
weakly outward if $Fx \cap \overline{O_X(x)} \neq \emptyset$ for each $x \in \operatorname{Bd} X$.

Note that, by replacing $\operatorname{Bd} X$ by $\delta_E(X)$, we can obtain more general boundary conditions.

For $p \in \{\operatorname{Re} h : h \in E^*\}$ and $U, V \subset E$, let

$$d_p(U, V) = \inf\{|p(u - v)| : u \in U, \ v \in V\}.$$

Further, motivated by the work of [121], we have the following much more general boundary conditions:

(iii) $d_p(Fx, \overline{I_X(x)}) = 0$ for each $x \in \delta_E(X)$,
$d_p(Fx, \overline{O_X(x)}) = 0$ for each $x \in \delta_E(X)$.

For this case, the domain X is not compact, and for maps $F : X \multimap E$ having certain continuity, boundary conditions, and certain compactness conditions, generalizations of the Kakutani theorem were obtained by Fan [68], Shih and Tan [242, 243], Jiang [121], Ding and Tan [50], Park [185, 188], and others.

All of the generalizations of the Brouwer and Kakutani theorems mentioned above were unified by Park [202] as follows:

A convex space X is a non-empty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A non-empty subset L of a convex space X is called a *c-compact set* if, for each finite set $S \subset X$, there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$ (see [145]).

Let $cc(E)$ denote the set of non-empty closed convex subsets of a t.v.s. E and $kc(E)$ the set of non-empty compact convex subsets of E.

The following is given in [202]:

Theorem 25. *Let X be a convex space, L a c-compact subset of X, K a non-empty compact subset of X, E a t.v.s. containing X as a subset, and F a generalized u.h.c. map satisfying either*

(A) *E^* separates points of E and $F : X \to kc(E)$; or*
(B) *E is locally convex and $F : X \to cc(E)$.*

(I) *Suppose that, for each $p \in \{\mathrm{Re}\, h : h \in E^*\}$;*
(1) *$p|_X$ is continuous on X;*
(2) *$d_p(Fx, \overline{I}_L(x)) = 0$ for every $x \in X \setminus K$; and*
(3) *$d_p(Fx, \overline{I}_X(x)) = 0$ for every $x \in K \cap \delta_E(X)$.*

Then there exists an $x \in X$ such that $x \in Fx$.

(II) *Suppose that, for each $p \in \{\mathrm{Re}\, h : h \in E^*\}$,*
(1)′ *$p|_X$ is continuous on X;*
(2)′ *$d_p(Fx, \overline{O}_L(x)) = 0$ for every $x \in X \setminus K$; and*
(3)′ *$d_p(Fx, \overline{O}_X(x)) = 0$ for every $x \in K \cap \delta_E(X)$.*

Then there exists an $x \in X$ such that $x \in Fx$. Further, if F is u.h.c., then $F(X) \supset X$.

The major particular forms of Theorem 25 can be adequately summarized by Table 1 which is an enlarged version of those in [180, 185].

In the table, class I stands for that of Euclidean spaces, II for normed vector spaces, III for locally convex Hausdorff topological vector spaces, and IV for topological vector spaces having sufficiently many linear functionals. Moreover, f stands for single-valued maps, F for set-valued maps, K for a non-empty compact convex subset of a space E, and X for a non-empty convex subset of E satisfying certain coercivity conditions with respect to $F : X \multimap E$ with certain boundary conditions.

In fact, Theorem 25 contains all the fixed point theorems in the diagram.

11. Fixed Point Theorems for Better Admissible Multimaps

It should be noted that the diagram does not contain fixed point theorems for compact maps and for condensing maps. One of the most well-known theorems generalizing the second Schauder theorem [234], Hukuhara's theorem [112], and most results on $f : K \to K$ and $F : K \multimap K$ in the diagram was the following due to Himmelberg [103]:

Theorem 26. [103] *Let X be a non-empty convex subset of a locally convex t.v.s. and $F : X \multimap X$ a compact u.s.c. map having non-empty closed convex values. Then F has a fixed point.*

This was applied to give generalizations of the von Neumann intersection lemma and the minimax theorem, and to various equilibrium problems by others.

The above theorem was extended to non-locally convex t.v.s. by Idzik [118], and to non-convex valued maps by Ben-El-Mechaiekh and Deguire [15, 16], Park [186, 187], and others.

For non-convex-valued multimaps, the author recently established the fixed point theory for "admissible" maps in very general classes of multimaps as follows:

Table 1.

E		$f:K\to K$		$F:K\multimap K$	
I	Brouwer	1912	Kakutani	1941
II	Schauder	1927, 1930	Bohnenblust and Karlin	1950
III	Tychonoff	1935	Fan Glicksberg	1952 1952
IV	Fan	1964	Granas and Liu	1986
I		$f:K\to E$		$F:K\multimap E$	
I	Knaster, Kuratowski and Mazurkiewicz	1929		
II	Rothe Halpern Fan Reich Sehgal and Singh	1937 1965 1969 1972 1983	Browder Fan Glebov Halpern	1968 1969, 1972 1969 1970
III			Cellina Reich Cornet Lasry and Robert Simons	1970 1972, 1978 1975 1975 1986
IV	Halpern and Bergman Kaczynski Roux and Singh Sehgal, Singh, and Whitfield	1968 1983 1989 1990	Granas and Liu Park	1986 1988, 1991
				$F:X\multimap E$	
II			Ding and Tan	1992
III			Fan Shih and Tan Jiang	1984 1987, 1988 1988
IV			Park	1992, 1993

In a t.v.s. E, any convex hulls of its finite subsets will be called *polytopes*.

Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X, Y)$ denotes the set of maps $F : X \multimap Y$ belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in $\mathbf{X}$.

A class $\mathcal{A}$ of maps is defined by the following properties:

(i) $\mathcal{A}$ contains the class $\mathbf{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathcal{A}_c$ is u.s.c. and non-empty compact-valued; and
(iii) for any polytope P, $F \in \mathcal{A}_c(P, P)$ has a fixed point, where the intermediate spaces are suitably chose

Examples of $\mathcal{A}$ are $\mathbf{C}$, the Kakutani maps $\mathbf{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbf{M}$ (with R_δ values), the acyclic maps $\mathbf{V}$ (with acyclic values), the Powers maps $\mathbf{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbf{N}$ (continuous with values consisting of one or m acyclic components, where m is fixed), the approachable maps $\mathbf{A}$ (whose domains and codomains are uniform spaces), admissible maps in the sense of Górniewicz [84], σ-selectionable maps of Haddad and Lasry [94], permissible maps of Dzedzej [54], and many others.

We introduce two more classes:

(1) $F \in \mathcal{A}_c^\sigma(X, Y) \Leftrightarrow$ for any σ-compact subset K of X, there is an $\tilde{F} \in \mathcal{A}_c(K, Y)$ such that $\tilde{F}x \subset Fx$ for each $x \in K$.

(2) $F \in \mathcal{A}_c^\kappa(X, Y) \Leftrightarrow$ for any compact subset K of X, there is an $\tilde{F} \in \mathcal{A}_c(K, Y)$ such that $\tilde{F}x \subset Fx$ for each $x \in K$.

Note that $\mathcal{A} \subset \mathcal{A}_c \subset \mathcal{A}_c^\sigma \subset \mathcal{A}_c^\kappa$. Any class belonging to $\mathcal{A}_c^\kappa$ is called *admissible*. These classes are all due to the author in his earlier works. Examples of $\mathcal{A}_c^\sigma$ are $\mathbf{K}_c^\sigma$ due to Lassonde [147], $\mathbf{V}_c^\sigma$ due to Park, Singh, and Watson [207] and others. Note that $\mathbf{K}_c^\sigma$ contains classes $\mathbf{K}$, the Fan–Browder-type maps, $\mathbf{T}$ in [147], approximable maps of Ben-El-Mechaiekh and Idzik [20], and many others. (For details, see [188, 189, 191, 199, 203]).

Motivated by the admissible class, for a convex space X and a topological space Y, Chang and Yen [39] defined the class of maps $T : X \multimap Y$ having the KKM property as follows:

(3) $T \in KKM(X, Y) \Leftrightarrow$ the family $\{Sx : x \in X\}$ has the finite intersection property whenever $S : X \multimap Y$ has closed values and $T(\mathrm{co}N) \subset S(N)$ for each finite subset N of X.

For a convex space X, it was noted that $\mathcal{A}_c^\kappa(X, Y) \subset KKM(X, Y)$ [191].

In order to improve the admissible class, Park [198] introduced the *better admissible class* $\mathcal{B}$ as follows:

(4) $F \in \mathcal{B}(X, Y) \Leftrightarrow F : X \multimap Y$ is a map such that, for any polytope P in X and any continuous map $f : F(P) \to P$, $f(F|_P) : P \multimap P$ has a fixed point.

(5) $F \in \mathcal{B}^\sigma(X, Y) \Leftrightarrow F : X \multimap Y$ is a map such that, for any σ-compact convex subset K of X, there is a closed map $\Gamma \in \mathcal{B}(K, Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

(6) $F \in \mathcal{B}^\kappa(X, Y) \Leftrightarrow F : X \multimap Y$ is a map such that, for any compact convex subset K of X, there is a closed map $\Gamma \in \mathcal{B}(K, Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

It is noted that, in the class of closed compact maps, two subclasses $\mathcal{B}$ and KKM coincide (see [198]). These classes of multimaps were used to generalize the KKM theory and the fixed point theory.

A non-empty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [132]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite-dimensional subspace L of E.

Note that every non-empty convex subset of a locally convex t.v.s. is admissible (see [112, 169]). Other examples of admissible t.v.s. are l^p, L^p, the Hardy spaces H^p for $0 < p < 1$, the space $S(0, 1)$ of equivalence classes of measurable functions on $[0, 1]$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others. Moreover, any locally convex subset of an F-normable t.v.s. is admissible, and every compact convex locally convex subset of a t.v.s. is admissible. Note that an example of a non-admissible, non-convex compact subset of the Hilbert space l^2 is known. (For details, see [95, 132, 268, 269] and references therein.)

Theorem 27. *Let X be an admissible convex subset of a t.v.s.*

(1) *If* $F \in \mathcal{B}^\kappa(X, X)$ *is compact, then* F *has a fixed point*;
(2) *If* $F \in \mathcal{B}(X, X)$ *is closed and compact, then* F *has a fixed point.*

This is recently due to Park [196, 197] and subsumes more than 60 particular forms including Ben-El-Mechaiekh et al. [13–15, 17–19], Bohnenblust and Karlin [25], Himmelberg [103], Hukuhara [112], Lassonde [145, 146], Mazur [163], Park et al. [186, 193, 195, 207], Powers [215], Rhee [223], Schauder [234], Simons [249], and Singbal [251].

There is another way of extending compact maps in certain situations using (generalizations of) the Kuratowski or other measures of non-compactness. This study was initiated by Darbo [44] (see the end of Sec. 5). In this direction we also have a very general theorem.

Let X be a closed convex subset of a t.v.s. E and C a lattice with a least element, which is denoted by 0. A function $\Phi : 2^X \to C$ is called a *measure of non-compactness* on X provided that the following conditions hold for any $A, B \in 2^X$:

(1) $\Phi(A) = 0$ if and only if A is relatively compact;
(2) $\Phi(\overline{\text{co}}\, A) = \Phi(A)$, where $\overline{\text{co}}$ denotes the convex closure of A; and
(3) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$.

The above notion is a generalization of the set-measure γ and the ball-measure χ of non-compactness defined either in terms of a family of seminorms or a norm. (For details, see [209, 210].)

Given a measure Φ of non-compactness on E and $X \subset E$, a map $T : X \multimap E$ is called Φ-*condensing* provided that, if $A \subset X$ and $\Phi(A) \leq \Phi(T(A))$, then A is relatively compact, that is, $\Phi(A) = 0$.

Note that each map defined on a compact set or each compact map is Φ-condensing. Especially, if E is locally convex, then a compact map $T : X \multimap E$ is γ- or χ-condensing whenever X is complete or E is quasi-complete.

The following is due to Park [196, 199]:

Theorem 28. *Let* X *be a closed convex subset of a locally convex t.v.s.* E*. Then any* Φ*-condensing map* $F \in \mathcal{B}^\kappa(X, X)$ *has a fixed point.*

This theorem extends earlier results of Daneš [42], Darbo [44], Ewert [56], Furi and Vignoli [76], Himmelberg, Porter, and Van Vleck [104], Lifšic and Sadovskii [150], Nussbaum [178], Petryshyn and Fitzpatrick [209, 210], Reich [217–219], Sadovskii [227], Su and Sehgal [256], and Tarafdar and Výborný [259].

Let C, D be subsets of a t.v.s. E, $T \in \mathcal{A}_c(C, D)$, and $\mathcal{M}$ the class of non-empty compact subsets of D consisting of the functional values of maps in $\mathcal{A}$. We say that T satisfies the *Schöneberg condition* if

$$tM \in \mathcal{M} \text{ for } t \in [0, 1] \text{ and } M \in \mathcal{M} \qquad \text{(Sö)}$$

holds (see [235]). For example, $\mathcal{M}$ can be the class of convex sets for $\mathcal{A} = \mathbf{K}$, acyclic sets for $\mathcal{A} = \mathbf{V}$, R_δ sets $\{X = \cap X_i : X_{i+1} \subset X_i, X_i \in \text{AR compact}, i \in N\}$ for $\mathcal{A} = \mathbf{M}$, and others.

For $U \subset D$, let $\text{Cl}_D U$ denote the closure of U in D and $\text{Bd}_D U$ the boundary of U in D. On the other hand, $\overline{}$ and Bd will denote the closure and boundary in the whole space E.

Now, we give some fixed point theorems due to [197] for compact maps satisfying the so-called Leray–Schauder condition:

Theorem 29. *Let D be a convex subset of a locally convex t.v.s., $0 \in D$, $U \subset D$ a neighborhood of* 0 (in D)*, and $F \in \mathcal{A}_c(\mathrm{Cl}_D U, D)$ a compact map satisfying* (Sö). *If*

$$Fy \cap \{ry : r > 1\} = \emptyset \text{ for all } y \in \mathrm{Bd}\, U, \tag{LS}$$

then the set of fixed points of F in $\mathrm{Cl}_D U$ is non-empty and compact.

This improves, unifies, and extends the results of Altman [4], Brouwer [29], Eilenberg and Montgomery [55], Furi and Martelli [75], Górniewicz, Granas, and Kryszewski [85], Granas [89], Hahn [96], Kaczynski and Wu [123], Kaniel [127], Knaster, Kuratowski, and Mazurkiewicz [133], Krasnoselskii [136], Leray and Schauder [149], Ma [155], Martelli [158], Petryshyn and Fitzpatrick [209], Potter [214], Powers [215], Reich [218, 220, 222], Rothe [225], Shinbrot [244], Su and Sehgal [256], and Yamamuro [270].

For Φ-condensing maps, we have the following in [197]:

Theorem 30. *Let D be a closed convex subset of a t.v.s. E on which E^* separates points, $0 \in D$, $U \subset D$ a neighborhood of* 0 *(in D), and $F \in \mathcal{A}_c(\mathrm{Cl}_D U, D)$ a Φ-condensing map satisfying* (Sö). *If the condition* (LS) *holds, then the set of fixed points of F in $\mathrm{Cl}_D U$ is non-empty and compact.*

This includes the results of Fitzpatrick and Petryshyn [70], Gatica and Kirk [80, 81], Lin [151], Martelli [158, 159], Petryshyn [208], Petryshyn and Fitzpatrick [209, 210], Reich [217, 220, 222], Su and Sehgal [256], and many others.

These Leray–Schauder-type theorems due to [197] are applied to

(i) the so-called Leray–Schauder principles of Browder [31], Leray and Schauder [149], Petryshyn and Fitzpatrick [209], Potter [214], and Schöneberg [235];
(ii) the Schaefer-type theorems due to Górniewicz, Granas, and Kryszewski [85], Granas [89], Martelli [159], Martelli and Vignoli [160], Reich [217, 218], Schaefer [231, 232], and Seda [236]; and
(iii) the Birkhoff–Kellogg-type theorems due to Birkhoff and Kellogg [22], Fournier and Martelli [72], Martelli [159], and Yamamuro [270].

12. Generalized Convex Spaces

In the last decade, there have also been advances in the KKM theory. Recently, Park [190, 191] obtained far-reaching generalizations of the KKM theorem, the Fan–Browder theorem, a matching theorem, an analytic alternative, the Ky Fan minimax inequalities, section properties of convex spaces, and other fundamental theorems in the theory from coincidence theorems on compositions of admissible maps. These new results extend, improve, and unify main theorems in more than 100 published works.

On the other hand, the concept of convex sets in a t.v.s. was extended to convex spaces by Lassonde [145], and further to C-spaces by Horvath [106–110]. Other authors also extended the concept of convexity for various purposes. Recently, Park and Kim [203, 205, 206] unified these concepts and introduced generalized convex spaces or

G-convex spaces. For these spaces, the foundations of the KKM theory with respect to admissible maps were established by Park and Kim [205], and some general fixed point theorems were obtained by Kim [131] and Park [201].

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a non-empty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that, for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\Phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\Phi_A(\Delta_J) \subset \Gamma(J)$. Note that $\Phi_A|_{\Delta_J}$ can be regarded as Φ_J.

Here, $\langle D \rangle$ denotes the set of all non-empty finite subsets of D, $\Delta_n = \text{co}\{e_i\}_{i=0}^n$ the standard n-simplex, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$, that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$ and $(X, \Gamma) = (X, X; \Gamma)$. (For details on G-convex spaces, see [204–206], where basic theory was extensively developed.)

There are many examples of G-convex spaces.

Example 1. If $X = D$ is a convex subset of a vector space and each Γ_A is the convex hull of $A \in \langle X \rangle$ equipped with the Euclidean topology, then (X, Γ) becomes a convex space due to Lassonde [14]. Note that any convex subset of a topological vector space is a convex space, but not conversely.

Example 2. If $X = D$ and Γ_A is assumed to be contractible or, more generally, infinitely connected (that is, n-connected for all $n \geq 0$) and if, for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then (X, Γ) becomes a C-space (or an H-space) due to Horvath [110].

Example 3. Other major examples of G-convex spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joo's pseudoconvex spaces, and so on (see [203, 204]). Recently, we found a number of new examples of G-convex spaces (see [200]). In particular, any continuous image of a G-convex space is a G-convex space, and any almost convex subset of a t.v.s. (see [103]) is a G-convex space.

For a G-convex space $(X, D; \Gamma)$, a map $F : D \multimap X$ is called a G-KKM map if $\Gamma_N \subset F(N)$ for each $N \in \langle D \rangle$. Now, the KKM theory is extended to the study of G-KKM maps on G-convex spaces. The following is basic in this theory:

Theorem 30. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, $S : D \multimap Y$, $T : X \multimap Y$ maps, and $F \in \mathcal{A}_c^\kappa(X, Y)$. Suppose*

(1) *for each $x \in D$, Sx is compactly open in Y;*
(2) *for each $y \in F(X)$, $M \in \langle S^- y \rangle$ implies $\Gamma_M \subset T^- y$;*
(3) *there exists a non-empty compact subset K of Y such that $\overline{F(X)} \cap K \subset S(D)$; and*
(4) *either*
  (i) *$Y \setminus K \subset S(M)$ for some $M \in \langle D \rangle$; or*
  (ii) *for each $N \in \langle D \rangle$, there exists a compact G-convex subset L_N of X containing N such that $F(L_N) \setminus K \subset S(L_N \cap D)$.*

Then there exists an $\bar{x} \in X$ such that $F\bar{x} \cap T\bar{x} \neq \emptyset$.

This was due to Park and Kim [204, 205], which was reformulated to more than a dozen foundational results in the KKM theory. The class $\mathcal{A}_c^\kappa$ in the above theorem can be replaced by the extended class $\mathcal{B}$ for G-convex spaces.

Finally, some fixed point theorems for the possible extended class $\mathcal{B}$ on G-convex spaces have appeared (see, for example, [201]).

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