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Short Communication

On Common Fixed Points for Three Commuting Mappings*

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1. In [2] (cf. [1]), we established some common fixed point theorems for a pair of commuting self-mappings on a complete metric space satisfying the so-called g-quasi-contraction and a metric condition of Fisher–Sessa type or Fisher–Iseki type. The case of three commuting mappings requires a new metric condition along these lines. More precisely, let (X, d) be a complete metric space, f_i, $i = 0, 1, 2$, three commuting self-mappings on X such that

(1) $f_i(X) \subset f_0(X), \quad i = 1, 2.$

(2) f_1 and f_2 satisfy the following g-quasi-contractive condition:

$$d(f_1x, f_2y) \leq g\big(\delta(\{f_0x, f_0y, f_1x, f_2y\})\big)\ \forall x, y \in X. \tag{1.1}$$

Here, $\delta(A) := \sup\{d(x, y) : x, y \in A\}$ for $A \subset X$ and $g: \mathbf{R}^+ \to \mathbf{R}^+$ is a function satisfying the following properties

$(g1)\, g$ is a non-decreasing function;
$(g2)\, g$ is right-continuous;
$(g3)\, g(t)\langle t\ \forall t\rangle 0$;
$(g4) \exists \lim_{t\to\infty} g(t)/t < 1$.
(Note that in the case $f_0 = \mathrm{id}_X$, (1.1) is nothing but the g-quasi-contraction treated in [2]).

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Let us introduce the following new conditions which generalize the metric conditions of Fisher–Sessa type or Fisher–Iseki type ([6, 7], cf. [1, 2]): There exists a point $x \in X$ such that $\forall y, y' \in \mathcal{O}_{f_0}(x)$

$$\sup\left\{d(f_i^{n+1}y, f_i^n y'),\quad n = 0, 1, 2, \ldots;\ i = 1, 2\right\} < \infty; \tag{1.2}$$

there exist a point $x \in X$ and a constant M such that $\forall y, y' \in \mathcal{O}_{f_0}(x)$

$$d(f_i^{n+1}y, f_i^n y') \leq (n+1)M \quad \text{for } n = 0, 1, 2, \ldots \text{ and } i = 1, 2. \tag{1.3}$$

(Here, $\mathcal{O}_{f_0}(x)$ denotes the orbit of x under f_0.)

Based on the approaches of Das–Naik [5] and ours mentioned above we can prove the following theorem.

Theorem 1. *Let (X, d) be a complete metric space, and $f_i,\ i = 0, 1, 2$ commuting self-mappings of X such that*

(i) $f_i,\ i = 0, 1, 2$ *satisfy conditions* (1.1), (1.3) *for a function* g *with properties* $(g1)$–$(g4)$;
(ii) $f_j(X) \subset f_0(X),\ j = 1, 2$;
(iii) f_0 is continuous.

Then there exists a unique common fixed point in X for f_0, f_1, f_2.

Corollary 1. *Let (X, d) be a complete metric space, and $f_i, i = 0, 1, 2$ commuting self-mappings of X such that*

(i) $f_i,\ i = 0, 1, 2$ *satisfy conditions* (1.1) *and* (1.2) *for a function* g *with properties* $(g1)$ *and* $(g4)$;
(ii) $f_j(X) \subset f_0(X),\ j = 1, 2$;
(iii) f_0 is continuous.

Then there exists a unique common fixed point in X for f_0, f_1, f_2.

Corollary 1 is an immediate consequence of Theorem 1 in view of the implication (1.2) $\Longrightarrow$ (1.3).

Corollary 2. [2, cf.1] *Let (X, d) be a complete metric space, and f_1, f_2 commuting self-mappings of X satisfying conditions (1.1) and (1.3) above with $f_0 = \mathrm{id}$ for a function g with properties (g1)–(g4). Then f_1, f_2 possess a unique common fixed point in X.*

Corollary 3. [2, cf.1] *Let (X, d) be a complete metric space, and f_1, f_2 commuting self-mappings of X satisfying conditions (1.1) and (1.2) above with $f_0 = \mathrm{id}$ for a function g with properties (g1)–(g4). Then f_1, f_2 possess a unique common fixed point in X.*

Corollaries 2 and 3 follow immediately from Theorem 1 and Corollary 1 by putting $f_0 = \mathrm{id}$, and as noted before (*loc. cit.*), they generalize and unify the results of [3, 4, 6, 7].

2. We can give various examples satisfying the conditions of Theorem 1 and Corollary 1. Let $X = [0, +\infty)$ with the usual metric. Consider the following self-mappings $f_0(x) = q_0 x,\ f_1(x) = q_1 x$ with $q_0 > q_1 > 0$, and $f_2(x) \equiv 0$. One checks easily that f_0, f_1, f_2 satisfy the conditions of Theorem 1 and Corollary 1 with function $g(t) = qt,\ q := q_1/q_0$. Hence, they have a unique common fixed point in X.

One can have more complicated examples. Let $X = \mathbf{N}$ be the set of positive integers which can be metrized as follows: $d(n, n) = 0,\ d(n, m) = d(m, n) = t_0 + 1/n^{\alpha}$ for $m > n$, where $t_0 \geq 0$ and $\alpha > 0$. If $t_0 > 0$, then X is complete with respect to d. Consider the following self-mappings of X : $f_0 = \text{id}$, $f_1(n) := n + 1$, $f_2(n) := n + 2$. Clearly f_0, f_1, and f_2 are commuting and have no common fixed points in X; since they satisfy the conditions of the above theorem and corollary for a function g with properties $(g1)$–$(g4)$, except for a "discontinuity" at $t = t_0$. These examples show that conditions $(g2)$ and $(g3)$ in the above theorem and corollary are essential. In order to remove the mentioned "discontinuity", one has to take $t_0 = 0$. But in this case, (X, d) is not complete. Clearly, we have a completion by adding the point ∞ to X with natural ordering $n < \infty$, $\forall n \in \mathbf{N}$, and $f_i,\ i = 0, 1, 2$ are well extended to the whole $X \cup \{\infty\} : f_i(\infty) = \infty$, i.e., ∞ is the unique common fixed point for f_0, f_1, and f_2.

It should be noted that the method here can be extended to the case of Menger probabilistic metric-spaces with a further application to the theory of random operator equations.

A detailed version with full proof of Theorem 1 will appear somewhere else.

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