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A Property of Entire Functions of Exponential Type*

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Abstract. In this paper, the existence and the concrete calculation of the limit $\lim_{m\to\infty} \|P^m(D)f\|_{N_\Phi}^{1/m}$ for any function $f \in N_\Phi(\mathbb{R}^n)$ with bounded spectrum are shown.

1. Introduction

Ha Huy Bang [1] has proved the following result: Let $\Phi(t)$ be an arbitrary Young function, $f(x) \in L_\Phi(\mathbb{R}^n)$, $P(\xi)$ a polynomial with constant coefficients, and $\operatorname{supp}\hat{f}$ bounded. Then there always exists the limit

$$d_f = \lim_{m\to\infty} \|P^m(D)f\|_{(\Phi)}^{1/m},$$

and moreover,

$$d_f = \sup\{|P(\xi)| : \xi \in \operatorname{supp}\hat{f}(\xi)\},$$

where $\hat{f}$ is the Fourier transform of the function f and $\|.\|_{(\Phi)}$ is the Luxemburg norm.

In this paper, by modifying the methods of [1], we prove this result for another norm generated by concave functions. Note that the Luxemburg norm is generated by convex functions and here we must overcome some difficulties due to the difference between convex and concave functions.

Let $\mathcal{L}$ denote the family of all non-zero concave functions $\Phi(t) : [0,\infty) \to [0,\infty]$, which are non-decreasing and satisfy $\Phi(0) = 0$. For $\Phi \in \mathcal{L}$, denote by $N_\Phi = N_\Phi(\mathbb{R}^n)$, the set of all measurable functions f such that

$$\|f\|_{N_\Phi} = \int_0^\infty \Phi(\lambda_f(y))dy,$$

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where $\lambda_f(y) = \text{mes}\{x : |f(x)| > y\}$, $(y \geq 0)$, and by $M_\Phi = M_\Phi(\mathbb{R}^n)$, the set of all measurable functions g such that

$$\|g\|_{M_\Phi} = \sup\left\{\frac{1}{\Phi(\text{mes}\,\Delta)}\int_\Delta |g(x)|dx : \Delta \subset \mathbb{R}^n,\ 0 < \text{mes}\,\Delta < \infty\right\} < \infty.$$

Then N_Φ and M_Φ are Banach spaces [5 – 6].

2. Result

We give the main theorem:

Theorem 1. *Let $\Phi \in \mathcal{L}$, $f(x) \in N_\Phi(\mathbb{R}^n)$, $P(\xi)$ be a polynomial with constant coefficients, and $\text{supp}\,\hat{f}$ bounded. Then there always exists the limit*

$$d_f = \lim_{m\to\infty} \|P^m(D)f\|_{N_\Phi}^{1/m},$$

and moreover,

$$d_f = \sup\{|P(\xi)| : \xi \in \text{supp}\,\hat{f}\}.$$

Note that Theorem 1 is a generalization of a result obtained in [3]. To prove Theorem 1, we need the following known result.

Let $m \in Z_+$. Denote by $W_{m,2}$ the usual Sobolev space, i.e., the set of all functions f such that

$$\|f\|_{m,2} = \Big(\sum_{|\alpha|\leq m} \|D^\alpha f\|_2^2\Big)^{1/2} < \infty.$$

We have the topological equality $H_{(m)} = W_{m,2}$ (see [4], (7.9)), where

$$H_{(m)} = \left\{f \in S' : \|f\|_{(m)} = \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^m |\hat{f}(\xi)|^2 d\xi\Big)^{1/2} < \infty\right\}.$$

Lemma 1. [6] *If $f \in N_\Phi$, $g \in M_\Phi$, then $fg \in L_1$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \|f\|_{N_\Phi}\|g\|_{M_\Phi}.$$

Lemma 2. [2] *If $f \in N_\Phi$, $u \in L_1$ then $f * u \in N_\Phi$ and*

$$\|f * u\|_{N_\Phi} \leq \|f\|_{N_\Phi}\|u\|_1.$$

Proof of Theorem 1. We shall begin by showing that

$$\underline{\lim}_{m\to\infty} \|P^m(D)f\|_{N_\Phi}^{1/m} \geq \sup_{\xi\in\text{sp}(f)} |P(\xi)|, \tag{1}$$

where we denote $\operatorname{supp}\hat{f}$ by $\operatorname{sp}(f)$ for simplicity.

Let $\xi^0 \in \operatorname{sp}(f)$ such that $|P(\xi^0)| = \sup_{\operatorname{sp}(f)} |P(\xi)|$. Without loss of generality, we may assume $P(\xi^0) > 0$. Further, we fix a number $0 < \epsilon < P(\xi^0)/4$ and choose a domain G such that $\xi^0 \in G$ and

$$P(\xi) > P(\xi^0) - \epsilon, \ \xi \in G. \tag{2}$$

Fix $\hat{u}, \hat{v}_0 \in C_0^\infty(G)$ such that $\xi^0 \in \operatorname{supp} \hat{u}\hat{f}$ and $< \hat{u}\hat{f}, \hat{v}_0 > \neq 0$. Let $\psi \in C_0^\infty(G)$ and $\psi = 1$ in some neighborhood of $\operatorname{supp}\hat{v}_0$. Then, for any $m \geq 1$, we obtain

$$\begin{aligned} |\langle \hat{u}\hat{f}, \hat{v}_0\rangle| &= \left|\langle \psi(\xi)P^{-m}(\xi)P^m(\xi)\hat{u}(\xi)\hat{f}(\xi), \hat{v}_0(\xi)\rangle\right| \\ &= \left|\langle P^m(\xi)\hat{u}(\xi)\hat{f}(\xi), \psi(\xi)P^{-m}(\xi)\hat{v}_0(\xi)\rangle\right| \\ &= |\langle F^{-1}P^m\hat{u}\hat{f}, FP^{-m}\hat{v}_0\rangle| \\ &= |\langle P^m(D)(u * f), F\hat{v}_m\rangle|, \end{aligned}$$

where $\hat{v}_m = P^{-m}\hat{v}_0(\xi)$. Therefore, by virtue of Lemmas 1 and 2, we obtain

$$| < \hat{u}\hat{f}, \hat{v}_0 > | \leq \|P^m(D)f\|_{N_\Phi}\|u\|_1\|F\hat{v}_m\|_{M_\Phi}, \ \forall m \geq 1. \tag{3}$$

Next we prove

$$\|F\hat{v}_m\|_{M_\Phi} \leq C(P(\xi^0) - \epsilon)^{-m}, \ m \geq 1. \tag{4}$$

Let $|\alpha| \leq 2n$. Since $P(\xi) \neq 0$ in G, we obtain by the Leibniz formula

$$D^\alpha\big(P^{-m}(\xi)\hat{v}_0(\xi)\big) = \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}\hat{v}_0(\xi)D^\beta P^{-m}(\xi), \tag{5}$$

$$D^\beta P^{-m}(\xi) = \sum_{\gamma^1+\cdots+\gamma^m=\beta} \frac{\beta!}{\gamma^1!\ldots\gamma^m!} D^{\gamma^1}P^{-1}(\xi)\cdots D^{\gamma^m}P^{-1}(\xi). \tag{6}$$

Therefore,

$$\begin{aligned} |x^\alpha F\hat{v}_m(x)| = \left|\int_G e^{-ix\xi}D^\alpha\big(P^{-m}(\xi)\hat{v}_0(\xi)\big)d\xi\right| &\leq \sum_{\beta\leq\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!} \\ \times \sum_{\gamma^1+\cdots+\gamma^m=\beta}\frac{\beta!}{\gamma^1!\ldots\gamma^m!}\int_G \left|D^{\alpha-\beta}\hat{v}_0(\xi)D^{\gamma^1}P^{-1}(\xi)\cdots D^{\gamma^m}P^{-1}(\xi)\right|d\xi \end{aligned} \tag{7}$$

for all $x \in \mathbb{R}^n$. By arguing as in [1], we obtain a constant $C_1 = C_1(P, \hat{v}_0, 2n)$ such that

$$|x^\alpha F\hat{v}_m(x)| \leq (2m)^{2n}C_1(P(\xi^0) - \epsilon)^{-m+2n}, \ \forall m \geq 2n,$$

where

$$C_1 = \max\big\{(P(\xi^0) - \epsilon)^{|\beta|-2n}\int_G \left|D^{\alpha-\beta}\hat{v}_0(\xi)D^{\gamma^1}P^{-1}(\xi)\cdots D^{\gamma^{|\beta|}}P^{-1}(\xi)\right|d\xi :$$

$$\beta \leq \alpha, |\alpha| \leq 2n, \gamma^1 + \cdots + \gamma^{|\beta|} = \beta\big\}.$$

Since

$$\lim_{m\to\infty}(2m)^{2n}\Big(\frac{P(\xi^0)-2\epsilon}{P(\xi^0)-\epsilon}\Big)^m=0,$$

we obtain a constant $C_2 = C_2(\epsilon)$ such that

$$\sup_{x\in\mathbb{R}^n}|x^\alpha F\hat{v}_m(x)|\le C_2(P(\xi^0)-2\epsilon)^{-m}$$

for all $|\alpha|\le 2n$ and $m\ge 2n$. Therefore,

$$\sup_{x\in\mathbb{R}^n}(1+x_1^2)\cdots(1+x_n^2)|F\hat{v}_m(x)|\le C_3(P(\xi^0)-2\epsilon)^{-m},\ \forall m\ge 2n.$$

We obtain

$$|F\hat{v}_m(x)|\le\frac{C_3(P(\xi^0)-2\epsilon)^{-m}}{(1+x_1^2)\cdots(1+x_n^2)},\ \forall m\ge 2n,\ \forall x\in\mathbb{R}^n.$$

By the definition of $\|.\|_{M_\Phi}$, we have

$$\|F\hat{v}_m(x)\|_{M_\Phi}\le C_3(P(\xi^0)-2\epsilon)^{-m}\sup\Big\{\frac{1}{\Phi(\text{mes}\,\Delta)}\int_\Delta\frac{dx}{(1+x_1^2)\cdots(1+x_n^2)}:\\ \Delta\subset\mathbb{R}^n, 0<\text{mes}\,\Delta<\infty\Big\}.$$

From $\Phi\in\mathcal{L}$, we see that $u/\Phi(u)$ increases as u increases [6]. Note that $\Phi(t)>0$ for $t>0$. We assume the contrary. Then there exists a number $t>0$ such that $\Phi(t)=0$. Since Φ is non-decreasing, then $\Phi(x)=0,\ \forall x\in[0,t]$. Put $t_1=\max\{t:\Phi(t)=0\}$. Then

$$0=\Phi(t_1)\ge\frac{1}{2}\Phi\Big(\frac{t_1}{2}\Big)+\frac{1}{2}\Phi\Big(\frac{3t_1}{2}\Big)>0,$$

a contradiction.

Therefore,

$$\sup\Big\{\frac{1}{\Phi(\text{mes}\,\Delta)}\int_\Delta\frac{dx}{(1+x_1^2)\cdots(1+x_n^2)}:\Delta\subset\mathbb{R}^n,\ 0<\text{mes}\,\Delta\le 1\Big\}$$
$$\le\sup\Big\{\frac{\text{mes}\,\Delta}{\Phi(\text{mes}\,\Delta)}:\Delta\subset\mathbb{R}^n, 0<\text{mes}\,\Delta\le 1\Big\}\le\frac{1}{\Phi(1)}<\infty,$$

and

$$\sup\Big\{\frac{1}{\Phi(\text{mes}\,\Delta)}\int_\Delta\frac{dx}{(1+x_1^2)\cdots(1+x_n^2)}:\Delta\subset\mathbb{R}^n,\ 1<\text{mes}\,\Delta<\infty\Big\}$$
$$\le\frac{1}{\Phi(1)}\int_{\mathbb{R}^n}\frac{dx}{(1+x_1^2)\cdots(1+x_n^2)}=\frac{\pi^n}{\Phi(1)}<\infty.$$

We obtain

$$\|F\hat{v}_m(x)\|_{M_\Phi}\le C(P(\xi^0)-2\epsilon)^{-m},$$

where

$$C=C_3\max\Big\{\frac{1}{\Phi(1)},\frac{\pi^n}{\Phi(1)}\Big\}=\frac{C_3\pi^n}{\Phi(1)}.$$

By combining (3) and (4), we obtain

$$\underline{\lim}_{m\to\infty} \|P^m(D)f\|_{N_\Phi}^{1/m} \geq P(\xi^0) - 2\epsilon.$$

Letting $\epsilon \to 0$, we obtain (1).

To complete the proof, it remains to show that

$$\overline{\lim}_{m\to\infty} \|P^m(D)f\|_{N_\Phi}^{1/m} \leq \sup_{\mathrm{sp}(f)} |P(\xi)|. \tag{8}$$

Given $\epsilon > 0$, we choose a domain $G \supset \mathrm{sp}(f)$ and a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ in some neighborhood of $\mathrm{sp}(f)$ and

$$\sup_G |P(\xi)| < \sup_{\mathrm{sp}(f)} |P(\xi)| + \epsilon. \tag{9}$$

We have for all $m \geq 0$,

$$\begin{aligned}\|P^m(D)f\|_{N_\Phi} &= \|F^{-1}\big(\varphi(\xi)P^m(\xi)\hat{f}(\xi)\big)\|_{N_\Phi}\\ &\leq \|F^{-1}\big(\varphi(\xi)P^m(\xi)\big)\|_1 \|f\|_{N_\Phi}.\end{aligned} \tag{10}$$

Putting $h_m(\xi) = \varphi(\xi)P^m(\xi)$, $m \geq 1$, and $s = [n/2] + 1$, we obtain from Holder's inequality that

$$\begin{aligned}\|F^{-1}h_m\|_1 &\leq \Big(\int |\hat{h}_m(\xi)|^2(1+|\xi|^2)^s d(\xi)\Big)^{1/2}\Big(\int (1+|\xi|^2)^{-s} d(\xi)\Big)^{1/2}\\ &= C_4\|h_m\|_{(s)},\end{aligned}$$

where C_4 is independent of m. Therefore, due to (10) and the topological equality $H_{(s)} = W_{s,2}$, we obtain

$$\|P^m(D)f\|_{N_\Phi} \leq C_5\|h_m\|_{s,2}\|f\|_{N_\Phi}. \tag{11}$$

On the other hand, it follows from the Leibniz formula that

$$D^\alpha h_m(\xi) = \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}\varphi(\xi) D^\beta P^m(\xi), \tag{12}$$

$$D^\beta P^m(\xi) = \sum_{\gamma^1+\cdots+\gamma^m=\beta} \frac{\beta!}{\gamma^1!\cdots\gamma^m!} D^{\gamma^1}P^1(\xi)\cdots D^{\gamma^m}P^1(\xi). \tag{13}$$

Further, we note that, for $|\beta| \leq s \leq m$ and $\gamma^1 + \cdots + \gamma^m = \beta$, there are at least $m - |\beta| \geq m - s$ multi-indices among $\gamma^1, \ldots, \gamma^m$ equal zero. Therefore, combining (9), (11)–(13), we obtain a constant $C_6 = C_6(P, \varphi, s)$ such that

$$\begin{aligned}\|P^m(D)f\|_{N_\Phi} &\leq C_5C_6(\sup_G |P(\xi)|)^{m-s}\|f\|_{N_\Phi}\\ &\leq C_5C_6(\sup_{\mathrm{sp}(f)} |P(\xi)| + \epsilon)^{m-s}\|f\|_{N_\Phi},\ \forall m \geq s.\end{aligned}$$

Hence,

$$\overline{\lim}_{m\to\infty} \|P^m(D)f\|_{N_\Phi}^{1/m} \leq \sup_{\mathrm{sp}(f)} |P(\xi)| + \epsilon.$$

Letting $\epsilon \to 0$, we obtain (8). The proof of Theorem 1 is complete. ■

3. An Application

From Theorem 1, we have

Theorem 2. *Let $f \in N_\Phi(\mathbb{R}^n)$. Then $sp(f) \subset B(0,r)$ if and only if*

$$\varlimsup_{m\to\infty} \|\Delta^m f\|_{N_\Phi}^{\frac{1}{m}} \le r^2.$$

Moreover, let $P(\xi)$ be a polynomial, $V \subset \mathbb{R}^n$, $\sigma = (\sigma_1, \dots, \sigma_n)$, $\sigma_j > 0$, and $r > 0$. We put

$$Q(V,P) = \{\xi \in \mathbb{R}^n : |P(\xi)| \le \sup_V |P(\xi)|\},$$
$$Q(V,P,\sigma) = Q(V,P) \cap \Delta_\sigma,$$
$$Q(V,P,r) = Q(V,P) \cap B(0,r).$$

It is easily seen that $V \subset Q(V,P)$, $Q(V,P)$ can be non-compact although V is compact, and $Q(V,P)$, $Q(V,P,\sigma)$, and $Q(V,P,r)$ can be non-convex.

By virtue Theorem 1, we have the following results:

Theorem 3. *Let $f \in N_\Phi(\mathbb{R}^n)$. Then $\mathrm{sp}(f) \subset Q(V,P,\sigma)$ if and only if*
(i) $\varlimsup_{m\to\infty} \|P^m(D) f\|_{N_\Phi}^{1/m} \le \sup_V |P(\xi)|$,
(ii) $\varlimsup_{m\to\infty} \|\partial^m/\partial x_j^m f\|_{N_\Phi}^{1/m} \le \sigma_j,\ j = 1, \dots, n.$

Theorem 4. *Let $f \in N_\Phi(\mathbb{R}^n)$. Then $\mathrm{sp}(f) \subset Q(V,P,r)$ if and only if*
(i) $\varlimsup_{m\to\infty} \|P^m(D) f\|_{N_\Phi}^{1/m} \le \sup_V |P(\xi)|$,
(ii) $\varlimsup_{m\to\infty} \|\Delta^m f\|_{N_\Phi}^{1/m} \le r^2.$

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