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Survey

Explicit Global Lipschitz Solutions to First-Order, Nonlinear Partial Differential Equations*

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Dedicated to Professor Le Van Thiem on the occasion of his 80th birthday

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Abstract. This paper presents some explicit formulas for global Lipschitz solutions of the Cauchy problem for first-order, nonlinear partial differential equations. The method used here is based on the technique of multivalued functions.

1. Introduction

The aim of this work is to present some formulas for explicit global Lipschitz solutions of the Cauchy problem for Hamilton–Jacobi equations of the form

$$\frac{\partial u}{\partial t} + f\left(t, \frac{\partial u}{\partial x}\right) = 0 \ \text{ in } \{t > 0,\ x \in \mathbb{R}^n\}, \tag{1.1}$$

$$u(0, x) = \phi(x) \ \text{ on } \{t = 0,\ x \in \mathbb{R}^n\}. \tag{1.2}$$

It is well-known that the Cauchy problem (1.1)–(1.2) has a locally unique C^2-solution if the Hamiltonian $f = f(t, p)$ and initial function $\phi = \phi(x)$ are of class C^2. However, there is generally no possibility of finding a global classical solution. One therefore needs to introduce a notion of generalized solutions and to develop theory and methods for constructing these solutions. During the past five decades, many mathematicians have obtained various global results by relaxing the smoothness conditions on the solutions. In particular, the global existence and uniqueness of (generalized) solutions for convex Hamilton–Jacobi equations were well studied by several approaches.

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If the Hamiltonian $f = f(p)$ is continuous and the initial function $\phi = \phi(x)$ is global Lipschitz continuous and convex with the Fenchel conjugate $\phi^* = \phi^*(p)$, Hopf [7] proved in 1965 that the formula

$$u(t,x) = \max_{p\in\mathbb{R}^n}\{\langle p, x\rangle - \phi^*(p) - tf(p)\} \tag{1.3}$$

determines a (generalized) solution of the Cauchy problem (1.1)–(1.2) in the sense that this solution satisfies (1.1) at every point where it is differentiable. Since the solution is locally Lipschitz continuous, the well-known Rademacher theorem [11, Theorem 1.18] shows that (1.1) is then satisfied almost everywhere.

If the Hamiltonian $f = f(p)$ is strictly convex with $\lim_{|p|\to+\infty} f(p)/|p| = +\infty$ and the initial function $\phi = \phi(x)$ is globally Lipschitz continuous, Hopf [7] also established

$$u(t,x) = \min_{y\in\mathbb{R}^n}\left\{\phi(y) + t\cdot f^*\left(\frac{x-y}{t}\right)\right\}. \tag{1.4}$$

These formulas are often associated with the name of Hopf, although (1.4) was actually first discovered for $n = 1$ by Lax [9] in 1957.

Step by step, certain more general cases of Hopf's formula (1.3) will thoroughly be dealt with in this paper under a standing hypothesis like (but somewhat stricter than) Carathéodory's condition on the Hamiltonian $f = f(t, p)$. Section 2 concerns the case of convex (but not necessarily global Lipschitz continuous) initial data. In Sec. 3, we consider the Cauchy problem with non-convex initial data: First, for the case where $\phi = \phi(x)$ can be represented as minimum of a family of convex functions, and second, for the case where $\phi = \phi(x)$ is a d.c. function (i.e., it can be represented as the difference of two convex functions). Finally, Sec. 4 discusses Hopf's formula (1.4) in case $\phi = \phi(x)$ is just continuous.

Most of the results presented here were originally published in [15–17]. Some of them have been revised and updated. Some materials are presented here for the first time. (For other results in the field, see, for example, [3–6, 10, 13, 14].) Our method is based on some techniques of multifunctions and convex functions.

Throughout, we use $\mathcal{D}$ to indicate the set $\{0 < t < +\infty,\ x \in \mathbb{R}^n\}$. Moreover, for any $G \subset \mathbb{R}$, put $\mathcal{D}_G \overset{\text{def}}{=} ((0,+\infty)\setminus G)\times\mathbb{R}^n = \{(t,x)\in\mathcal{D} : t \notin G\}$. The notation $\partial/\partial x$ will denote the gradient $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Let $|\cdot|$ and $\langle .,. \rangle$ be the Euclidean norm and scalar product in $\mathbb{R}^n$, respectively. Further, we define $\mathrm{Lip}(\overline{\mathcal{D}}) \overset{\text{def}}{=} \mathrm{Lip}(\mathcal{D}) \cap C(\overline{\mathcal{D}})$, where $\mathrm{Lip}(\mathcal{D})$ is the set of all locally Lipschitz continuous functions $u = u(t,x)$ defined on $\mathcal{D}$.

Definition 1.1. *A function $u = u(t,x)$ in $\mathrm{Lip}(\overline{\mathcal{D}})$ is called a global Lipschitz solution of the Cauchy problem (1.1)–(1.2) if it satisfies (1.1) almost everywhere in $\Omega_T \overset{\text{def}}{=} \{(t,x), 0 < t < T, x \in \mathbb{R}^n\}$ and if $u(0,x) = \phi(x)$ for all $x \in \mathbb{R}^n$.*

2. The Cauchy Problem with Convex Initial Data

In this section, we consider the Cauchy problem (1.1)–(1.2), with $\phi = \phi(x)$ a finite convex function on $\mathbb{R}^n$. Denote by $\phi^* = \phi^*(p)$ the Fenchel conjugate function of $\phi = \phi(x)$,

$$\phi^*(p) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}^n} \{\langle p, x \rangle - \phi(x)\} \text{ for } p \in \mathbb{R}^n,$$

and by E the effective domain of $\phi^* = \phi^*(p)$,

$$E \stackrel{\text{def}}{=} \operatorname{dom} \phi^* = \{p \in \mathbb{R}^n : \phi^*(p) < +\infty\}.$$

We assume the following two hypotheses:

(E.I) The Hamiltonian $f = f(t, p)$ is continuous in $\{(t, p) : t \in (0, +\infty) \setminus G,\ p \in \mathbb{R}^n\}$ for some closed set $G \subset \mathbb{R}$ of Lebesgue measure 0. Moreover, to each positive number N, there corresponds a function $g_N = g_N(t)$ in $L^\infty_{\text{loc}}(\mathbb{R})$ such that

$$\sup_{|p| \le N} |f(t, p)| \le g_N(t) \text{ for almost all } t \in (0, +\infty).$$

(E.II) For every bounded subset V of $\overline{\mathcal{D}}$, there exists a positive number $N(V)$ so that

$$\langle p, x \rangle - \phi^*(p) - \int_0^t f(\tau, p) d\tau < \max_{|q| \le N(V)} \{\langle q, x \rangle - \phi^*(q) - \int_0^t f(\tau, q) d\tau\} \quad (2.1)$$

whenever $(t, x) \in V$, $|p| > N(V)$.

Hypothesis (E.I) implies the t-measurability and p-continuity of $f = f(t, p)$ on $\{t > 0,\ p \in \mathbb{R}^n\}$. Moreover, since $\phi = \phi(x)$ is finite on $\mathbb{R}^n$, this hypothesis allows us to define an upper semicontinuous function $\varphi = \varphi(t, x, p)$ from $\overline{\mathcal{D}} \times \mathbb{R}^n$ into $[-\infty, +\infty)$ by

$$\varphi(t, x, p) \stackrel{\text{def}}{=} \langle p, x \rangle - \phi^*(p) - \int_0^t f(\tau, p) d\tau, \quad (2.2)$$

which, for each $p \in E$, is actually finite and continuous in (t, x) on $\overline{\mathcal{D}}$.

The next theorem will be fundamental in this section.

Theorem 2.1. *Let $\phi = \phi(x)$ be a finite convex function on $\mathbb{R}^n$. Assume (E.I)–(E.II) hold. Then a global Lipschitz solution $u = u(t, x)$ of the Cauchy problem (1.1)–(1.2) is given by*

$$\begin{aligned} u(t, x) &\stackrel{\text{def}}{=} \sup_{p \in \mathbb{R}^n} \varphi(t, x, p) \\ &= \sup_{p \in \mathbb{R}^n} \left\{\langle p, x \rangle - \phi^*(p) - \int_0^t f(\tau, p) d\tau\right\} \textit{ for } (t, x) \in \overline{\mathcal{D}}. \end{aligned} \quad (2.3)$$

Remark 2.1. The requirement (E.II) could be in a sense regarded as a compatible condition between Hamiltonian and initial data for the existence of global Lipschitz solutions of the Cauchy problem (1.1)–(1.2). To see this, we first rewrite (E.II) in an alternative form that is essentially equivalent (by a standard compactness argument) but seemingly more amenable to verification:

(E.II)$'$ For every $(t^0, x^0) \in \overline{\mathcal{D}}$, there exist positive numbers $r(t^0, x^0)$ and $N(t^0, x^0)$ so that

$$\varphi(t, x, p) < \max_{|q| \le N(t^0, x^0)} \varphi(t, x, q) \textit{ whenever } (t, x) \in V(t^0, x^0),\ |p| > N(t^0, x^0), \tag{2.1$'$}$$

where $V(t^0, x^0) \stackrel{\text{def}}{=} \{(t, x) \in \overline{\mathcal{D}} : |t - t^0| + |x - x^0| < r(t^0, x^0)\}$.

We proceed now to consider, for example, the Cauchy problem

$$\frac{\partial u}{\partial t} - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = 0 \text{ in } \{t > 0,\ x \in \mathbb{R}\},$$

$$u(0, x) = \frac{x^2}{2} \text{ on } \{t = 0,\ x \in \mathbb{R}\}.$$

Then the method of characteristics gives the unique classical solution

$$u = u(t, x) \stackrel{\text{def}}{=} -\frac{x^2}{2(t-1)} \tag{2.4}$$

in $\{0 \le t < 1,\ x \in \mathbb{R}\}$. This solution cannot be extended continuously beyond the time $t = 1$. We should note that in this case the set of all $(t^0, x^0) \in \overline{\mathcal{D}}$ such that (2.1$'$) holds for some $r(t^0, x^0) > 0$ and $N(t^0, x^0) > 0$ is precisely the domain $\{0 \le t < 1,\ x \in \mathbb{R}\}$. Moreover, if we try to apply Hopf's formula (1.3), ignoring the fact that the initial function here, $\phi = x^2/2$, is not globally Lipschitz continuous, then we also obtain the same solution as (2.4) in $\{0 \le t < 1,\ x \in \mathbb{R}\}$.

Remark 2.2. Let $\phi = \phi(x)$ be a finite convex function on $\mathbb{R}^n$. Assume (E.I). Then (E.II) is satisfied if $\varphi(t, x, p)$ tends to $(-\infty)$ locally uniformly in $(t, x) \in \overline{\mathcal{D}}$ as $|p|$ tends to $(+\infty)$, i.e., if the following holds:

(E.II)$''$ For any $\lambda \in \mathbb{R}$ and any bounded subset V of $\overline{\mathcal{D}}$, there exists positive number $N(\lambda, V)$ so that $\varphi(t, x, p) < \lambda$ whenever $(t, x) \in V$, $|p| > N(\lambda, V)$.

Indeed, fix an arbitrary $q^0 \in E \stackrel{\text{def}}{=} \operatorname{dom} \phi^* \ne \emptyset$ [12, Theorem 12.2]. Since, the finite function $\overline{\mathcal{D}} \ni (t, x) \mapsto \varphi(t, x, q^0)$ is continuous, it follows that

$$\lambda_V \stackrel{\text{def}}{=} \inf_{(t,x) \in V} \varphi(t, x, q^0) > -\infty$$

for any bounded subset V of $\overline{\mathcal{D}}$. Under hypothesis (E.II)$''$, we find a finite number $N_V \ge |q^0|$ (for each such V) so that

$$\varphi(t, x, p) < \lambda_V \le \varphi(t, x, q^0) \le \max_{|q| \le N_V} \varphi(t, x, q)$$

as $(t, x) \in V$ and $|p| > N_V$, thus obtaining the validity of (E.II).

Remark 2.3. The condition "$g_N = g_N(t) \in L^{\infty}_{\rm loc}$" in the hypothesis (E.I) could not be replaced by "$g_N = g_N(t) \in L^1_{\rm loc}$". To see this, consider the following Cauchy problem:

$$\frac{\partial u}{\partial t} + \frac{\partial u/\partial x}{|t-1|^{1/2}} = 0 \ \text{ in } \{t > 0,\ x \in \mathbb{R}\},$$
$$u(0, x) = x \ \text{ on } \{t = 0,\ x \in \mathbb{R}\}.$$

Here, $n \overset{\text{def}}{=} 1$,

$$f(t, p) \overset{\text{def}}{=} \frac{p}{|t-1|^{1/2}} \ \text{ and } \ \phi(x) \overset{\text{def}}{=} x$$

for $x \in \mathbb{R}$, $p \in \mathbb{R}$, and $t > 0$. Then

$$\phi^*(p) = \begin{cases} +\infty & \text{if } p \neq 1, \\ 0 & \text{if } p = 1, \end{cases}$$

hence, (E.II) holds when $N(V) \geq 1$. All the assumptions of Theorem 2.1 with $L^1_{\rm loc}$ in place of $L^{\infty}_{\rm loc}$ for hypothesis (E.I) are therefore satisfied here. However, in this case, (2.3) gives the function

$$u = u(t, x) \overset{\text{def}}{=} x\ 2 - 2|t-1|^{1/2}\,\text{sign}(t-1),$$

which is not Lipschitz continuous in any neighborhood of a point $(1, x^0)$.

For the proof of Theorem 2.1, we need some preparations. We first recall that a directional derivative is defined as follows: Let $\psi = \psi(\xi)$ be a finite-valued function ξ near a point $\xi^0 \in \mathbb{R}^m$ and let $0 \neq e \in \mathbb{R}^m$. Denote

$$\partial_e^+ \psi(\xi^0) \overset{\text{def}}{=} \limsup_{\delta \downarrow 0} \frac{\psi(\xi^0 + \delta e) - \psi(\xi^0)}{\delta}$$

and

$$\partial_e^- \psi(\xi^0) \overset{\text{def}}{=} \liminf_{\delta \downarrow 0} \frac{\psi(\xi^0 + \delta e) - \psi(\xi^0)}{\delta}.$$

If $+\infty > \partial_e^+ \psi(\xi^0) = \partial_e^- \psi(\xi^0) > -\infty$ for all non-zero $e \in \mathbb{R}^m$, then $\psi = \psi(\xi)$ is said to be *directionally differentiable at* ξ^0, and $\partial_e \psi(\xi^0) \overset{\text{def}}{=} \partial_e^+ \psi(\xi^0) = \partial_e^- \psi(\xi^0)$ will be called its *derivative at* ξ^0 *in the direction e.*

Lemma 2.2. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^m$ and $\omega = \omega(\xi, p)$ an upper semicontinuous function from $\mathcal{O} \times \mathbb{R}^n$ to $[-\infty, +\infty)$ with the following two properties:*

(i) *there exists a non-empty set $E \subset \mathbb{R}^n$ such that $\omega = \omega(\xi, p)$ is finite on $\mathcal{O} \times E$ and that $\omega(\xi, p)|_{(\xi,p)\in\mathcal{O}\times E^c} \equiv -\infty$ where $E^c \overset{\text{def}}{=} \mathbb{R}^n \setminus E$. Moreover, to each bounded subset V of $\mathcal{O}$, there corresponds a positive number $N(V)$ so that*

$$\omega(\xi, p) < \max_{|q| \leq N(V)} \omega(\xi, q) \ \text{ whenever } \ \xi \in V,\ |p| > N(V);$$

(ii) *For every fixed $p \in E$, the function $\omega = \omega(\xi, p)$ is differentiable in $\xi \in \mathcal{O}$. Besides that, the gradient $\partial\omega/\partial\xi = \partial\omega(\xi, p)/\partial\xi$ is continuous on $\mathcal{O} \times E$.*

Then

(a) $\psi = \psi(\xi) \stackrel{\text{def}}{=} \sup\{\omega(\xi, p) : p \in \mathbb{R}^n\}$ *is a locally Lipschitz continuous function in the domain $\mathcal{O}$;*

(b) $\psi = \psi(\xi)$ *is directionally differentiable in $\mathcal{O}$ with*

$$\partial_e\psi(\xi) = \max_{p\in L(\xi)} \left\langle \partial\omega\frac{(\xi, p)}{\partial\xi}, e \right\rangle \quad (\xi \in \mathcal{O},\ 0 \neq e \in \mathbb{R}^m),$$

where

$$L(\xi) \stackrel{\text{def}}{=} \{p \in \mathbb{R}^n : \omega(\xi, p) = \psi(\xi)\} \subset E. \tag{2.5}$$

We shall also need the following:

Lemma 2.3. *Let property (i) in Lemma 2.2 hold for a given function $\omega = \omega(\xi, p)$ which we assume to be continuous in $\xi \in \mathcal{O}$ (whenever $p \in E$) and upper semicontinuous with respect to the whole (ξ, p) on $\mathcal{O} \times \mathbb{R}^n$. Then (2.5) determines a non-empty-valued, closed, and locally bounded multifunction $L = L(\xi)$ of $\xi \in \mathcal{O}$.*

Remark 2.4. Lemma 2.3 implies that $L = L(\xi)$ is a compact-valued and upper semicontinuous multifunction. In this lemma, the set $\mathcal{O} \subset \mathbb{R}^m$ is not necessarily open.

Proof of Lemma 2.3. For any bounded subset V of $\mathcal{O}$, denote $N_V \stackrel{\text{def}}{=} N(V)$ (see (i)), then $L(\xi) \subset \overline{B}(0, N_V)$ as $\xi \in V$. This means that $L = L(\xi)$ is locally bounded. Moreover, with $\omega = \omega(\xi, p)$ being upper semicontinuous, we can deduce (also from (i)) that the supremum $\psi(\xi) = \sup\{\omega(\xi, p) : p \in \mathbb{R}^n\}$ should always be finite and attained. Consequently, $L = L(\xi) \subset E$ is a non-empty-valued multifunction of $\xi \in \mathcal{O}$.

To complete the proof, one need only check the closedness of $L = L(\xi)$. For this purpose, let $\{(\xi^k, p^k)\}_{k=1}^{+\infty}$ be a sequence convergent to a point (ξ^0, p^0) in $\mathcal{O} \times \mathbb{R}^n$ such that $p^k \in L(\xi^k)$ as $k = 1, 2, 3, \ldots$. By the definitions of $\psi = \psi(\xi)$ and $L = L(\xi)$, we have

$$\omega(\xi^k, p^k) \geq \omega(\xi^k, p) \ \text{ for all } p, k. \tag{2.6}$$

Since $\omega = \omega(\xi, p)$ is upper semicontinuous in (ξ, p) and continuous in ξ, (2.6) shows that

$$\omega(\xi^0, p^0) \geq \limsup_{k\to+\infty} \omega(\xi^k, p^k) \geq \lim_{k\to+\infty} \omega(\xi^k, p) = \omega(\xi^0, p) \ \text{ for all } p.$$

Thus, $p^0 \in L(\xi^0)$, and the multifunction $L = L(\xi)$ is therefore closed. ■

Proof of Lemma 2.2. (a) Let V be an arbitrary compact convex subset of $\mathcal{O}$ and let $N \stackrel{\text{def}}{=} N(V)$. For any two points $\xi^1, \xi^2 \in V$, pick up an element p^1 in the non-empty set $L(\xi^1) \subset E \cap \overline{B}(0, N)$ (cf. Lemma 2.3). Then

$$\psi(\xi^1) - \psi(\xi^2) \leq \omega(\xi^1, p^1) - \omega(\xi^2, p^1),$$

hence, the mean-value theorem gives

$$\psi(\xi^1) - \psi(\xi^2) \le \left| \partial\omega \frac{(\xi^*, p^1)}{\partial\xi} \right| \cdot |\xi^1 - \xi^2| \le \lambda\,|\xi^1 - \xi^2|,$$

where $\xi^* \in [\xi^1, \xi^2] \subset V$ and λ is a (finite) upper bound of $|\partial\omega(\xi, p)/\partial\xi|$ over $(\xi, p) \in V \times \overline{B}(0, N)$. Analogously, $\psi(\xi^2) - \psi(\xi^1) \le \lambda\,|\xi^1 - \xi^2|$. The function $\psi = \psi(\xi)$ is thus locally Lipschitz continuous in $\mathcal{O}$.

(b) For any $\xi^0 \in \mathcal{O}$ and $0 \ne e \in \mathbb{R}^m$, we find two sequences $\{\alpha_k\}_{k=1}^{+\infty}$, $\{\beta_k\}_{k=1}^{+\infty}$ of positive numbers convergent to zero such that

$$\partial_e^- \psi(\xi^0) = \lim_{k\to+\infty} \frac{\psi(\xi^0 + \alpha_k e) - \psi(\xi^0)}{\alpha_k}$$

and

$$\partial_e^+ \psi(\xi^0) = \lim_{k\to+\infty} \frac{\psi(\xi^0 + \beta_k e) - \psi(\xi^0)}{\beta_k}.$$

Let us take an arbitrary $p \in L(\xi^0 \subset E$ and apply the mean-value theorem to obtain

$$\frac{\psi(\xi^0 + \alpha_k e) - \psi(\xi^0)}{\alpha_k} \ge \frac{\omega(\xi^0 + \alpha_k e, p) - \omega(\xi^0, p)}{\alpha_k} = \left\langle \partial\omega \frac{(\xi^0 + \overline{\alpha}_k e, p)}{\partial\xi}, e \right\rangle,$$

where $\overline{\alpha}_k \in (0, \alpha_k)$. A passage to the limit as $k \to +\infty$ shows that

$$\partial_e^- \psi(\xi^0) \ge \left\langle \partial\omega \frac{(\xi^0, p)}{\partial\xi}, e \right\rangle$$

for any $p \in L(\xi^0)$. Hence

$$\partial_e^- \psi(\xi^0) \ge \sup_{p\in L(\xi^0)} \left\langle \partial\omega \frac{(\xi^0, p)}{\partial\xi}, e \right\rangle. \tag{2.7}$$

Now choose an element $p^k \in L(\xi^0 + \beta_k e)$ for each $k = 1, 2, \ldots$. Since the multifunction $L = L(\xi)$ is closed and locally bounded (Lemma 2.3), by taking a subsequence if necessary, we can assume that $p^k \underset{(k\to+\infty)}{\longrightarrow} p^0 \in L(\xi^0)$. Therefore, a passage to the limit (similar to the above) in the inequality

$$\frac{\psi(\xi^0 + \beta_k e) - \psi(\xi^0)}{\beta_k} \le \frac{\omega(\xi^0 + \beta_k e, p^k) - \omega(\xi^0, p^k)}{\beta_k} = \left\langle \partial\omega \frac{(\xi^0 + \overline{\beta}_k e, p^k)}{\partial\xi}, e \right\rangle$$

(where $\overline{\beta}_k \in (0, \beta_k)$) gives

$$\partial_e^+ \psi(\xi^0) \le \left\langle \partial\omega \frac{(\xi^0, p^0)}{\partial\xi}, e \right\rangle \le \sup_{p\in L(\xi^0)} \left\langle \partial\omega \frac{(\xi^0, p)}{\partial\xi}, e \right\rangle. \tag{2.8}$$

Finally, combining (2.7)–(2.8) yields

$$\partial_e^- \psi(\xi^0) = \partial_e^+ \psi(\xi^0) = \max_{p\in L(\xi^0)} \left\langle \partial\omega \frac{(\xi^0, p)}{\partial\xi}, e \right\rangle$$

for any $\xi^0 \in \mathcal{O}, 0 \ne e \in \mathbb{R}^m$. This implies the directional differentiability of $\psi = \psi(\xi)$ and completes the proof. ■

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. It can be verified that the function $\omega = \omega(\xi, p) \stackrel{\text{def}}{=} \varphi(t, x, p)$ (see (2.2)) satisfies all the assumptions of Lemma 2.3 where $E \stackrel{\text{def}}{=} \operatorname{dom}\phi^* \neq \emptyset$ [12, Theorem 12.2] and $m \stackrel{\text{def}}{=} n+1$, $\xi \stackrel{\text{def}}{=} (t, x)$. Here, we put $\mathcal{O} \stackrel{\text{def}}{=} \overline{\mathcal{D}}$ and conclude that the definition

$$L(t,x) \stackrel{\text{def}}{=} \{p \in E : \varphi(t,x,p) = u(t,x)\} \tag{2.8a}$$

determines a non-empty-valued, locally bounded (and closed) multifunction $L = L(t, x)$ of $(t, x) \in \overline{\mathcal{D}}$.

Our proof starts with the claim that $u = u(t, x)$ is in $\operatorname{Lip}(\overline{\mathcal{D}})$. To this end, arbitrarily take an $r \in (0, +\infty)$ and denote $V_r \stackrel{\text{def}}{=} \{(t, x) \in \overline{\mathcal{D}} : t+|x| < r\}$, $N_r \stackrel{\text{def}}{=} N(V_r)$ (cf. (E.II)). Let $g_{N_r} = g_{N_r}(t)$ be as in hypothesis (E.I). Then for any two points (t^1, x^1), $(t^2, x^2) \in V_r$, we may choose an element p^1 of the non-empty set $L(t^1, x^1) \subset \overline{B}(0, N_r)$ and obtain

$$\begin{aligned} u(t^1,x^1) - u(t^2,x^2) &\le \varphi(t^1,x^1,p^1) - \varphi(t^2,x^2,p^1) \\ &= \langle p^1, x^1 - x^2\rangle + \int_{t^1}^{t^2} f(\tau, p^1)\,d\tau \\ &\le N_r\,|x^1 - x^2| + s_r\,|t^1 - t^2|, \end{aligned}$$

where $s_r \stackrel{\text{def}}{=} \operatorname{ess\,sup}_{t\in(0,r)} g_{N_r}(t)$. Analogously, $u(t^2, x^2) - u(t^1, x^1) \le N_r\,|x^1 - x^2| + s_r\,|t^1 - t^2|$. Thus, $u = u(t, x)$ belongs to $\operatorname{Lip}(\overline{\mathcal{D}})$.

Next, let $e^0 \stackrel{\text{def}}{=} (1, 0, 0, \ldots, 0)$, $e^1 \stackrel{\text{def}}{=} (0, 1, 0, \ldots, 0)$, $\ldots$, $e^n \stackrel{\text{def}}{=} (0, 0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. It follows from Lemma 2.2 (we now replace $\mathcal{O} \stackrel{\text{def}}{=} \mathcal{D}_G$, the set G being as in (E.I)) that $u = u(t, x)$ is directionally differentiable in $\mathcal{D}_G$ with

$$\begin{aligned} \partial_{e^0} u(t,x) &= \max\{-f(t,p) : p \in L(t,x)\}, \\ \partial_{-e^0} u(t,x) &= \max\{f(t,p) : p \in L(t,x)\} \end{aligned} \tag{2.9}$$

and (for $1 \le i \le n$)

$$\begin{aligned} \partial_{e^i} u(t,x) &= \max\{p_i : p \in L(t,x)\}, \\ \partial_{-e^i} u(t,x) &= \max\{-p_i : p \in L(t,x)\}. \end{aligned} \tag{2.10}$$

On the other hand, according to Rademacher's Theorem [11, Theorem 1.18], there exists a set $\mathcal{Q} \subset \mathcal{D}$ of ($(n+1)$-dimensional) Lebesgue measure 0 such that $u = u(t, x)$ is (totally) differentiable with

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \partial_{e^0} u(t,x) = -\partial_{-e^0} u(t,x), \\ \frac{\partial u(t,x)}{\partial x_i} &= \partial_{e^i} u(t,x) = -\partial_{-e^i} u(t,x) \end{aligned} \tag{2.11}$$

at any point $(t, x) \in \mathcal{D} \setminus \mathcal{Q}$. Hence, (2.10)–(2.11) show that the equalities (for $1 \le i \le n$)

$$\frac{\partial u(t,x)}{\partial x_i} = \max\{p_i : p \in L(t,x)\} = \min\{p_i : p \in L(t,x)\}$$

hold outside the null set $\mathcal{P} \stackrel{\text{def}}{=} (G \times \mathbb{R}^n) \cup \mathcal{Q}$, i.e., $L(t,x)$ is precisely the singleton $\{\partial u(t,x)/\partial x\}$ except on $\mathcal{P}$. Consequently, (2.9), together with (2.11), implies that (1.1) must be satisfied almost everywhere in $\mathcal{D}$.

Further, by a well-known property of Frenchel conjugate functions [12, Theorem 12.2], (9.7) gives

$$u(0,x) = \max_{p \in \mathbb{R}^n} \{\langle p, x\rangle - \phi^*(p)\} = \phi^{**}(x) = \phi(x) \quad \text{for all } x \in \mathbb{R}^n.$$

From what has already been proved, we conclude that $u = u(t,x)$ is a global Lipschitz solution of (1.1)–(1.2). ■

Theorem 2.4. *The function $u(t,x)$ defined by (2.3) is continuously differentiable in an open set $V \subset \mathcal{D}_G$ if and only if the multifunction $L(t,x)$ defined by (2.8a) is single-valued in V.*

Proof. In the proof of Theorem 2.1, we have seen that if $u(t,x)$ is differentiable at (t,x), then $L(t,x)$ is single-valued. Conversely, suppose now that $L(t,x)$ is a single-valued function for all (t,x) in an open set $V \subset \mathcal{D}_G$. By virtue of Lemma 2.3, we see that $V \ni (t,x) \mapsto L(t,x)$ is continuous. Note that, for each fixed x, $\partial_{e_0} u(t,x)$ (resp. $\partial_{e_i} u(t,x)$) is the right-hand partial derivative of $u(t,x)$ with respect to t (resp. x_i). Using Maximum Theorem [2, p. 38] together with the formulas (2.9)–(2.11), we have that $u_t(t,x)$ and $u_{x_i}(t,x)$ exist and are continuous in V. Theorem 2.4 is then proved. ■

Corollary 2.5. *Let $\phi = \phi(x)$ be a finite convex function on $\mathbb{R}^n$. Under hypothesis (E.I), suppose that* $\inf_{p \in \mathbb{R}^n} f(t,p)/(1+|p|)$ *is locally essentially bounded from below in $t \in [0, \infty)$. Then (2.3) determines a global Lipschitz solution of the Cauchy problem (1.1)–(1.2).*

Proof. Only (E.II) needs verifying. Given any $r \in (0, +\infty)$, denote $V_r \stackrel{\text{def}}{=} \{(t,x) \in \overline{\mathcal{D}} : t + |x| < r\}$ and $s_r \stackrel{\text{def}}{=} \operatorname{ess\,inf}_{t \in (0,r)} \inf_{p \in \mathbb{R}^n} f(t,p)/(1+|p|)$. Then

$$\begin{aligned} \varphi(t,x,p) &= \langle p, x\rangle - \phi^*(p) - \int_0^t f(\tau,p)d\tau \\ &\le r|p| - \phi^*(p) - ts_r(1+|p|) \\ &\le r(1+2|s_r|)\max\{1,|p|\} - \phi^*(p) \end{aligned} \tag{2.12}$$

for all $(t,x) \in V_r$, $p \in \mathbb{R}^n$. On the other hand, it being known (cf. Remark 4.2) that $\lim_{|p| \to +\infty} \phi^*(p)/|p| = +\infty$, there exists a finite number $N_r \ge 1$ such that $\phi^*(p)/|p| \ge r(1+2|s_r|)+1$ whenever $|p| \ge N_r$. Thus, (2.12) implies

$$\varphi(t,x,p) \le -|p|, \quad \text{provided} \quad (t,x) \in V_r, \; |p| \ge N_r.$$

Finally, since $r \in (0, +\infty)$ is arbitrarily chosen, (E.II)$''$ holds, and hence, so does (E.II). ■

Corollary 2.6. *Let $f = f(t,p)$ be continuous on $\overline{\mathcal{D}}$ and let $\phi = \phi(x)$ be convex and globally Lipschitz continuous on $\mathbb{R}^n$. Then (2.3) determines a Lipschitz solution of the Cauchy problem (1.1)–(1.2).*

Proof. Since $\phi = \phi(x)$ is convex and globally Lipschitz continuous, $E \stackrel{\text{def}}{=} \operatorname{dom}\phi^*$ should be bounded [12, 13.3] (and non-empty). Independent of $(t, x) \in \overline{D}$, it follows that

$$\varphi(t, x, p) = \langle p, x\rangle - \phi^*(p) - \int_0^t f(\tau, p)d\tau = -\infty$$

whenever $|p|$ is large enough. Hypothesis (E.II) therefore holds while (E.I) is trivially satisfied. ∎

Remark 2.5. As we have mentioned in the introduction, Corollary 2.6 was proved by Hopf [7] in a different way for the case where $f = f(p)$ depends only on p.

Example 2.1. Investigate the smoothness of the global Lipschitz solution defined by (2.3) of the following problem:

$$\frac{\partial u}{\partial t} + H\left(\frac{\partial u}{\partial x}\right) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R},$$

$$u(0, x) = \frac{x^2}{2},$$

where

$$H(q) = \begin{cases} 0, & q \leq 0, \\ q, & 0 < q \leq 1, \\ 1, & q > 1. \end{cases}$$

Note that $u(0, x) = x^2/2$ is not globally Lipschitz continuous and we cannot use Hopf's formula (1.3). But by Theorem 2.1 and formula (2.3), a global Lipschitz solution of this problem is

$$u(t, x) = \begin{cases} \frac{x^2}{2}, & (t, x) \in D_1 \stackrel{\text{def}}{=} \{t \geq 0;\ x \leq 0\} \\ 0, & (t, x) \in D_2 \stackrel{\text{def}}{=} \{t \geq \max\{x, \frac{x^2}{2}\}, x \geq 0\} \\ \frac{x^2}{2} - t\,, & (t, x) \in D_4 \stackrel{\text{def}}{=} \{t, 0, x \in \mathbb{R}\}(D_1 \cup D_2 \cup D_3\} \\ \frac{x^2}{2} + \frac{t^2}{2} - xt, & (t, x) \in D_3 \stackrel{\text{def}}{=} \{\max\{0, 2(x-1)\} \leq t \leq x, 0 \leq x \leq 2\}. \end{cases}$$

The singularities of this solution is the curve

$$C: \ \frac{x^2}{2} - t = 0, \ \ t > 2. \tag{2.13}$$

This fact can be foreseen by applying Theorem 2.4. Indeed, $L(t, x) = \{0, x\}$ if $t = x^2/2, x \geq 2$, and $L(t, x) = \{x - t, x\}$ if $t = 2(x - 1), 1 \leq x \leq 2$, while $L(t, x)$ is a singleton if $(t, x) \notin (C)$. Direct computation shows that $L(t, x) = \{x\}$ for $(t, x) \in D_1$, $L(t, x) = \{0\}$ for $(t, x) \in D_2\backslash(C)$, $L(t, x) = \{x - t\}$ for $(t, x) \in D_3\backslash(C)$, and $L(t, x) = \{x\}$ for $(t, x) = D_4\backslash(C)$. Thus, Theorem 2.4 provides a method for investigating the smoothness of the global Lipschitz solutions.

3. The Case of Non-Convex Initial Data

In this section we consider the Cauchy problem (1.1)–(1.2) under the more general assumptions that $f = f(t, p)$ is still t-measurable and p-continuous as in Sec. 2, while $\phi = \phi(x)$ is now a non-convex function: First for the case where $\phi = \phi(x)$ can be represented as the minimum of a family of convex functions, and second, for the case where $\phi = \phi(x)$ is a d.c. function, i.e., the difference of two finite convex functions on $\mathbb{R}^n$. The class $DC(\mathbb{R}^n)$ of d.c. functions plays an important role in the theory of global optimization. This class is rather large; it contains all the semiconcave functions. We emphasize that it also contains all functions of class C^2 (of the whole $\mathbb{R}^n$) with second derivatives bounded either from below or above. The reader is referred to [1] and [8] for a sufficiently complete study of d.c. functions.

We first prove the following result.

Theorem 3.1. *Let I be an arbitrary non-empty set and $u_\alpha = u_\alpha(t, x)$ a global Lipschitz solution of the Cauchy problem for the same Hamilton–Jacobi equation*

$$\frac{\partial u}{\partial t} + f\left(t, \frac{\partial u}{\partial x}\right) = 0 \quad in \quad \{t > 0,\ x \in \mathbb{R}^n\} \tag{1.1}$$

with the initial condition

$$\partial u(0, x) = \phi_\alpha(x) \quad on \quad \{t = 0,\ x \in \mathbb{R}^n\} \tag{1.2α}$$

for each $\alpha \in I$. Suppose that, to each bounded subset V of $\overline{\mathcal{D}}$, there correspond a set $W(V) \subset V$ of Lebesgue measure 0, a non-negative number $M(V)$, and a subset $J(V)$ of I such that all the functions $u_\alpha = u_\alpha(t, x)$ for $\alpha \in J(V)$ are Lipschitz continuous in V with a common Lipschitz constant $M(V)$ and satisfy (1.1) at every point of $V \setminus W(V)$ and that $\inf_{\alpha \in I} u_\alpha(t, x) = \min_{\alpha \in J(V)} u_\alpha(t, x)$ for $(t, x) \in V$. Then the function $u = u(t, x) \stackrel{\text{def}}{=} \inf_{\alpha \in I} u_\alpha(t, x)$ is a global Lipschitz solution of the Cauchy problem (1.1)–(1.2) where $\phi = \phi(x) \stackrel{\text{def}}{=} \inf_{\alpha \in I} \phi_\alpha(x)$.

Proof. By assumption, $u(t, x) = \min_{\alpha \in J(V)} u_\alpha(t, x)$ on each bounded subset V of $\overline{\mathcal{D}}$. Moreover, $|u_\alpha(t^1, x^1) - u_\alpha(t^2, x^2)| \le M(V)\,(|t^1 - t^2| + |x^1 - x^2|)$ for any $\alpha \in J(V)$ and any fixed $(t^1, x^1), (t^2, x^2) \in V$. Assume $u(t^1, x^1) \ge u(t^2, x^2) = u_{\alpha^0}(t^2, x^2)$ for some $\alpha^0 \in J(V)$. Then

$$\begin{aligned} 0 \le u(t^1, x^1) - u(t^2, x^2) &\le u_{\alpha^0}(t^1, x^1) - u_{\alpha^0}(t^2, x^2) \\ &\le M(V)\,(|t^1 - t^2| + |x^1 - x^2|). \end{aligned}$$

This means that $u = u(t, x)$ is in $\mathrm{Lip}(\overline{\mathcal{D}})$.

Now, denote $V_k \stackrel{\text{def}}{=} \{(t, x) \in \overline{\mathcal{D}} : t + |x| < k\}$, $J_k \stackrel{\text{def}}{=} J(V_k)$, $W_k \stackrel{\text{def}}{=} W(V_k)$ for each $k = 1, 2, \ldots$ and let $W_0 \subset \mathcal{D}$ be a set of Lebesgue measure 0 such that $u = u(t, x)$ is differentiable at every point of $\mathcal{D} \setminus W_0$ (Rademacher's theorem). It will be shown that $u = u(t, x)$ satisfies (1.1) except on the null set $\mathcal{Q} \stackrel{\text{def}}{=} \bigcup_{k=0}^{+\infty} W_k$.

Indeed, given any $(t^0, x^0) \in \mathcal{D} \setminus \mathcal{Q}$, we choose a positive integer $k > t^0 + |x^0|$ and some index $\alpha^0 \in J_k$ so that $u(t^0, x^0) = u_{\alpha^0}(t^0, x^0)$. Obviously,

$$u(t, x) - u(t^0, x^0) \le u_{\alpha^0}(t, x) - u_{\alpha^0}(t^0, x^0) \tag{3.1}$$

for all (t, x) close enough to (t^0, x^0). Since $u = u(t, x)$ and $u_{\alpha^0} = u_{\alpha^0}(t, x)$ are both differentiable at (t^0, x^0), (3.1) implies

$$\frac{\partial u(t^0, x^0)}{\partial t} = \frac{\partial u_{\alpha^0}(t^0, x^0)}{\partial t}$$

and

$$\frac{\partial u(t^0, x^0)}{\partial x} = \frac{\partial u_{\alpha^0}(t^0, x^0)}{\partial x}.$$

But $u_{\alpha^0} = u_{\alpha^0}(t, x)$ satisfies (1.1) at (t^0, x^0), and so does $u = u(t, x)$.

On the other hand, it is clear from the hypotheses that

$$u(0, x) = \inf_{\alpha \in I} u_\alpha(0, x) = \inf_{\alpha \in I} \phi_\alpha(x) = \phi(x) \quad \text{for all } x \in \mathbb{R}^n.$$

The function $u = u(t, x)$ is thus a global Lipschitz solution of (1.1)–(1.2). ■

Now, suppose that $\phi = \phi(x)$ is given in the form

$$\phi(x) \stackrel{\text{def}}{=} \inf_{\alpha \in I} \phi_\alpha(x) \quad \text{for } x \in \mathbb{R}^n, \tag{3.2}$$

with $\phi_\alpha = \phi_\alpha(x)$ a finite convex function for every $\alpha \in I$. Combining Theorems 2.1 and 3.1, we obtain the following first results for the representation of a global Lipschitz solution in the case of non-convex initial data.

Corollary 3.2. *Assume (E.I)–(E.II) hold for each problem (1.1)–(1.2α), with $\phi_\alpha = \phi_\alpha(x)$ a finite convex function, $\alpha \in I$. Furthermore, assume that all the hypotheses of Theorem 3.1 hold for the solutions*

$$u_\alpha = u_\alpha(t, x) \stackrel{\text{def}}{=} \max_{p \in \mathbb{R}^n} \left\{ \langle p, x \rangle - \phi_\alpha^*(p) - \int_0^t f(\tau, p) d\tau \right\}$$

of those problems. Then $u = u(t, x) \stackrel{\text{def}}{=} \inf_{\alpha \in I} u_\alpha(t, x)$ is a global Lipschitz solution of the Cauchy problem (1.1)–(1.2) where $\phi = \phi(x)$ is defined by (3.2).

Corollary 3.3. *Let $\phi = \phi(x) \stackrel{\text{def}}{=} \min_{i \in \{1,2,\ldots,k\}} \phi_i(x)$, with $\phi_1 = \phi_1(x)$, $\phi_2 = \phi_2(x), \ldots, \phi_k = \phi_k(x)$, be some convex and globally Lipschitz continuous functions. If $f = f(t, p)$ is continuous on $\overline{\mathcal{D}}$, then a global Lipschitz solution $u = u(t, x)$ of the Cauchy problem (1.1)–(1.2) can be found in the form*

$$u(t, x) \stackrel{\text{def}}{=} \min_{i \in \{1,2,\ldots,k\}} \max_{p \in \mathbb{R}^n} \left\{ \langle p, x \rangle - \phi_i^*(p) - \int_0^t f(\tau, p) d\tau \right\}$$

for $(t, x) \in \overline{\mathcal{D}}$.

Proof. Since $I \stackrel{\text{def}}{=} \{1, 2, \ldots, k\}$ is a finite set, the conclusion is straightforward from Corollary 2.6 and Theorem 3.1. ■

Example 3.1. Consider the Cauchy problem

$$\frac{\partial u}{\partial t} + \left| \left(\frac{\partial u}{\partial x} \right)^2 - 1 \right| = 0 \text{ in } \{0 < t < +\infty, \ x \in \mathbb{R}\},$$

$$u(0, x) = \exp(-|x|) = \min\{\exp(x), \exp(-x)\} \text{ on } \{t = 0, \ x \in \mathbb{R}\}.$$

By Corollary 3.3, a global Lipschitz solution of the problem is

$$u = u(t, x) \stackrel{\text{def}}{=} \min_{i=1,2} \max_{p \in \mathbb{R}} \{px - h_i(p) - t|p^2 - 1|\},$$

where

$$h_1(p) \stackrel{\text{def}}{=} \begin{cases} p\ln p - p & \text{if } p > 0, \\ 0 & \text{if } p = 0, \\ +\infty & \text{if } p < 0, \end{cases}$$

and

$$h_2(p) \stackrel{\text{def}}{=} \begin{cases} +\infty & \text{if } p > 0, \\ 0 & \text{if } p = 0, \\ -p\ln(-p) + p & \text{if } p < 0. \end{cases}$$

The solution can also be rewritten as

$$\begin{aligned} u(t, x) &= \min\{\max_{p \geq 0}\{px - p\ln p + p - t|p^2 - 1|\}, \max_{p \leq 0}\{px + p\ln(-p) - p - t|p^2 - 1|\}\} \\ &= \max_{p \geq 0}\{-p|x| - p\ln p + p - t|p^2 - 1|\}, \end{aligned}$$

in which we adopt the convention that $p\ln p = 0$ if $p = 0$.

Example 3.2. Let $h = h(\alpha)$ be a finite-valued continuous function of α on a given compact set $K \subset \mathbb{R}^n$. We put

$$\phi(x) \stackrel{\text{def}}{=} \min_{\alpha \in K}\{h(\alpha) + |\alpha| \cdot |x|\} \text{ for } x \in \mathbb{R}^n.$$

If $f = f(t, p)$ belongs to $C(\overline{\mathcal{D}})$, depends only on t and $|p|$, and is decreasing with respect to $|p|$, then it follows from Corollary 3.2 that a global Lipschitz solution $u = u(t, x)$ of the Cauchy problem (1.1)–(1.2) can be found by the formula

$$\begin{aligned} u(t, x) &\stackrel{\text{def}}{=} \min_{\alpha \in K} \left\{ h(\alpha) + \max_{|p| \leq |\alpha|} \left\{ \langle p, x \rangle - \int_0^t f(\tau, p) d\tau \right\} \right\} \\ &= \min_{\alpha \in K} \left\{ h(\alpha) + |\alpha| \cdot |x| - \int_0^t f(\tau, \alpha) d\tau \right\} \text{ for } (t, x) \in \overline{\mathcal{D}}. \end{aligned}$$

We now consider the Cauchy problem (1.1)–(1.2) in the main case of this section where $\phi = \phi(x)$ belongs to the class $DC(\mathbb{R}^n)$, i.e., it has a representation of the form

$$\phi(x) \equiv \sigma_1(x) - \sigma_2(x) \text{ on } \mathbb{R}^n \tag{3.2a}$$

for some finite convex functions $\sigma_1 = \sigma_1(x)$ and $\sigma_2 = \sigma_2(x)$. We call (3.2a) a *d.c. representation* of $\phi = \phi(x)$. (Of course, there are always an infinite number of such representations for each d.c. function $\phi = \phi(x)$.) The notations $\sigma_1^* = \sigma_1^*(p)$ and $\alpha_2^* = \sigma_2^*(p)$ will signify the Frenchel conjugate functions of $\sigma_1 = \sigma_1(x)$ and $\sigma_2 = \sigma_2(x)$. The effective domains of the theses conjugates are denoted by E_1 and E_2, respectively.

Besides hypothesis (E.I), we shall also assume the following ones.

(E.III) To any bounded sets $V \subset \overline{\mathcal{D}}$ and $E \subset \mathbb{R}^n$, *there corresponds a positive number* $N(V, E)$ *so that*

$$\varphi_\alpha(t, x, p) < \max_{|q| \le N(V,E)} \varphi_\alpha(t, x, q) \ \ as \ \ (t, x) \in V, \ \alpha \in E, \ |p| > N(V, E).$$

Here,

$$\varphi_\alpha(t, x, p) \overset{\text{def}}{=} \langle p, x\rangle - \sigma_1^*(p + \alpha) - \int_0^t f(\tau, p) d\tau. \tag{3.3}$$

(E.IV) All the multifunctions $L_\alpha = L_\alpha(t, x)$ *of* $(t, x) \in \overline{\mathcal{D}}$ *defined by*

$$L_\alpha(t, x) \overset{\text{def}}{=} \left\{p \in \mathbb{R}^n : \varphi_\alpha(t, x, p) = \max_{q \in \mathbb{R}^n} \varphi_\alpha(t, x, q)\right\} \tag{3.4}$$

are actually single-valued in $\mathcal{D} \setminus \mathcal{Q}$ *where* $\mathcal{Q}$ *is a certain closed set of* ($(n+1)$*-dimensional) Lebesgue measure* 0 *and is independent of* $\alpha \in \mathbb{R}^n$.

Remark 3.1. Let $\sigma_1 = \sigma_1(x)$ be a finite convex function on $\mathbb{R}^n$, and $f = f(t, p)$ a p-convex function on $\{t > 0, \ p \in \mathbb{R}^n\}$. Assume (E.I) and (E.III) hold. Then it can be proved that (E.IV) is satisfied with $\mathcal{Q} \overset{\text{def}}{=} G \times \mathbb{R}^n$, the set G being as in (E.I), if one of the following two conditions holds:

(E.IV)′ $f = f(t, p)$ *is strictly p-convex, i.e., it is strictly convex with respect to p on* $\mathbb{R}^n$ *for almost every fixed* $t \in (0, +\infty)$.

(E.IV)″ $\sigma_1^* = \sigma_1^*(p)$ *is strictly convex on its effective domain* $E_1 \overset{\text{def}}{=} \operatorname{dom} \sigma_1^*$.

Theorem 3.4. *Let* $\phi = \phi(x)$ *be in the class* $DC(\mathbb{R}^n)$ *with a d.c. representation (3.2a) such that* $\sigma_2 = \sigma_2(x)$ *is globally Lipschitz continuous on* $\mathbb{R}^n$. *Under hypotheses (E.I), (E.III), and (E.IV), the formula*

$$u(t, x) \overset{\text{def}}{=} \min_{\alpha \in E_2} \left\{\sigma_2^*(\alpha) + \max_{p \in \mathbb{R}^n} \varphi_\alpha(t, x, p)\right\} \ \ on \ \ \overline{\mathcal{D}}, \tag{3.5}$$

in which $E_2 \overset{\text{def}}{=} \operatorname{dom} \sigma_2^*$ *determines a global Lipschitz solution* $u = u(t, x)$ *of the Cauchy problem (1.1)–(1.2).*

Proof. Let $\phi_\alpha(x) \overset{\text{def}}{=} \sigma_1(x) - \langle\alpha, x\rangle + \sigma_2^*(\alpha)$ as $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^n$. Then $\phi_\alpha = \phi_\alpha(x)$ are obviously convex functions. For each $\alpha \in E_2$, consider the Cauchy problem (1.1)–(1.2). By (E.I) and (E.III), it follows from Theorem 2.1 that the formula

$$u(t, x) \overset{\text{def}}{=} \sigma_2^*(\alpha) + \max_{p \in \mathbb{R}^n} \varphi_\alpha(t, x, p) \ \text{ on } \overline{\mathcal{D}} \tag{3.6}$$

determines a global Lipschitz solution $u_\alpha = u_\alpha(t, x)$ of this problem. Moreover, we may assume $\mathcal{Q} \supset G \times \mathbb{R}^n$, the sets $\mathcal{Q}$ and G are as in (E.I) and (E.IV), and then see that all the solutions $u_\alpha = u_\alpha(t, x)$ satisfy (1.1) at every point of $\mathcal{D} \setminus \mathcal{Q}$. (For the smoothness of such $u_\alpha = u_\alpha(t, x)$, see Theorem 2.4.)

Now, since $\sigma_2 = \sigma_2(x)$ is globally Lipschitz continuous on $\mathbb{R}^n$, the (non-empty) set $E_2 = \operatorname{dom} \sigma_2^*$ should be bounded [12, 13.3]. Given any $r \in (0, +\infty)$, denote $V_r \overset{\text{def}}{=} \{(t, x) \in \overline{\mathcal{D}} : t + |x| < r\}$ and $N_r \overset{\text{def}}{=} N(V_r, E_2)$ (cf. (E.III)). For any (t^1, x^1), (t^2, x^2) in V_r, we can then choose $p^\alpha \in L_\alpha(t^1, x^1) \subset \overline{B}(0, N_r)$ and deduce from (3.3)–(3.4), (3.6) that

$$\begin{aligned} u_\alpha(t^1, x^1) - u_\alpha(t^2, x^2) &\le \varphi_\alpha(t^1, x^1, p^\alpha) - \varphi_\alpha(t^2, x^2, p^\alpha) \\ &= \langle p^\alpha, x^1 - x^2 \rangle + \int_0^t f(\tau, p^\alpha) d\tau \\ &\le N_r |x^1 - x^2| + s_r |t^1 - t^1|, \end{aligned}$$

where $s_r \overset{\text{def}}{=} \operatorname{ess\,sup}_{t \in (0,r)}, g_{N_r}(t)$ (cf. (E.I)). The solutions $u_\alpha = u_\alpha(t, x)$ therefore satisfy a Lipschitz condition on V_r with constants N_r and s_r, which are independent of $\alpha \in E_2$.

Next, rewrite (3.6) as

$$u_\alpha(t, x) = \sigma_2^*(\alpha) + \max_{p \in \mathbb{R}^n} \varphi_\alpha(t, x, p - \alpha) \tag{3.7}$$

and fix temporarily $(t, x) \in \overline{\mathcal{D}}$. By (3.3) and hypothesis (E.I), $\varphi_\alpha(t, x, p - \alpha)$ is continuous in $\alpha \in \mathbb{R}^n$. Hence, by [12], the right side of (3.7), being the supremum of a family of continuous functions, actually determines a lower semicontinuous function from the whole $\mathbb{R}^n$ into $(-\infty, +\infty)$ whose effective domain is precisely the non-empty bounded set $E_2 \subset \mathbb{R}^n$. It follows that

$$\begin{aligned} +\infty > &\inf_{\alpha \in E_2} u_\alpha(t, x) \\ &= \inf_{\alpha \in \overline{E}_2} \{\sigma_2^*(\alpha) + \max_{p \in \mathbb{R}^n} \varphi_\alpha(t, x, p - \alpha)\} \\ &= \min_{\alpha \in \overline{E}_2} \{\sigma_2^*(\alpha) + \max_{p \in \mathbb{R}^n} \varphi_\alpha(t, x, p - \alpha)\} \\ &= \min_{\alpha \in E_2} \{\sigma_2^*(\alpha) + \max_{p \in \mathbb{R}^n} \varphi_\alpha(t, x, p - \alpha)\} \\ &= \min_{\alpha \in E_2} u_\alpha(t, x) \ (> -\infty). \end{aligned}$$

Finally, since $\sigma_2(x) = \max_{\alpha \in E_2}\{\langle x, \alpha \rangle - \sigma_2^*(\alpha)\}$ (see [7, p. 964]), one has

$$\begin{aligned} \min_{\alpha \in E_2} \phi_\alpha(x) &= \sigma_1(x) + \min_{\alpha \in E_2}\{-\langle \alpha, x \rangle - \sigma_2^*(\alpha)\} \\ &= \sigma_1(x) - \max_{\alpha \in E_2}\{\langle x, \alpha \rangle - \sigma_2^*(\alpha)\} \\ &= \sigma_1(x) - \sigma_2(x) = \phi(x) \quad \text{for all } x \in \mathbb{R}^n. \end{aligned}$$

As a consequence of Theorem 3.1, $u = u(t, x) \overset{\text{def}}{=} \min_{\alpha \in E_2} u_\alpha(t, x)$ is therefore a global Lipschitz solution of the Cauchy problem (1.1)–(1.2). ∎

Corollary 3.5. *Let $f = f(t, p)$ be of class C^0 on $\overline{\mathcal{D}}$ and let $\phi = \phi(x)$ have a d.c. representation (3.2a) such that $\sigma_1 = \sigma_1(x)$ and $\sigma_2 = \sigma_2(x)$ are globally Lipschitz continuous on $\mathbb{R}^n$. Assume (E.IV) holds. Then the function $u = u(t, x)$ given by (3.5) is a global Lipschitz solution of problem (1.1)–(1.2).*

Proof. Since the non-empty set $E_1 \stackrel{\text{def}}{=} \operatorname{dom} \sigma_1^*$ is bounded [12, 13.3], hypothesis (E.III) must hold while (E.I) is trivially satisfied. Hence, the conclusion is immediate from Theorem 3.4. ■

Example 3.3. Consider the Cauchy problem

$$\frac{\partial u}{\partial t} + f\left(\frac{\partial u}{\partial x}\right) = 0 \text{ in } \{0 < t < +\infty,\ x \in \mathbb{R}\},$$
$$u(0, x) = \phi(x) \text{ on } \{t = 0,\ x \in \mathbb{R}\},$$

with $f(p) \stackrel{\text{def}}{=} (1 + |p|^3)^{1/3}$ (for $p \in \mathbb{R}$) and

$$\phi(x) \stackrel{\text{def}}{=} \begin{cases} \frac{x^3}{3} & \text{if } x \in [-1, 1], \\ x - \frac{2}{3}\operatorname{sign} x & \text{if } x \notin [-1, 1]. \end{cases}$$

We first note that neither the formula (1.3) of Hopf nor the formula (2.3) of Theorem 2.1 works in this case since the initial function here is not convex. Although the present Hamiltonian $f = f(p)$ is in fact convex, we should also mention that Hopf's formula (1.4) could not be applied directly to the problem because

$$\lim_{|p| \to +\infty} \frac{f(p)}{|p|} = 1 \neq +\infty.$$

In this case, however, it is easy to check the validity of d.c. representation (3.2a) where

$$\sigma_1 = \sigma_1(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x < 0, \\ \frac{x^3}{3} & \text{if } x \in [0, 1], \\ x - \frac{2}{3} & \text{if } x > 1, \end{cases}$$

and $\sigma_2 = \sigma_2(x) \stackrel{\text{def}}{=} \sigma_1(-x)$ are globally Lipschitz continuous on $\mathbb{R}$. Further, we may invoke either (E.IV)$''$ or (E.IV)$'$ to deduce that (E.IV) holds. Therefore, by Corollary 3.5, a global Lipschitz solution $u = u(t, x)$ of the problem can be found in the form

$$u(t, x) \stackrel{\text{def}}{=} \min_{\alpha \in [-1, 0]} \max_{p \in [0, 1]} \left\{ x(p - \alpha) - \frac{2}{3}\left(|p|^{3/2} - |\alpha|^{3/2}\right) - t(1 + |p - \alpha|^3)^{1/3} \right\}.$$

As we have seen, Theorem 3.4 and its Corollary 3.5 concern the Cauchy problem (1.1)–(1.2) in the case where initial function $\phi = \phi(x)$ has a d.c. representation $\phi(x) \equiv \sigma_1(x) - \sigma_2(x)$ such that $\operatorname{dom} \sigma_2^*$ is bounded in $\mathbb{R}^n$. The following will be devoted to the case where $\operatorname{dom} \sigma_2^* = \mathbb{R}^n$.

Theorem 3.6. *Let $\phi = \phi(x)$ be in the class $DC(\mathbb{R}^n)$ with a d.c. representation (3.2a) such that $\lim_{|x| \to +\infty} \sigma_2(x)/|x| = +\infty$. Under hypotheses (E.I), (E.III), and (E.IV), suppose there exists a function $g = g(t)$ in $L^1_{\text{loc}}(\mathbb{R})$ with the property that $\sup\{f(t, p) : p \in \mathbb{R}^n\} \leq g(t)$ for almost all $t \in (0, +\infty)$. Then (3.5) determines a global Lipschitz solution of the Cauchy problem (1.1)–(1.2).*

Proof. Since $\sigma_2 = \sigma_2(x)$ is a finite convex function on $\mathbb{R}^n$ with $\lim_{|x|\to+\infty} \sigma_2(x)/|x| = +\infty$, so is its Fenchel conjugate function $\sigma_2^* = \sigma_2^*(p)$; in particular, $E_2 \overset{\text{def}}{=} \operatorname{dom} \sigma_2^* = \mathbb{R}^n$ (cf. Remarks 4.1 and 4.2).

We shall continue using the notation $u_\alpha(t, x)$ introduced in the proof of Theorem 3.4. Let $r \in (0, +\infty)$, $V_r \overset{\text{def}}{=} \{(t, x) \in \overline{\mathcal{D}} : t + |x| < r\}$, $\mu_r \overset{\text{def}}{=} \sup_{|x|<r} |\sigma_1(x)|$, and $s_r \overset{\text{def}}{=} \int_0^t |g(\tau)|d\tau$. Since $\lim_{|p|\to+\infty} \sigma_2^*(p)/|p| = +\infty$, to any $M \in (0, +\infty)$, there corresponds a finite number $N_{r,M} \geq 1$ so that

$$\sigma_2^* \frac{(\alpha)}{|\alpha|} \geq r + s_r + \mu_r + M \text{ as } |\alpha| > N_{r,M}. \tag{3.8}$$

By (3.3), it follows that, if $(t, x) \in V_r$, then $\varphi_\alpha(t, x, p-\alpha) \geq \langle p, x\rangle - \sigma_1^*(p) - r|\alpha| - \sigma_r$. Therefore, (3.7) and (3.8) imply

$$\begin{aligned} u_\alpha(t, x) &\geq \sigma_2^*(\alpha) + \max_{p\in\mathbb{R}^n} \{\langle p, x\rangle - \sigma_1^*(p)\} - r|\alpha| - s_r \\ &= \sigma_2^*(\alpha) + \sigma_1(x) - r|\alpha| - s_r \\ &\geq (r + s_r + \mu_r + M) \cdot |\alpha| - \mu_r - r|\alpha| - s_r > M, \end{aligned}$$

provided $(t, x) \in V_r$ and $|\alpha| > N_{r,M}$. This means that

$$\lim_{|\alpha|\to+\infty} u_\alpha(t, x) = +\infty \text{ locally uniformly in } (t, x) \in \overline{\mathcal{D}}.$$

Hence (cf. Remark 2.2), we may find a positive number N_r for each $r \in (0, +\infty)$ such that

$$\inf_{x\in\mathbb{R}^n} u_\alpha(t, x) = \min_{|\alpha|\leq N_r} u_\alpha(t, x) \text{ whenever } (t, x) \in V_r.$$

(It should be noted that $u_\alpha(t, x)$ is lower semicontinuous in the whole $\mathbb{R}^n$.)

Moreover, the analysis similar to that in the proof of Theorem 3.4 shows that the solutions $u_\alpha = u_\alpha(t, x)$ satisfy a Lipschitz condition on V_r with constants depending on r but independent of α for $|\alpha| \leq N_r$, and that they satisfy (1.1) except the common set of Lebesgue measure 0. The proof is thus complete in view of Theorem 3.1. ∎

4. Equation with Convex Hamiltonian $f = f(p)$

We now consider the Cauchy problem

$$\frac{\partial u}{\partial t} + f\left(\frac{\partial u}{\partial x}\right) = 0 \text{ in } \mathcal{D} = \{t > 0,\ x \in \mathbb{R}^n\}, \tag{4.1}$$

$$u(0, x) = \phi(x) \text{ on } \{t = 0,\ x \in \mathbb{R}^n\} \tag{4.2}$$

under the following two hypotheses.

(F.I) The initial function $\phi = \phi(x)$ is of class C^0 and the Hamiltonian $f = f(p)$ is strictly convex on $\mathbb{R}^n$ with $\lim_{|p|\to+\infty} f(p)/|p| = +\infty$.

(F.II) For every bounded subset V of $\mathcal{D}$, there exists a positive number $N(V)$ so that

$$\min_{|w|\le N(V)} \left\{\phi(w) + t\cdot f^*\Big(\frac{x-w}{t}\Big)\right\} < \phi(y) + t\cdot f^*\Big(\frac{x-y}{t}\Big)$$

whenever $(t,x) \in V$, $|y| > N(V)$. Here, $f^* = f^*(z)$ denotes the Frenchel conjugate function of $f = f(p)$.

In the sequel, we use the notation

$$\zeta(t,x,y) \stackrel{\text{def}}{=} \phi(y) + t\cdot f^*\Big(\frac{x-y}{t}\Big), \tag{4.3}$$

where $(t,x) \in \mathcal{D}$, $y \in \mathbb{R}^n$, and shall prove the following theorem.

Theorem 4.1. *Assume (F.I)–(F.II). Then the formula*

$$u(t,x) \stackrel{\text{def}}{=} \inf_{y\in\mathbb{R}^n} \zeta(t,x,y) = \inf_{y\in\mathbb{R}^n}\left\{\phi(y) + t\cdot f^*\Big(\frac{x-y}{t}\Big)\right\} \text{ for } (t,x)\in\mathcal{D} \tag{4.4}$$

determines a global Lipschitz solution of the Cauchy problem (4.1)–(4.2).

The next auxiliary lemma is known [7; 12, Theorems 23.5, 25.5, and 26.3], but what we would like to insist here is its simple proof by the use of Lemmas 2.2 and 2.3.

Lemma 4.2. *Let $f = f(p)$ be strictly convex on $\mathbb{R}^n$ with $\lim_{|p|\to+\infty} f(p)/|p| = +\infty$. Then $f^* = f^*(z)$ is everywhere continuously differentiable; moreover,*

$$f^*(z) = \Big\langle z, \frac{\partial f^*(z)}{\partial z}\Big\rangle - f\Big(\frac{\partial f^*(z)}{\partial z}\Big) \text{ for all } z \in \mathbb{R}^n. \tag{4.5}$$

Proof. The strict convexity on $\mathbb{R}^n$ of the function $f = f(p)$ says that this function is everywhere finite and that

$$f(\lambda p^1 + (1-\lambda)p^2) \le f(p^1) + (1-\lambda)f(p^2)$$

for any $p^1, p^2 \in \mathbb{R}^n$, $\lambda \in [0,1]$; the sign of equality holds if and only if $p^1 = p^2$ or $\lambda \in \{0,1\}$. Accordingly, $f = f(p)$ is continuous.

It is a simple matter to check that $\omega = (z,p) \stackrel{\text{def}}{=} (z,p) - f(p)$ satisfies all the conditions of Lemmas 2.2 and 2.3 where we put $E \stackrel{\text{def}}{=} \mathbb{R}^n$, $m \stackrel{\text{def}}{=} n$, $\xi \stackrel{\text{def}}{=} z$, $\mathcal{O} \stackrel{\text{def}}{=} \mathbb{R}^m = \mathbb{R}^n$, and shall deal with the function

$$\psi = \psi(z) \stackrel{\text{def}}{=} \sup\{\omega(z,p) : p \in \mathbb{R}^n\} = f^*(z).$$

Indeed, since $\lim_{|p|\to+\infty} f(p)/|p| = +\infty$, condition (i) in Lemma 2.2 holds while the others are almost ready.

As $f = f(p)$ is strictly convex, it can be verified that the multifunction $L = L(z)$ defined by

$$L(z) \stackrel{\text{def}}{=} \{p \in \mathbb{R}^n : \omega(z,p) = f^*(z)\}$$

is actually single-valued on the whole $\mathbb{R}^n$. Therefore, by Lemma 2.2(b), all the partial derivatives $\partial f^*(z)/\partial z_i$ exist, and $L(z) = \{\partial f^*(z)/\partial z\}$. Property (4.5) thus comes from the definitions of $f^* = f^*(z)$ and $L = L(z)$. Further, Lemma 2.3 and Remark 2.4 imply the continuity of $\partial f^*/\partial z = \partial f^*(z)/\partial z$. ∎

Remark 4.1. Consider a convex and lower semicontinuous function $f = f(p)$ on $\mathbb{R}^n$. Assume $\operatorname{dom} f \neq \emptyset$ and $\operatorname{im} f \subset (-\infty, +\infty]$ (the function $f = f(p)$ is then called *proper*). It will be shown that

$$\lim_{|p|\to+\infty} \frac{f(p)}{|p|} = +\infty \text{ if and only if } \operatorname{dom} f^* = \mathbb{R}^n.$$

In fact, if $\lim_{|p|\to+\infty} f(p)/|p| = +\infty$, then for each $z \in \mathbb{R}^n$, the supremum

$$f^*(z) = \sup_{p\in\mathbb{R}^n} \{\langle z, p\rangle - f(p)\}$$

is essentially taken over all elements p of just a compact set $K_z \subset \mathbb{R}^n$, and is hence finite. Conversely, let there exist an $M \in \mathbb{R}$ and non-zero points $p^1, p^2, \ldots$ in $\mathbb{R}^n$ such that $f(p^k) \leq M|p^k|$ for $k = 1, 2, \ldots$ and that $|p^k| \to +\infty$ as $k \to +\infty$. Since $\mathbb{R}^n$ is locally compact, we may suppose $p^k/|p^k| \to z^0 \in \mathbb{R}^n$. Putting $z \stackrel{\text{def}}{=} (M+1)z^0$, we thus obtain

$$\begin{aligned} f^*(z) &\geq \sup_k\{\langle z, p^k\rangle - f(p^k)\} \geq \sup_k\{|p^k| \cdot [(M+1)\Big\langle z^0, \frac{p^k}{|p^k|}\Big\rangle - M]\} \\ &\geq \lim_{k\to+\infty} |p^k| = +\infty. \end{aligned}$$

Remark 4.2. Consider a finite convex function $\phi = \phi(x)$ on $\mathbb{R}^n$ with the Frenchel conjugate $\phi^* = \phi^*(p)$. Let $\phi^{**} = \phi^{**}(x)$ be the Frenchel conjugate of $\phi^* = \phi^*(p)$. Then it is known [7] that $\phi^* = \phi^*(p)$ is proper, convex, and lower semicontinuous on $\mathbb{R}^n$ and that $\phi^{**} = \phi$. Accordingly, $\operatorname{dom}\phi^{**} = \operatorname{dom}\phi = \mathbb{R}^n$, hence, Remark 4.1 implies $\lim_{|p|\to+\infty} \phi^*(p)/|p| = +\infty$.

Proof of Theorem 4.1. By (F.I)–(F.II) and Lemma 4.2, (4.3) determines a continuous function $\zeta = \zeta(t, x, y)$ whose derivatives

$$\frac{\partial\zeta(t,x,y)}{\partial t}, \frac{\partial\zeta(t,x,y)}{\partial x_1}, \ldots, \frac{\partial\zeta(t,x,y)}{\partial x_n}$$

exist and are continuous on the whole $\mathcal{D} \times \mathbb{R}^n$; moreover, one may apply Lemma 2.2 to the function $\omega = \omega(\xi, p) \stackrel{\text{def}}{=} -\zeta(t, x, y)$ where $p \stackrel{\text{def}}{=} y$, $E \stackrel{\text{def}}{=} \mathbb{R}^n$, and $m \stackrel{\text{def}}{=} n+1$, $\xi \stackrel{\text{def}}{=} (t, x)$, $\mathcal{O} \stackrel{\text{def}}{=} \mathcal{D}$. Consequently, $u = u(t, x)$ defined by (4.4) is locally Lipschitz continuous and directionally differentiable in $\mathcal{D}$ with

$$\begin{aligned} &\partial_e u(t,x) \\ &= \min_{y\in L(t,x)} \Big\langle f^*\Big(\frac{x-y}{t}\Big) - \Big\langle \frac{x-y}{t}, \frac{\partial f^*((x-y)/t)}{\partial z}\Big\rangle, \frac{\partial f^*((x-y)/t)}{\partial z}, e\Big\rangle. \end{aligned} \tag{4.6}$$

Here, $\mathbb{R}^{n+1} \ni e \neq 0$, $L(t, x) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : \xi(t, x, y) = u(t, x)\} \neq \emptyset$ (Lemma 2.3.) But, according to Rademacher's Theorem, $u = u(t, x)$ is (totally) differentiable at any point outside a null set $\mathcal{Q} \subset \mathcal{D}$. Therefore, suitable choices of e in (4.6) give

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \min_{y\in L(t,x)} \Big\{f^*\Big(\frac{x-y}{t}\Big) - \Big\langle\frac{x-y}{t}, \frac{\partial f^*((x-y)/t)}{\partial z}\Big\rangle\Big\} \\ &= \max_{y\in L(t,x)} \Big\{f^*\Big(\frac{x-y}{t}\Big) - \Big\langle\frac{x-y}{t}, \frac{\partial f^*((x-y)/t)}{\partial z}\Big\rangle\Big\} \end{aligned} \tag{4.7}$$

and

$$\frac{\partial u(t,x)}{\partial x_i} = \min_{i\in L(t,x)} \frac{\partial f^*((x-y)/t)}{\partial z_i} = \max_{y\in L(t,x)} \frac{\partial f^*((x-y)/t)}{\partial z_i}, \tag{4.8}$$

provided $(t,x) \in \mathcal{D} \setminus \mathcal{Q}$ and $i \in \{1, 2, \dots, n\}$.

Now, given any $(t,x) \in \mathcal{D} \setminus \mathcal{Q}$, we pick up some $y \in L(t,x)$. Then it follows from (4.5) and (4.7)–(4.8) that

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= f^*\Big(\frac{x-y}{t}\Big) - \Big\langle \frac{x-y}{t}, \frac{\partial f^*((x-y)/t)}{\partial z}\Big\rangle \\ &= -f\Big(\frac{\partial f^*((x-y)/t)}{\partial z}\Big) = -f\Big(\frac{\partial u(t,x)}{\partial x}\Big).\end{aligned}$$

The Equation (4.1) is thus satisfied almost everywhere in $\mathcal{D}$.

As the next step, we claim that

$$\lim_{\mathcal{D}\ni(t,x)\to(0,x^0)} u(t,x) = \phi(x^0) \tag{4.9}$$

for each fixed $x^0 \in \mathbb{R}^n$. Indeed, on the other hand, the definition (4.4) clearly forces $u(t,x) \le \phi(x) + t\cdot f^*(0)$, hence,

$$\limsup_{\mathcal{D}\ni(t,x)\to(0,x^0)} u(t,x) \le \phi(x^0). \tag{4.10}$$

On the other hand, let us first take a sequence $\left\{(t^k, x^k)\right\}_{k=1}^{+\infty} \subset \mathcal{D}$ converging to $(0, x^0)$ such that $\liminf_{\mathcal{D}\ni(t,x)\to(0,x^0)} u(t,x) = \lim_{k\to+\infty} u(t^k, x^k)$, and second, choose arbitrary points $y^k \in L(t^k, x^k)$ (for $k = 1, 2, \ldots$). Then it will be shown that $y^k \longrightarrow_{(k\to+\infty)} x^0$. By contrast, suppose without loss of generality that $y^k \longrightarrow_{(k\to+\infty)} y^0 \in \mathbb{R}^n$, where $y^0 \neq x^0$. (We emphasize here that the sequence $\{y^k\}_{k=1}^{+\infty} \subset \mathbb{R}^n$ is bounded by Lemma 2.3.) Since $\lim_{z\to+\infty} f^*(z)/|z| = +\infty$ (cf. Remark 4.2), (4.10) and a passage to the limit as $k \to +\infty$ in the equality

$$u(t^k, x^k) = \phi(y^k) + t^k \cdot f^*\Big(\frac{x^k - y^k}{t^k}\Big) \tag{4.11}$$

would yield

$$\phi(x^0) \ge \liminf_{\mathcal{D}\ni(t,x)\to(0,x^0)} u(t,x) = \lim_{k\to+\infty} u(t^k, x^k) = \phi(y^0) + (+\infty) = +\infty,$$

a contradiction. This shows that $\lim_{k\to+\infty} y^k = x^0$. But the continuous function, $f^* = f^*(z)$, is bounded from below since again $\lim_{|z|\to+\infty} f^*(z)/|z| = +\infty$. Therefore, a passage to the limit as $k \to +\infty$, also in (4.11), implies

$$\liminf_{\mathcal{D}\ni(t,x)\to(0,x^0)} u(t,x) = \lim_{k\to+\infty} u(t^k, x^k) \ge \phi(x^0). \tag{4.12}$$

Finally, combining (4.10) and (4.12) gives (4.9), which says that $u = u(t,x)$ has a (unique) continuous extension over the whole $\overline{\mathcal{D}}$ satisfying (4.2). The proof is thus complete. ∎

Remark 4.3. Assume (F.I). Then (F.II) is satisfied if $\lim_{|y|\to+\infty}\zeta(t,x,y) = +\infty$ uniformly in (t,x) on each bounded subset of $\mathcal{D}$.

In fact, let $V \subset \mathcal{D}$ be bounded, say $V \subset (0,r)\times B(0,r)$ for some $r \in (0,+\infty)$. Put $M \overset{\text{def}}{=} r\cdot|f^*(0)| + \max_{|x|\le r}\phi(x) < +\infty$. It follows from (4.3) that $\min_{|w|\le r}\zeta(t,x,w) \le \zeta(t,x,x) = \phi(x) + t\cdot f^*(0) \le M$ whenever $(t,x)\in V$. Hence, if $\lim_{|y|\to+\infty}\zeta(t,x,y) = +\infty$ uniformly in (t,x) on each such V, then for a suitable number $N(V)\ge r$, we have

$$\begin{aligned}\min_{|w|\le N(V)}\zeta(t,x,w) &\le \min_{|w|\le r}\zeta(t,x,w) \le M\\ &< \zeta(t,x,y) \text{ as } (t,x)\in V\cdot|y|>N(V),\end{aligned}$$

i.e., (F.II) is satisfied.

Corollary 4.3. *Under hypothesis (F.I), suppose*

$$\liminf_{|x|\to+\infty}\phi\frac{(x)}{|x|} > -\infty. \tag{4.13}$$

Then (4.4) determines a global Lipschitz solution of the Cauchy problem (4.1)–(4.2).

Proof. By Remark 4.3, it suffices to prove that $\lim_{|y|\to+\infty}\zeta(t,x,y) = +\infty$ uniformly in (t,x) on each bounded subset V of $\mathcal{D}$. To this end, let $V\subset\mathcal{D}$ be bounded, say $V\subset(0,r)\times B(0,r)$, for some $r\in(0,+\infty)$ and let $M\in(0,+\infty)$ be arbitrarily given. Condition (4.13) says that there exist numbers $\lambda, N\in(0,+\infty)$ such that

$$\phi(y)\ge-\lambda|y| \text{ whenever } |y|\ge N.$$

But we can certainly find a positive number with the property where

$$f^*\frac{(z)}{|z|}\ge 2(M+\lambda) \text{ as } |z|\ge\nu.$$

Putting $N(V)\overset{\text{def}}{=}\max\{1,N,2r,r(1+\nu)\}$, we therefore deduce from (4.3) that if $(t,x)\in V$ and $|y|\ge N(V)$, then

$$\begin{aligned}\zeta(t,x,y) &\ge \left[-\lambda+\frac{f^*((x-y)/t)}{|(x-y)/t|}\cdot\frac{|x-y|}{|y|}\right]\cdot|y|\\ &\ge\left[-\lambda+2(M+\lambda)\cdot\frac{1}{2}\right]\cdot|y|\ge M\end{aligned}$$

because $|(x-y)/t|\ge[r(1+\nu)-r]/r=\nu$, $|x-y|/|y|\ge(|y|-r)/|y|\ge1/2$. ■

If $\phi=\phi(x)$ is globally Lipschitz continuous on $\mathbb{R}^n$, then (4.13) clearly holds. The following result of [7] can thus be considered as a consequence of Corollary 4.3.

Corollary 4.4. *If the initial function $\phi = \phi(x)$ is globally Lipschitz continuous and if the Hamiltonian $f = f(p)$ is strictly convex on $\mathbb{R}^n$ with $\lim_{|p|\to+\infty|} f(p)/|p| = +\infty$, then (4.4) determines a global Lipschitz solution of (4.1)–(4.2).*

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