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Stability of Linear Infinite-Dimensional Systems Under Affine and Fractional Perturbations

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Abstract. In this paper the robust stability of the class of positive linear discrete-time systems in Banach space under affine multi-perturbations and nonlinear fractional perturbations is studied via the notion of stability radii. We show that for this class the complex and the real stability radii coincide and can be computed by simple formulae. An example is provided to illustrate the result.

1. Introduction

In this paper we study the robust stability of infinite-dimensional, positive, discrete-time systems subjected to affine and nonlinear fractional perturbations. Our aim is to generalize some results of [5] and [10] to systems in Banach spaces.

Recall that the main problem in the study of robust stability of the dynamical system $x(k+1) = Ax(k)$ under affine perturbations is to characterize and compute its stability radius which can be defined as the smallest (in norm) complex or real perturbation Δ_i for which the perturbed system

$$x(k+1) = (A + \sum_{i=1}^{N} D_i \Delta_i E_i)x(k),$$

is unstable. Here, D_i, E_i are given matrices defining the structure of perturbation. We mention that in the case of *single affine perturbation* (that is, when $N = 1$), the robust stability of infinite-dimensional, positive systems was considered first in [1] for discrete-time systems and, quite recently, in [2] for continuous-time systems. In this paper we shall deal with the more general case when $N = \infty$. It will be shown, as in the case of single perturbation, that for positive systems, real and complex stability radii coincide and can be computed by a simple formula which extends the formula of [5] to positive systems in Banach space.

The following notations will be used throughout this paper. Let $\mathbf{C}$ and $\mathbf{R}$ be the set of all complex and real numbers, respectively. Define $\mathbf{N} = \{1, 2, ...\}$, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mathbf{R}_+ = \{t \in \mathbf{R} | t \geq 0\}$. Let X, Y be real or complex Banach spaces and X^* the dual space of X. Then $\mathcal{L}(X, Y)$ stands for the Banach space of bounded linear operators $A : X \to Y$ endowed with the norm $\|A\| = \sup_{\|x\|=1} \|Ax\|$, $\mathcal{L}(X) := \mathcal{L}(X, X)$, and I (or I_X) is the identity operator on X. For $A \in \mathcal{L}(X)$, the spectrum $\sigma(A)$ of A is defined by $\sigma(A) = \mathbf{C} \backslash \{s \in \mathbf{C} | (sI - A)^{-1} \in \mathcal{L}(X)\}$ and the spectral radius of A is denoted by $\rho(A) := \sup\{|s| \mid s \in \sigma(A)\}$. Sequences of elements in a Banach space will be denoted by $(u_i)_{i \in \mathbf{N}}$ or, more briefly, (u_i).

2. Robust Stability of Linear Systems in Banach Spaces

In this section we extend to Banach spaces the general framework developed in [5] for the study of robust stability of discrete-time systems under affine multi-perturbations (see [13] for a similar development). We consider the discrete-time linear system

$$x(k+1) = Ax(k), \quad k \in \mathbf{N}, \quad x \in X, \tag{1}$$

where X is a Banach space over $\mathbf{K}$, $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and A is a bounded linear operator on X, that is, $A \in \mathcal{L}(X)$. We say that the system (1) is *Schur stable* if

$$\rho(A) < 1.$$

Then it is well known that the system (1) is Schur stable if and only if there exist $c \geq 1$ and $0 < \alpha < 1$ such that

$$\|A^k\| \leq c\alpha^k, \quad k \in \mathbf{N}_0,$$

(see, e.g., [12]).

Assume U, Y are complex Banach spaces and A is Schur stable and subjected to affine perturbations of the form

$$A \rightsquigarrow A + D\Delta E, \quad \Delta \in \mathcal{D}. \tag{2}$$

Here, $D \in \mathcal{L}(U, X)$ and $E \in \mathcal{L}(X, Y)$ are given bounded linear operators and $\mathcal{D} \subset \mathcal{L}(Y, U)$ is a given subset of disturbance operators. As in [5], we introduce the following definition of stability radius:

Definition 2.1. *Given a subset $\mathcal{D} \subset \mathcal{L}(Y, U)$, the stability radius of the system (1) with respect to perturbations of the form (2) is defined by*

$$r_{\mathcal{D}}(A, D, E) = \inf\{\|\Delta\| \mid \Delta \in \mathcal{D}, \quad \rho(A + D\Delta E) \geq 1\}. \tag{3}$$

We now proceed by showing how the above general definition will be specified for some perturbation classes of particular interest.

Let Y_i, U_i $(i \in \mathbf{N})$ be Banach spaces. Assume that $E_i \in \mathcal{L}(X, Y_i)$, $D_i \in \mathcal{L}(U_i, X)$ $(i \in \mathbf{N})$ are given bounded linear operators. We consider perturbations of the following type:

$$A \rightsquigarrow A + \sum_{i=1}^{\infty} D_i \Delta_i E_i , \tag{4}$$

where $\Delta_i \in \mathcal{L}(Y_i, U_i)$, $i \in \mathbf{N}$ are unknown bounded linear operators defining the scaling of the parameter uncertainty. The uncertainty model

$$A \rightsquigarrow A + \sum_{i=1}^{N} D_i \Delta_i E_i \tag{5}$$

is clearly the particular case of (4) where $U_i = Y_i = \{0\}$, $i > N$. To ensure convergence of the series in (4), we assume

$$\sum_{i=1}^{\infty} \|D_i\| < \infty, \quad \sup_{i \in \mathbf{N}} \|E_i\| < \infty, \tag{6}$$

and the perturbation operators Δ_i are uniformly bounded:

$$\sup_{i \in \mathbf{N}} \|\Delta_i\| < \infty. \tag{7}$$

We show that the above perturbation class can be represented in the form of (2). To this end, let us consider the vector spaces of all bounded sequences

$$U = \{(u_i) \in \prod_{i \in \mathbf{N}} U_i \,|\, u_i \in U_i, \sup_{i \in \mathbf{N}} \|u_i\| < \infty\}, \tag{8}$$

$$Y = \{(y_i) \in \prod_{i \in \mathbf{N}} Y_i \,|\, y_i \in Y_i, \sup_{i \in \mathbf{N}} \|y_i\| < \infty\}. \tag{9}$$

It is easy to check that U, Y are Banach spaces with respect to the norms $\|(u_i)\|_U = \sup_{i \in \mathbf{N}} \|u_i\|$ and $\|(y_i)\|_Y = \sup_{i \in \mathbf{N}} \|y_i\|$. Let us define the linear operators $E : X \longrightarrow Y$, $D : U \longrightarrow X$ by setting

$$Ex = (E_i x)_{i \in \mathbf{N}}, \ \forall x \in X, \quad D(u_i)_{i \in \mathbf{N}} = \sum_{i=1}^{\infty} D_i u_i, \ \forall (u_i)_{i \in \mathbf{N}} \in U. \tag{10}$$

It follows from (6) that $E \in \mathcal{L}(X, Y)$, $D \in \mathcal{L}(U, X)$. Further, as in [13], for each sequence of operators $\Delta_i \in \mathcal{L}(Y_i, U_i)$, $i \in \mathbf{N}$, we define an operator $\Delta : Y \longrightarrow U$ by putting

$$\Delta((y_i)_{i \in \mathbf{N}}) = (\Delta_i y_i)_{i \in \mathbf{N}}, \quad (y_i)_{i \in \mathbf{N}} \in Y, \tag{11}$$

and we denote $\Delta = \prod_{i \in \mathbf{N}} \Delta_i$. It is easy to see that, under the assumption (7), $\Delta \in \mathcal{L}(Y, U)$ and

$$\|\Delta\| = \sup_{i \in \mathbf{N}} \|\Delta_i\|.$$

Using the above definitions, the perturbation model (4) can be rewritten in the following form:

$$A \rightsquigarrow A + D\Delta E, \ \ \Delta \in \mathcal{D}_I,$$

where

$$\mathcal{D}_I := \{\Delta = \prod_{i \in \mathbf{N}} \Delta_i \,|\, \ \Delta_i \in \mathcal{L}(Y_i, U_i), \sup_{i \in \mathbf{N}} \|\Delta_i\| < \infty\}.$$

Therefore, in this case, the stability radius of the system (1) is defined by

$$\begin{aligned} r_{\mathcal{D}_I}(A, D, E) &:= r(A; (D_i, E_i)_{i\in\mathbf{N}}) \\ &= \inf\{\sup_{i\in\mathbf{N}} \|\Delta_i\| \mid \Delta_i \in \mathcal{L}(Y_i, U_i), i \in \mathbf{N}, \sup_{i\in\mathbf{N}} \|\Delta_i\| < \infty, \\ &\qquad \rho(A + \sum_{i=1}^{\infty} D_i \Delta_i E_i) \geq 1\}. \end{aligned} \tag{12}$$

Consider now another perturbation class of interest. Let $A, A_i \in \mathcal{L}(X)$ $(i \in \mathbf{N})$ be given bounded linear operators such that A is Schur stable and

$$\sum_{i=1}^{\infty} \|A_i\| < \infty. \tag{13}$$

Suppose A is subjected to affine perturbation of the form

$$A \rightsquigarrow A + \sum_{i=1}^{\infty} \delta_i A_i\,, \tag{14}$$

where $(\delta_i)_{i\in\mathbf{N}}$ is a sequence of unknown scalar perturbation satisfying the following condition:

$$\sup_{i\in\mathbf{N}} |\delta_i| < \infty. \tag{15}$$

The affine perturbations (14) can also be represented in the form (2). To see this, denote by Y the Banach space of all bounded sequences in X:

$$Y = \{(x_i) \mid x_i \in X, i \in \mathbf{N}, \sup_{i\in\mathbf{N}} \|x_i\| < \infty\}$$

(with the norm $\|(x_i)\| = \sup_{i\in\mathbf{N}} \|x_i\|$) and define operators $E \in \mathcal{L}(X, Y)$, $D \in \mathcal{L}(Y, X)$, $\Delta \in \mathcal{L}(Y, Y)$ by setting

$$\begin{aligned} Ex &= (x_i),\ x_i = x,\ \forall i \in \mathbf{N},\ \forall x \in X \\ D(x_i) &= \sum_{i=1}^{\infty} A_i x_i,\ \forall (x_i) \in Y \qquad (16) \\ \Delta(x_i) &= (\delta_i x_i). \end{aligned}$$

Then (14) can be rewritten in the form:

$$A \rightsquigarrow A + D\Delta E,\ \Delta \in \mathcal{D}_{II},$$

where the perturbation class $\mathcal{D}_{II}$ is given by

$$\mathcal{D}_{II} = \{\prod_{i\in\mathbf{N}} \delta_i I \mid \delta_i \in \mathbf{C},\ i \in \mathbf{N},\ \sup_{i\in\mathbf{N}} |\delta_i| < \infty\}. \tag{17}$$

Thus, in this case, Definition 2.1 of the stability radius is reduced to

$$\begin{aligned} r_{\mathcal{D}_{II}}(A, D, E) &:= r(A; (A_i)_{i\in\mathbf{N}}) \\ &= \inf\{\sup_{i\in\mathbf{N}} |\delta_i| \mid \delta_i \in \mathbf{C},\ \sup_{i\in\mathbf{N}} |\delta_i| < \infty,\ \rho(A + \sum_{i=1}^{\infty} \delta_i A_i) \geq 1\}. \end{aligned} \tag{18}$$

We note that (14) can also be represented in the form (4) by setting $Y_i = U_i = X$, $D_i = A_i$, $E_i = I$, and $\Delta_i u_i = \delta_i u_i$ for all $i \in \mathbf{N}$.

Motivated by the above examples we introduce the following:

Definition 2.2. *Let Y_i, U_i ($i \in \mathbf{N}$) be Banach spaces. Let Banach spaces of sequences Y, U be defined by (8) and (9). A subset $\mathcal{D} \subset \mathcal{L}(Y, U)$ is called a perturbation class of block-diagonal operators if there exists a subset $J \subset \mathbf{N}$ such that $Y_i = U_i$ for $i \in \mathbf{N} \setminus J$ and*

$$\mathcal{D} = \{ \prod_{i \in \mathbf{N}} \Delta_i \mid \Delta_i \in \mathcal{D}_i,\ i \in \mathbf{N},\ \sup_{i \in \mathbf{N}} \|\Delta_i\| < \infty \}, \tag{19}$$

where

$$\mathcal{D}_i = \begin{cases} \mathcal{L}(Y_i, U_i) & \text{if } i \in J \\ \mathbf{C} I_{U_i} & \text{if } i \in \mathbf{N} \setminus J. \end{cases}$$

Thus, the classes $\mathcal{D}_I$ and $\mathcal{D}_{II}$ considered above correspond to the two extreme cases where $J = \mathbf{N}$ and $J = \{\emptyset\}$, respectively. As in [5], given a block-diagonal class $\mathcal{D} \subset \mathcal{L}(Y, U)$ we define the $\mu_{\mathcal{D}}$-function on $\mathcal{L}(U, Y)$ by setting, for each $M \in \mathcal{L}(U, Y)$,

$$\mu_{\mathcal{D}}(M) = [\inf\{\|\Delta\| \mid \Delta \in \mathcal{D},\ \det(I_U - M\Delta) = 0\}]^{-1}. \tag{20}$$

Then the following theorem is the extension of Proposition 3.7 in [5] to systems in Banach spaces and can be proved similarly (see, e.g., [13]).

Theorem 2.3. *Suppose the system (1) is Schur stable and subjected to affine perturbation of the form (2), where $\mathcal{D}$ is a perturbation class of block-diagonal operators. Let $G(s) = E(sI - A)^{-1} D$ be the transfer function defined on the resolvent set $\mathbf{C} \setminus \sigma(A)$. Then*

$$r_{\mathcal{D}}(A; D, E) = [\sup_{|s|=1} \mu_{\mathcal{D}}(G(s))]^{-1}. \tag{21}$$

3. Positive Systems in Banach Spaces

In the finite-dimensional case (i.e., when $X = \mathbf{R}^n$), the system (1) is called positive if and only if the matrix A is nonnegative, i.e., $A \in \mathbf{R}_+^{n \times n}$. Positive systems in $\mathbf{R}^n$ admit an interesting feature that their real stability radius is equal to the complex one and can be computed easily in certain cases (for instance, in the case of single affine perturbations and the case of block-diagonal perturbations of class $\mathcal{D}_{II}$) (see, e.g., [5]). Our objective is to generalize these results to positive systems in Banach spaces. For the convenience of the reader, we briefly summarize some notions and basic facts we need about Banach lattices and positive operators. We refer to [6, 7, 9] for more details.

Let $X_{\mathbf{R}}$ be a real Banach lattice, $X := X_{\mathbf{R}} + \imath X_{\mathbf{R}}$, the complex Banach lattice obtained by complexification of $X_{\mathbf{R}}$ and $X_{\mathbf{R}}^+ = \{x \in X_{\mathbf{R}} | x \geq 0\}$. Then $X_{\mathbf{R}}^+$ is a closed convex cone in $X_{\mathbf{R}}$. The modulus of $x \in X_{\mathbf{R}}$ defined by $|x| := \sup\{x, -x\}$ satisfies the lattice norm property, that is,

$$|x| \leq |y| \ \Rightarrow\ \|x\| \leq \|y\|,\quad \forall x, y \in X_{\mathbf{R}} \tag{22}$$

which implies $\| |x| \| = \|x\|$ for all $x \in X_{\mathbf{R}}$. For $z = x + \imath y \in X$, the modulus of z defined by $|z| = \sup\{|(\cos\phi)x + (\sin\phi)y| : 0 \le \phi \le 2\pi\} \in X_{\mathbf{R}}$ satisfies $\|z\| = \| |z| \|$. Let X, Y be complex Banach lattices which are complexifications of $X_{\mathbf{R}}, Y_{\mathbf{R}}$, respectively. Then the bounded linear operator $A \in \mathcal{L}(X, Y)$ is called real if $AX_{\mathbf{R}} \subset Y_{\mathbf{R}}$. A is called positive ($A \ge 0$) if A is real and $AX_{\mathbf{R}}^{+} \subset Y_{\mathbf{R}}^{+}$. The set of all real and positive operators in $\mathcal{L}(X, Y)$ are denoted, respectively, by $\mathcal{L}^{\mathbf{R}}(X, Y)$ and $\mathcal{L}^{+}(X, Y)$. If $\sup_{|z| \le x} |Az| \in Y_{\mathbf{R}}$, for every $x \in X_{\mathbf{R}}^{+}$, then there exists a unique operator $|A| \in \mathcal{L}^{+}(X, Y)$ such that

$$|A|x = \sup_{|z| \le x} |Az|, \quad x \in X_{\mathbf{R}}^{+}.$$

The following monotonicity property of positive operators will be used frequently: Let $A \in \mathcal{L}^{+}(X, Y)$ and $B \in \mathcal{L}(X, Y)$ such that $|B|$ exists, then

$$0 \le |B| \le A \Rightarrow \|B\| \le \| |B| \| \le \|A\| \Rightarrow \rho(B) \le \rho(|B|) \le \rho(A) \tag{23}$$

(the second implication is a consequence of the Gelfand–Beurlin formula for spectral radius). Let $(Y_{\mathbf{R}i})_{i \in \mathbf{N}}$ be a sequence of real Banach lattices. Then it is easy to show that the vector space of all bounded sequences

$$Y_{\mathbf{R}} := \{(y_i)_{i \in \mathbf{N}} \in \prod_{i \in \mathbf{N}} Y_{\mathbf{R}i} \,|\, \sup_{i \in \mathbf{N}} \|y_i\| < \infty\}$$

endowed with the norm $\|(y_i)\| = \sup_{i \in \mathbf{N}} \|y_i\|$ and the order relation

$$(y_i) \le (\tilde{y}_i) \Leftrightarrow y_i \le \tilde{y}_i \quad (i \in \mathbf{N})$$

is also a real Banach lattice satisfying

$$\sup\{(y_i), (\tilde{y}_i)\} = (\sup\{y_i, \tilde{y}_i\}) \quad \forall (y_i), (\tilde{y}_i) \in Y_{\mathbf{R}}. \tag{24}$$

In particular, we have

$$|(y_i)| = (|y_i|), \quad \forall (y_i) \in Y_{\mathbf{R}}. \tag{25}$$

Let $Y := Y_{\mathbf{R}} + \imath Y_{\mathbf{R}}$ be the complex Banach lattice obtained by the complexification of $Y_{\mathbf{R}}$. Then it is easily seen that Y is isometrically isomorphic to the Banach space of all bounded sequences

$$\prod_{i \in \mathbf{N}} Y_i := \{(z_i) |\, z_i \in Y_i, i \in \mathbf{N}, \sup \|z_i\| < \infty\},$$

where $Y_i = Y_{\mathbf{R}i} + \imath Y_{\mathbf{R}i}$ is the complexification of $Y_{\mathbf{R}i}$. Therefore, in what follows, given a sequence of complex Banach lattices Y_i, we shall identify, without loss of generality, $\prod_{i \in \mathbf{N}} Y_i$ with the complex Banach lattice Y.

Let X be a complex Banach lattice. We consider the discrete-time linear system

$$x(k+1) = Ax(k) \quad k \in \mathbf{N}, \quad x(k) \in X, \tag{26}$$

where $A \in \mathcal{L}(X)$ and A is Schur stable. System (26) is called positive if $A \in \mathcal{L}^{+}(X)$.

Suppose Y_i, U_i, $(i \in \mathbf{N})$ are complex Banach lattices and $E_i \in \mathcal{L}(X, Y_i)$, $D_i \in \mathcal{L}(U_i, X)$, $(i \in \mathbf{N})$ are given bounded linear operators satisfying the conditions (6). We define the Banach spaces of bounded sequences U, Y by (8) and (9) and the operators $E \in \mathcal{L}(X, Y)$, $D \in \mathcal{L}(U, X)$, $\Delta := \prod_{i\in\mathbf{N}} \Delta_i \in \mathcal{L}(Y, U)$ by (10) and (11). Let $\mathcal{D} \subset \mathcal{L}(Y, U)$ be a perturbation class of block-diagonal operators. We define the following classes of real and positive perturbations:

$$\mathcal{D}_{\mathbf{R}} = \left\{ \prod_{i\in\mathbf{N}} \Delta_i \mid \Delta_i \in \mathcal{L}^{\mathbf{R}}(Y_i, U_i),\ i \in \mathbf{N},\ \sup_{i\in\mathbf{N}} \|\Delta_i\| < \infty \right\}, \tag{27}$$

$$\mathcal{D}_{+} = \left\{ \prod_{i\in\mathbf{N}} \Delta_i \mid \Delta_i \in \mathcal{L}^{+}(Y_i, U_i),\ i \in \mathbf{N},\ \sup_{i\in\mathbf{N}} \|\Delta_i\| < \infty \right\}. \tag{28}$$

It is clear that $\mathcal{D}_{\mathbf{R}}$, $\mathcal{D}_+$ are linear subspaces of $\mathcal{L}(Y, U)$ and

$$\mathcal{D}_+ \subset \mathcal{D}_{\mathbf{R}} \subset \mathcal{D}.$$

Therefore, from Definition 2.1, we have

$$r_{\mathcal{D}}(A; D, E) \le r_{\mathcal{D}_{\mathbf{R}}}(A; D, E) \le r_{\mathcal{D}_+}(A; D, E). \tag{29}$$

The following lemma gives a technical criterion under which equalities of the stability radii hold.

Lemma 3.1. *If $\mathcal{D} \subset \mathcal{L}(Y, U)$ is a perturbation class of block-diagonal operators, then for all $\Delta = \prod_{i\in\mathbf{N}} \Delta_i \in \mathcal{D}$ and $y = (y_i) \in Y$, there exists $\tilde{\Delta} = \prod_{i\in\mathbf{N}} \tilde{\Delta}_i \in \mathcal{D}$, satisfying*

$$\tilde{\Delta} y = \Delta y,\ |\tilde{\Delta}| \in \mathcal{D}_+$$

and

$$\| |\tilde{\Delta}| \| \le \|\Delta\|,$$

where $|\tilde{\Delta}| = \prod_{i\in\mathbf{N}} |\tilde{\Delta}_i|$.

Proof. The proof is similar to the one in [5]. Suppose $\Delta = \prod_{i\in\mathbf{N}} \Delta_i \in \mathcal{D}$, where $\Delta_i \in \mathcal{L}(Y_i, U_i)$, $i \in \mathbf{N}$, and $y = (y_i) \in Y$, $y \ne 0$. Assume that $\Delta y = (u_i) = (\Delta_i y_i)$. Then $u_i = \Delta_i y_i$, $i \in \mathbf{N}$ and we define

$$\tilde{\Delta} = \prod_{i\in\mathbf{N}} \tilde{\Delta}_i,$$

where

$$\tilde{\Delta}_i = \begin{cases} \Delta_i & \text{if } i \in \mathbf{N} \setminus J \\ 0 & \text{if } i \in J,\ y_i = 0 \\ \frac{u_i f_i}{\|y_i\|} & \text{if } i \in J,\ y_i \ne 0, \end{cases}$$

$f_i \in (Y_i)^*$ is a linear functional on Y_i such that $\|f_i\| = 1$ and $f_i(y_i) = \|y_i\|$ (by the Hahn–Banach theorem). Clearly, $\tilde{\Delta}_i y_i = u_i$, $\|\tilde{\Delta}_i\| \le \|\Delta_i\|$, $i \in \mathbf{N}$ and so $\tilde{\Delta} y = \Delta y$. It is obvious that, for every $i \in \mathbf{N} \setminus J$, $|\tilde{\Delta}_i|$ exists and $\| |\tilde{\Delta}_i| \| = \|\tilde{\Delta}_i\|$. Since $\tilde{\Delta}_i$ is of rank one for $i \in J$, it follows from Lemma 3.4 in [1] that $|\tilde{\Delta}_i|$ exists and $\| |\tilde{\Delta}_i| \| = \|\tilde{\Delta}_i\|$. So we have $|\tilde{\Delta}| = \prod_{i\in\mathbf{N}} |\tilde{\Delta}_i| \in \mathcal{D}_+$ and $\| |\tilde{\Delta}| \| = \sup_{i\in\mathbf{N}} \| |\tilde{\Delta}_i| \| = \sup_{i\in\mathbf{N}} \|\tilde{\Delta}_i\| \le \sup_{i\in\mathbf{N}} \|\Delta_i\| = \|\Delta\|$. This completes the proof. ■

We are now in a position to prove the following result.

Theorem 3.2. *Let the systems (1) be positive and Schur stable. Let $E_i \in \mathcal{L}^+(X, Y_i)$, $D_i \in \mathcal{L}^+(U_i, X)$, $i \in \mathbf{N}$, be given such that (6) is satisfied, and let D, E be given by (10). Assume A and, for each $i \in \mathbf{N}$, either D_i or E_i are compact operators. If $\mathcal{D}$ is a perturbation class of block-diagonal operators, then*

$$r_{\mathcal{D}}(A, D, E) = r_{\mathcal{D}_{\mathbf{R}}}(A, D, E) = r_{\mathcal{D}_+}(A, D, E).$$

Proof. The case $r_{\mathcal{D}}(A, D, E) = \infty$ is trivial. Suppose $r_{\mathcal{D}}(A, D, E) < \infty$, then by (29), it suffices to show that $r_{\mathcal{D}_+}(A, D, E) \leq r_{\mathcal{D}}(A, D, E)$. Let $\Delta = \prod_{i\in\mathbf{N}} \Delta_i \in \mathcal{D}$ be a destabilizing disturbance, that is, $\rho(A + D\Delta E) \geq 1$. Since $A + D\Delta E = A + \sum_{i=1}^{\infty} D_i \Delta_i E_i$ is a compact operator, there exists a complex number α and a non-zero vector $x_0 \in X$ such that $(A + D\Delta E)x_0 = \alpha x_0$, $|\alpha| = \rho(A + D\Delta E)$. By Lemma 3.1, there exists $\tilde{\Delta} = \prod_{i\in\mathbf{N}} \tilde{\Delta}_i \in \mathcal{D}$ such that $\tilde{\Delta} E x_0 = \Delta E x_0$, $|\tilde{\Delta}| = \prod_{i\in\mathbf{N}} |\tilde{\Delta}_i| \in \mathcal{D}_+$, and $\||\tilde{\Delta}|\| \leq \|\Delta\|$. We will show that $|\tilde{\Delta}|$ is also a destabilizing disturbance. Indeed, we have

$$|(A + D\tilde{\Delta}E)x| = |(A + \sum_{i=1}^{\infty}(D_i \tilde{\Delta}_i E_i))x| \leq |Ax| + |\sum_{i=1}^{\infty}(D_i \tilde{\Delta}_i E_i)x|, \quad \forall x \in X.$$

Since X^+ is closed, it follows that $|\sum_{i=1}^{\infty}(D_i \tilde{\Delta}_i E_i)x| \leq \sum_{i=1}^{\infty} |(D_i \tilde{\Delta}_i E_i)x|$, $\forall x \in X$. Since A, D_i and E_i $(i \in \mathbf{N})$ are positive operators, we have

$$\begin{aligned} |(A + D\Delta E)x| &\leq A|x| + \sum_{i=1}^{\infty}(D_i |\tilde{\Delta}_i| E_i)|x| \\ &= (A + \sum_{i=1}^{\infty}(D_i |\tilde{\Delta}_i| E_i))|x| \\ &= (A + D|\tilde{\Delta}|E)|x| \end{aligned}$$

for every $x \in X$. It follows from Lemma 3.5 in [1] that $\rho(A + D|\tilde{\Delta}|E) \geq \rho(A + D\tilde{\Delta}E)$. Moreover, since $(A + D\tilde{\Delta}E)x_0 = (A + D\Delta E)x_0 = \alpha x_0$, we have $\rho(A + D|\tilde{\Delta}|E) \geq |\alpha| \geq 1$. Therefore, by definition, we have

$$r_{\mathcal{D}_+}(A; D, E) \leq r_{\mathcal{D}}(A; D, E),$$

completing the proof. ∎

As it will be shown in Theorem 4.4 below (when $M = 0$), in the case where $N = 1$, the above theorem remains valid without the compactness assumption (see also [1] for another proof of this fact). This is also true for the case where $\mathcal{D} = \mathcal{D}_{II}$ which is defined by (17). To see this, let X be a complex Banach lattice, let $A, A_i \in \mathcal{L}^+(X)$ $(i \in \mathbf{N})$ be given such that A is Schur stable, and $\sum_{i=1}^{\infty} \|A_i\| < \infty$. Suppose A is subjected to perturbations of the form

$$A \rightsquigarrow A + \sum_{i=1}^{\infty} \delta_i A_i\,, \tag{30}$$

where $(\delta_i)_{i\in\mathbf{N}}$ are unknown scalar sequences satisfying $\sup_{i\in\mathbf{N}} |\delta_i| < \infty$. Let Y be the Banach space of bounded sequences in X:

$$Y = \{(x_i) : x_i \in X,\ i \in \mathbf{N},\ \|(x_i)\| := \sup_{i\in\mathbf{N}} \|x_i\| < \infty\}.$$

As noted at the beginning of this section, Y can be considered as a complex Banach lattice which is the complexification of $Y_{\mathbf{R}} := \{(x_i) : x_i \in X_{\mathbf{R}}\}$. Then, as shown in the previous section, (30) can be represented in the form

$$A \rightsquigarrow A + D\Delta E,\ \Delta \in \mathcal{D}_{II},$$

where $E \in \mathcal{L}(X, Y)$, $D \in \mathcal{L}(Y, X)$, $\Delta \in \mathcal{L}(Y, Y)$ are defined by (16) and the perturbation class $\mathcal{D}_{II}$ is given by

$$\mathcal{D}_{II} = \{\prod_{i\in\mathbf{N}} \delta_i I \mid \delta_i \in \mathbf{C},\ i \in \mathbf{N},\ \sup_{i\in\mathbf{N}} |\delta_i| < \infty\}.$$

It is easy to verify that E, D are positive operators: $E \in \mathcal{L}^+(X, Y)$, $D \in \mathcal{L}^+(Y, X)$.

Denote by $\mathcal{D}_{II}^{\mathbf{R}}$ and $\mathcal{D}_{II}^{+}$ the subclasses of real and nonnegative perturbations obtained by putting, respectively, $\delta_i \in \mathbf{R}$ and $\delta_i \geq 0$ in the formula defining $\mathcal{D}_{II}$. Let

$$G(s) = E(sI - A)^{-1}D$$

be the corresponding transfer function. Since $A \in \mathcal{L}^+(X)$, it is well known from the spectral theory of positive operators (see, e.g., [6]) that $\rho(A) \in \sigma(A)$ and $(sI - A)^{-1} \in \mathcal{L}^+(X)$ if and only if $s > \rho(A)$. This implies $G(s) \in \mathcal{L}^+(Y, Y)$ for all $s > \rho(A)$. Moreover, the following monotonicity property holds:

$$t_2 \geq t_1 > \rho(A) \Rightarrow G(t_1) \geq G(t_2) \geq 0 \text{ and } \|G(t_1)\| \geq \|G(t_2)\| \tag{31}$$

(see [5, Lemma 4.1] and [1, Proposition 3.9]).

Theorem 3.3. *Let A, $A_i \in \mathcal{L}^+(X)$, $i \in \mathbf{N}$, and A is Schur stable. Suppose A is subject to affine perturbations of the form (14) and denote by $r_{\mathbf{C}}$, $r_{\mathbf{R}}$, r_+ the stability radii of A corresponding to perturbation classes $\mathcal{D}_{II}$, $\mathcal{D}_{II}^{\mathbf{R}}$, $\mathcal{D}_{II}^{+}$ of complex, real, and nonnegative perturbations, respectively. Then*

$$r_{\mathbf{C}} = r_{\mathbf{R}} = r_+ = \frac{1}{\rho(G(1))}. \tag{32}$$

Proof. It is clear that

$$r_{\mathbf{C}} \leq r_{\mathbf{R}} \leq r_+ .$$

Suppose $r_{\mathbf{C}} < +\infty$ and let $\Delta = \prod_{i\in\mathbf{N}} \delta_i I \in \mathcal{D}_{II}$ be a destabilizing operator, that is,

$$\rho(A + D\Delta E) = \rho(A + \sum_{i=1}^{\infty} \delta_i A_i) \geq 1.$$

Set

$$\tilde{\Delta} = \prod_{i\in\mathbf{N}} |\delta_i| I \in \mathcal{D}_{II}^{+}.$$

Then we have

$$|(A + D\Delta E)x| \leq |Ax| + |(D\Delta E)x| .$$

Since $A,\ A_i \in \mathcal{L}^+(X)$ and $X^+_{\mathbf{R}}$ is closed in X, we can derive

$$|(A+D\Delta E)x| \le A|x| + \sum_{i=1}^{\infty} |\delta_i|\,|A_i x| \le A|x| + \sum_{i=1}^{\infty}(|\delta_i|\,A_i)|x|$$
$$= (A+\sum_{i=1}^{\infty}|\delta_i|\,A_i)|x| = (A+D\tilde{\Delta}E)|x|,\ \forall x\in X. \quad (33)$$

Therefore, from Lemma 3.5 in [1], it follows that

$$\rho(A+D\tilde{\Delta}E) \ge \rho(A+D\Delta E) \ge 1$$

so that $\tilde{\Delta}$ is destabilizing. Moreover, since $\|\tilde{\Delta}\| = \|\Delta\| = \sup_{i\in\mathbf{N}}|\delta_i|$, we obtain by definition that

$$r_+ \le r_{\mathbf{C}}\,.$$

This establishes the first two equalities in (32). Assume $r_+ < +\infty$, otherwise there is nothing to show. By definition, for every $\varepsilon > 0$, there exists $\Delta \in \mathcal{D}^+_{II}$ such that

$$\|\Delta\| < r_+ + \varepsilon,\ \ s_0 := \rho(A+D\Delta E) \ge 1.$$

Since $A + D\Delta E \in \mathcal{L}^+(X)$, it follows that $s_0 \in \sigma(A + D\Delta E)$ (Perron–Frobenius theorem), and moreover, s_0 is contained in the approximate point spectrum of $A+D\Delta E$ (see, e.g., [7, Proposition 2.2]). Therefore, there exists a sequence $x_n \in X$ such that $\|x_n\| = 1,\ \ \|s_0x_n - (A+D\Delta E)x_n\| \to 0$ as $n \to \infty$. Since $(s_0I - A)$ is invertible, we can derive that

$$\|x_n - (s_0I - A)^{-1}D\Delta E x_n\| \to 0,\ \ n\to\infty \quad (34)$$

and hence, $\|Ex_n - G(s_0)\Delta E x_n\| \to 0$ as $n \to \infty$. Since $\|x_n\| = 1$, from (34), it follows that there exist $c > 0$ and a subsequence x_{n_k} of x_n such that $\|Ex_{n_k}\| \ge c$. Setting $y_k = Ex_{n_k}/\|Ex_{n_k}\|$, we obtained that $\|y_k - G(s_0)\Delta y_k\| \to 0$ as $k \to \infty$, which means

$$1 \in \sigma(G(s_0)\Delta)$$

and therefore, $\rho(G(s_0)\Delta) \ge 1$. On the other hand, since $\Delta \in \mathcal{D}^+_{II}$, for any $y \in Y^+_{\mathbf{R}} := \{(x_i) \mid x_i \in X^+_{\mathbf{R}}\}$, we have

$$\Delta y = (\delta_i x_i) \le \sup_{i\in\mathbf{N}} |\delta_i|(x_i) = \|\Delta\|y\,.$$

Since $s_0 \ge 1 > \rho(A)$, it follows, in view of (31), that

$$\|\Delta\|G(1) \ge \|\Delta\|G(s_0) \ge G(s_0)\Delta \ge 0,$$

which, by (23), yields

$$\|\Delta\|\rho(G(1)) = \rho(\|\Delta\|G(1)) \ge \rho(G(s_0)\Delta)) \ge 1.$$

Hence, $r_+ \ge \|\Delta\| - \varepsilon \ge [\rho(G(1))]^{-1} - \varepsilon$, which implies

$$r_+ \ge \frac{1}{\rho(G(1))}.$$

On the other hand, since $G(1) \geq 0$, it follows that $s_1 := \rho(G(1))$ is contained in the approximate spectrum of $G(1)$. Therefore, there exists a sequence y_n in Y such that $\|y_n\| = 1$ and $\|s_1 y_n - G(1)y_n\| \to 0$ as $n \to \infty$. Define the disturbance operator $\Delta_1 \in \mathcal{D}_{II}^+$ by setting

$$\Delta_1(x_i) = s_1^{-1}(x_i), \quad (x_i) \in Y,$$

then we have $\|y_n - G(1)\Delta_1 y_n\| \to 0$ as $n \to \infty$. This implies $1 \in \sigma(G(1)\Delta_1)$ which is equivalent to $1 \in \sigma(A + D\Delta_1 E)$. Thus, Δ_1 is a destabilizing operator, and hence, by definition,

$$\|\Delta_1\| = \frac{1}{\rho(G(1))} \geq r_+.$$

Consequently, $r_+ = [\rho(G(1))]^{-1}$, concluding the proof. ■

We illustrate the above result by a simple example. Consider the linear discrete time system

$$x(k+1) = Ax(k), \quad k \in \mathbf{N}, \quad x(k) \in l_2,$$

where the operator $A : l_2 \longrightarrow l_2$ is defined by the infinite matrix (a_{nm}) with $a_{nm} = 0$, $n \neq m$ and $a_{nn} = 1/2^n$, $n, m \in \mathbf{N}$. It is clear that $A \in \mathcal{L}(l_2)$, A is a compact operator, and $\sigma(A) = \{1/2^n \,|\, n \in \mathbf{N}\}$. Therefore, the system is Schur stable. We consider the perturbed system

$$x(k+1) = (A + \sum_{i=1}^{p} \delta_i A_i)x(k), \quad k \in \mathbf{N},$$

where $A_i = (a_{nm}^{(i)})$ (with $a_{nm}^{(i)} = 1$ if $n = m = i$, otherwise, $a_{nm}^{(i)} = 0$) are given infinite matrices and $p \in \mathbf{N}$ is given. By Theorem 3.3, the perturbed system remains Schur stable for all $(\delta_i) \in \mathbf{C}^p$ such that $\max_{1 \leq i \leq p} |\delta_i| < 1/\rho(G(1)) = 1/2$. Finally, we consider the perturbed system

$$x(k+1) = (A + \sum_{i=1}^{\infty} \delta_i A_i)x(k), \quad k \in \mathbf{N},$$

where $A_i = (a_{nm}^{(i)})$ (with $(a_{nm}^{(i)}) = (1/2)i$ if $n = m = i$, otherwise, $a_{nm}^{(i)} = 0$). Then this system is Schur stable for all sequences (δ_i), $\delta_i \in \mathbf{C}$ such that $\sup_{i \in \mathbf{N}} |\delta_i| < 1/\rho(G(1)) = 1$.

4. Systems Under Fractional Perturbations

Let X, Y, U be a complex Banach lattice with real parts $X_\mathbf{R}$, $Y_\mathbf{R}$, $U_\mathbf{R}$, respectively. Assume the linear infinite-dimensional system

$$x(k+1) = Ax(k), \quad k \in \mathbf{N}, \quad x \in X \tag{35}$$

is subjected to non-linear fractional perturbations of the form

$$A \rightsquigarrow A(\Delta) := A + D\Delta(I - M\Delta)^{-1}E, \tag{36}$$

where $D \in \mathcal{L}(U,X)$, $E \in \mathcal{L}(X,Y)$, $M \in \mathcal{L}(U,Y)$ are given bounded linear operators and $\Delta \in \mathcal{L}(Y,U)$ are unknown bounded linear perturbation operators. Our consideration in this section is motivated by the recent results due to Shafai et al. [10] for finite-dimensional systems. It turns out that some of the ideas behind the proof of results in [2, 3, 5] can be used to treat the case of fractional perturbations.

Let $A \in \mathcal{L}(X)$ such that $\rho(A) < 1$. We define the complex and real stability radii of A by

$$r_{\mathbf{C}}(A; D, E, M) := \inf\{\|\Delta\| \mid \Delta \in \mathcal{L}(Y,U), \quad \text{either } 1 \in \sigma(M\Delta) \text{ or } \rho(A(\Delta)) \geq 1\},$$
$$r_{\mathbf{R}}(A; D, E, M) := \inf\{\|\Delta\| \mid \Delta \in \mathcal{L}_{\mathbf{R}}(Y,U), \text{ either } 1 \in \sigma(M\Delta) \text{ or } \rho(A(\Delta)) \geq 1\},$$

where we set $\inf \emptyset := \infty$. Obviously,

$$r_C(A; D, E, M) \leq r_R(A; D, E, M). \tag{37}$$

Let $G(s) := E(sI - A)^{-1}D \in \mathcal{L}(U,Y)$, $s > \rho(A)$, denote the transfer function associated with A, D, E. We set $\tilde{G}(s) := M + G(s)$, $s > \rho(A)$. It is clear that $\tilde{G}(s)$ is analytic on $\mathbf{C}\backslash\overline{\mathbf{D}}, \mathbf{D} := \{s \in \mathbf{C} : |s| < 1\}$. Therefore, it follows that, for each $f \in Y^*$ and $u \in U$, the function $s \longrightarrow f(\tilde{G}(s)u)$ is analytic on $\mathbf{C} \setminus \overline{\mathbf{D}}$. Since $\lim_{|s|\to\infty} |f(\tilde{G}(s)u)|$ exists, we obtain, by the maximum principle, that

$$\max_{|s|\geq 1} |f(\tilde{G}(s)u)| = \max_{|s|=1} |f(\tilde{G}(s)u)|, \tag{38}$$

and consequently,

$$\max_{|s|\geq 1} \|\tilde{G}(s)\| = \max_{|s|=1} \|\tilde{G}(s)\|. \tag{39}$$

In order to give the characterization of stability radii of system (35), we need the following two lemmas.

Lemma 4.1. *Let Y, U be Banach lattices, $M \in \mathcal{L}(U,Y)$. Then*

$$\inf\{\|\Delta\| \mid \Delta \in \mathcal{L}(Y,U),\ 1 \in \sigma(M\Delta)\} = \frac{1}{\|M\|}.$$

Proof. It is easy to see that, if $\Delta \in \mathcal{L}(Y,U)$ satisfies $\|\Delta\| < 1/\|M\|$, then $(I - M\Delta)$ is invertible. It follows that

$$\inf\{\|\Delta\| \mid \Delta \in \mathcal{L}(Y,U),\ 1 \in \sigma(M\Delta)\} \geq \frac{1}{\|M\|}.$$

Fixing $\varepsilon > 0$, from the formula $\|M\| = \sup_{u\in U, \|u\|=1} \|Mu\|$, there exists $u_0 \in U$, $\|u_0\| = 1$ such that $1/\|Mu_0\| < 1/\|M\| + \varepsilon$. By the Hahn–Banach theorem, there exists a linear functional $f \in Y^*$ such that $f(Mu_0) = \|Mu_0\|$, $\|f\| = 1$. Set $\Delta_1 := f(.)u_0/\|Mu_0\| \in \mathcal{L}(Y,U)$. We have $\|\Delta_1\| = 1/\|Mu_0\| < 1/\|M\| + \varepsilon$ and $\Delta_1 M u_0 = u_0$. Hence, $1 \in \sigma(M\Delta_1)$ and $\|\Delta_1\| < 1/\|M\| + \varepsilon$, completing the proof. ■

From the above proof, we observe that, if $M \in \mathcal{L}^+(U,Y)$, then for any $\varepsilon > 0$, one can choose $\Delta \in \mathcal{L}^+(Y,U)$ such that $\|\Delta\| < (1/\|M\|) + \varepsilon$ and $1 \in \sigma(M\Delta)$. Indeed, since $M \in \mathcal{L}^+(U,Y)$, there exists $u_0 \in U^+$ such that $\|u_0\| = 1$ and $1/\|Mu_0\| < 1/\|M\| + \varepsilon$. Proceeding further as in the proof of the previous lemma where linear functional f is chosen, by the Krein theorem [9], to be positive (i.e., $f(y) \geq 0$ for all $y \in Y^+$), we obtain $\Delta \in \mathcal{L}^+(Y,U)$ satisfying the required property.

Lemma 4.2. *Let $\Delta \in \mathcal{L}(Y, U)$ be such that $(I - M\Delta)$ is invertible. Then $\sigma(A(\Delta)) \subset \mathbf{D}$ if and only if $(I - (M + G(s))\Delta)$ is invertible for every $s \in \mathbf{C}$, $|s| \geq 1$.*

Proof. Assume the system (35) is Schur stable, that is, $\sigma(A) \subset \mathbf{D}$. It follows that $(sI - A)$ is invertible for every $s \in \mathbf{C}$, $|s| \geq 1$. Therefore, we can write the following obvious equivalent relations:

$$\begin{aligned}
\sigma(A(\Delta)) \subset \mathbf{D} &\Leftrightarrow sI - A - D\Delta(I - M\Delta)^{-1}E \text{ is invertible for } s \in \mathbf{C}, |s| \geq 1\\
&\Leftrightarrow I - (sI - A)^{-1}(D\Delta(I - M\Delta)^{-1})E \text{ is invertible for } s \in \mathbf{C}, |s| \geq 1\\
&\Leftrightarrow I - E(sI - A)^{-1}D\Delta(I - M\Delta)^{-1} \text{ is invertible for } s \in \mathbf{C}, |s| \geq 1\\
&\Leftrightarrow I - M\Delta - E(sI - A)^{-1}D\Delta \text{ is invertible for } s \in \mathbf{C}, |s| \geq 1\\
&\Leftrightarrow I - (M + G(s))\Delta \text{ is invertible for } s \in \mathbf{C}, |s| \geq 1.
\end{aligned}$$

This concludes the proof. ∎

Now, we are prepared to give the following characterization of the complex stability radius of positive systems (35) under the perturbations of the form (36).

Theorem 4.3. *Let the linear discrete-time system (35) be Schur stable. Assume $D \in \mathcal{L}(U, X)$, $E \in \mathcal{L}(X, Y)$, $M \in \mathcal{L}(U, Y)$ are given and A is subjected to nonlinear fractional perturbations of the form (36). Then we have*

$$r_{\mathbf{C}}(A; D, E, M) = \min\left\{\frac{1}{\|M\|}, \frac{1}{\sup_{|s|=1}\|M + G(s)\|}\right\}. \tag{40}$$

Proof. Suppose $\Delta \in \mathcal{L}(Y, U)$ is a destabilizing perturbation operator. If $I - M\Delta$ is not invertible, then by Lemma 4.1, we have

$$\|\Delta\| \geq \frac{1}{\|M\|} \geq \min\left\{\frac{1}{\|M\|}, \frac{1}{\sup_{|s|=1}\|M + G(s)\|}\right\}.$$

Otherwise, from Lemma 4.2, there exists a complex number s_0, $|s_0| \geq 1$ such that $I - (M + G(s_0))\Delta$ is not invertible and so, again by Lemma 4.1, we have

$$\begin{aligned}
\|\Delta\| &\geq \frac{1}{\|M + G(s_0)\|} \geq \frac{1}{\sup_{|s|\geq 1}\|M + G(s)\|}\\
&= \frac{1}{\sup_{|s|=1}\|M + G(s)\|} \geq \min\left\{\frac{1}{\|M\|}, \frac{1}{\sup_{|s|=1}\|M + G(s)\|}\right\}.
\end{aligned}$$

Consequently, by the definition $r_{\mathbf{C}}(A; D, E, M)$, we have

$$r_{\mathbf{C}}(A; D, E, M) \geq \min\left\{\frac{1}{\|M\|}, \frac{1}{\sup_{|s|=1}\|M + G(s)\|}\right\}.$$

Furthermore, by the definition of supremum, for any $\varepsilon > 0$, there exists $s_1 \in \mathbf{C}$, $|s_1| = 1$ such that

$$\frac{1}{\|M + G(s_1)\|} < \frac{1}{\sup_{|s|=1}\|M + G(s)\|} + \frac{\varepsilon}{2}.$$

This implies, in combination with Lemma 4.1, that there exists an operator $\Delta_1 \in \mathcal{L}(Y, U)$ such that $I - (M + G(s_1))\Delta_1$ is not invertible and satisfying

$$\frac{1}{\|M + G(s_1)\|} \le \|\Delta_1\| < \frac{1}{\|M + G(s_1)\|} + \frac{\varepsilon}{2} < \frac{1}{\sup_{|s|=1} \|M + G(s)\|} + \varepsilon . \quad (41)$$

If $(I - M\Delta_1)$ is not invertible (or, equivalently, $1 \in \sigma(M\Delta_1)$), then it follows from the definition that $r_{\mathbf{C}}(A; D, E, M) \le \|\Delta_1\|$. If, on the contrary, $(I - M\Delta_1)$ is invertible, then by Lemma 4.2, $\sigma(A(\Delta_1)) \not\subset \mathbf{D}$, which means that Δ_1 is a destabilizing operator, and hence, by the definition, we have again $r_{\mathbf{C}}(A; D, E, M) \le \|\Delta_1\|$. Thus, from (41) and in view of arbitrariness of $\varepsilon > 0$, we obtain

$$r_{\mathbf{C}}(A; D, E, M) \le \frac{1}{\sup_{|s|=1} \|M + G(s)\|} .$$

Since the inequality $r_{\mathbf{C}}(A; D, E, M) \le \|M\|^{-1}$ is obvious from the definition and Lemma 4.1, we finally obtain

$$r_{\mathbf{C}}(A; D, E, M) \le \min \left\{ \frac{1}{\|M\|}, \frac{1}{\sup_{|s|=1} \|M + G(s)\|} \right\}.$$

The proof is complete. ∎

In the case of positive systems, the following theorem provides a computable formula for the real stability radius.

Theorem 4.4. *Let $A \in \mathcal{L}^+(X)$ and let the linear discrete-time system (35) be Schur stable. Assume $D \in \mathcal{L}^+(U, X)$, $E \in \mathcal{L}^+(X, Y)$, $M \in \mathcal{L}^+(U, Y)$ are given positive operators and A is subject to nonlinear fractional perturbations of the form (36). Then we have*

$$r_{\mathbf{C}}(A; D, E, M) = r_{\mathbf{R}}(A; D, E, M) = \frac{1}{\|M + G(1)\|} .$$

Proof. Since $\rho(A) < 1$, we have, by (31), $G(1) \ge 0$ and so $G(1) + M \ge M \ge 0$. This implies, by virtue of (23), $\|M + G(1)\| \ge \|M\|$. Therefore, by Theorem 4.3, we have

$$r_{\mathbf{C}}(A; D, E, M) = \frac{1}{\sup_{|s|=1} \|M + G(s)\|} . \quad (42)$$

Since $M + G(1) \ge 0$, from the observation we have made after the proof of Lemma 4.1, it follows that, for any $\varepsilon > 0$, there exists $\Delta_0 \in \mathcal{L}^+(Y, U)$ such that $\|\Delta_0\| < [\|M + G(1)\|]^{-1} + \varepsilon$ and $1 \in \sigma((M + G(1))\Delta_0)$. Suppose $(I - M\Delta_0)$ is invertible, then, by Lemma 4.2, $\sigma(A(\Delta_0)) \not\subset \mathbf{D}$. This implies by definition that

$$r_{\mathbf{R}}(A; D, E, M) \le \|\Delta_0\| < \frac{1}{\|M + G(1)\|} + \varepsilon$$

which implies

$$r_{\mathbf{R}}(A; D, E, M) \le \frac{1}{\|M + G(1)\|} . \quad (43)$$

Further, since $\rho(A) < 1$, for any $s \in \mathbf{C}$, $|s| = 1$, the operator $G(s) = E(sI - A)^{-1}D$ can be expanded as

$$G(s) = \frac{1}{s}\sum_{i=0}^{\infty} E\frac{A^i}{s^i}D.$$

We derive

$$|(M + G(s))u| \leq |Mu| + \sum_{i=0}^{\infty} |E\frac{A^i}{s^i}Du| = M|u| + \sum_{i=0}^{\infty} E\,|\frac{A}{s}|^i\,D|u|$$
$$= M|u| + \sum_{i=0}^{\infty}(EA^iD)|u| = (M + G(1))|u|$$

for every $u \in U$. This yields

$$\|M + G(s)\| \leq \|M + G(1)\|$$

for all $s \in \mathbf{C}$, $|s| = 1$. Therefore, by definition and (42) and (43),

$$\frac{1}{\|M + G(1)\|} \geq r_{\mathbf{R}}(A; D, E, M) \geq r_{\mathbf{C}}(A; D, E, M) \geq \frac{1}{\|M + G(1)\|},$$

concluding the proof. ■

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