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On a Positive Bounded Solution of the n-Competing Species Problem

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Abstract. We consider the n-dimensional, non-autonomous Lotka–Volterra competition equations. Conditions for the existence and uniqueness of a solution defined on $(-\infty, +\infty)$ whose components are bounded above and below by positive constants are given.

1. Introduction

Consider the Lotka–Volterra equations for n-competing species

$$\dot{u}_i = u_i\Big[b_i(t) - \sum_{j=1}^{n} a_{ij}(t)\,u_j\Big], \quad 1 \le i \le n, \tag{1.1}$$

where $n \geq 2$ and $b_i : R \to R, a_{ij} : R \to R_+, R := (-\infty, +\infty), R_+ := (0, +\infty)$, are continuous and bounded. The case that b_i, a_{ij} are continuous and bounded above and below by positive constants was also considered in [1–6]. It was shown in [7] that if

(i) $b_i, a_{ij} : R \to R_+$ $(1 \le i, j \le n)$ are continuous, bounded above and below by positive constants;

(ii) there exists a positive number ε_1 such that

$$b_i(t) \geq \sum_{j\in J_i} a_{ij}(t)\,U_j^0(t) + \varepsilon_1, \quad 1 \le i \le n,\ t \in R, \tag{1.2}$$

where $J_i = \{1, \dots, i-1, i+1, \dots, n\}$ and $U_j^0(t)$ is the unique solution to the logistic equation

$$\dot{U} = U\big[b_j(t) - a_{jj}(t)\,U\big] \tag{1.3j}$$

which is defined on $(-\infty, +\infty)$ and is bounded above and below by positive constants;

(iii) there are positive numbers $\varepsilon_2,\ \alpha_1,\ \alpha_2, \dots,\ \alpha_n$ such that

$$\alpha_i\, a_{ii}(t) \geq \sum_{j\in J_i} a_{ji}(t)\,\alpha_j + \varepsilon_2\,, \quad 1 \leq i \leq n,\ \ t \in R, \tag{1.4}$$

hold;

then the system (1.1) has a unique solution $u^0(t) = (u_1^0(t), ..., u_n^0(t))$ defined on $(-\infty, +\infty)$, whose components are bounded above and below by positive constants, and moreover, $u_i(t) - u_i^0(t) \to 0$ as $t \to +\infty$ $(1 \leq i \leq n)$ for any solution $u(t) = (u_1(t), ..., u_n(t))$ of (1.1) with $u_i(t_0) > 0$ for some $t_0 \in R$ and for all $i : 1 \leq i \leq n$.

In this paper, we prove a result which is more general than the one above. Our main result is as follows:

Theorem 1. *Suppose*

(i) $\liminf_{t\to\pm\infty} b_i(t) > 0,\ \liminf_{t\to\pm\infty} a_{ij}(t) > 0\ (1 \leq i,\ j \leq n);$ (1.5)

(ii) $\liminf_{t\to\pm\infty}\left[b_i(t) - \sum_{j\in J_i} a_{ij}(t)\, U_j^0(t)\right] > 0 \quad (1 \leq i \leq n),$ (1.6)

where $U_j^0(t)$ $(1 \leq j \leq n)$ is the unique solution to (1.3) defined on $(-\infty, +\infty)$ which is bounded above and below by positive constants;

(iii) *there are $2n$ positive constants $\alpha_1^{\pm},\ \alpha_2^{\pm}, ..., \alpha_n^{\pm}$ such that*

$$\liminf_{t\to\pm\infty}\left[\alpha_i^{\pm}\, a_{ii}(t) - \sum_{j\in J_i} a_{ji}(t)\,\alpha_j^{\pm}\right] > 0 \quad (1 \leq i \leq n) \tag{1.7}$$

hold.

Then the system (1.1) has a unique solution $u^0(t) = (u_1^0(t), ..., u_n^0(t))$ defined on $(-\infty, +\infty)$, whose components are bounded above and below by positive constants, and moreover, $u_i(t) - u_i^0(t) \to 0$ as $t \to +\infty$ $(1 \leq i \leq n)$ for any solution $u(t) = (u_1(t), ..., u_n(t))$ to (1.1) with $u_i(t_0) > 0,\ 1 \leq i \leq n$ for some $t_0 \in R$.

The ecological significance of such a system is discussed in [4,5].

2. Preliminaries

It is easy to see that the Cauchy problem for (1.1) with the initial condition $u(t_0) = (u_{10}, ..., u_{n0}) \in R_+^n := \{(u_1, ..., u_n) \in R^n : u_i \geq 0,\ 1 \leq i \leq n\}$, $t_0 \in R$, has a unique solution. Moreover, R_+^n and $\mathrm{int}\,(R_+^n)$ are positively invariant.

Lemma 1. *Let + $a, b : R \to R$ be continuous bounded functions such that $\liminf_{t\to\pm\infty} a(t) > 0$, $\liminf_{t\to\pm\infty} b(t) > 0$, and $b(t) > 0$ for all $t \in R$. Then the logistic equation*

$$x = x\,[a(t) - b(t)\,x] \tag{2.1}$$

has a unique solution $x^0(t)$ defined on $(-\infty, +\infty)$, which is bounded above and below by positive constants. Moreover, $\lim_{t\to+\infty} |x(t) - x^0(t)| = 0$ for any solution $x(t)$ to (2.1) with $x(t_0) > 0$ for some $t_0 \in R$.

Proof. *Existence.* It is an easy matter to show that there exist positive numbers T, δ, Δ such that $a(t) > 0, b(t) > 0$, and $\delta < a(t)/b(t) < \Delta$ for $|t| \geq T$. For each $k = 1, 2, ...,$ let $x_k(t)$ be the solution to (2.1) with $x_k(-k - T) = \Delta$. Let $v_k(t) = \Delta$ and $u_k(t) = \delta$ for all $-k - T \leq t \leq -T$. Then

$$v_k(t)\big[a(t) - b(t)v_k(t)\big] = \Delta\big[a(t) - b(t)\Delta\big] < 0,$$

and

$$u_k(t)\big[a(t) - b(t)u_k(t)\big] = \delta\big[a(t) - b(t)\delta\big] > 0 \text{ for } -k - T \leq t \leq -T.$$

By the Comparison Lemma (see, for example, [6, p. 135]) we have that

$$\delta = u_k(t) < x_k(t) < v_k(t) = \Delta \text{ for } -k - T \leq t \leq -T.$$

By passing to a subsequence if necessary, we can assume $x_k(-T) \to \eta \in [\delta, \Delta]$ as $k \to \infty$.

Let $x^0(t)$ be the solution to (2.1) with $x^0(-T) = \eta$. It follows that $x_k(t) \to x^0(t)$ uniformly with respect to t on any compact subinterval of $(-\infty, -T]$. Therefore, $x^0(t)$ is defined on $(-\infty, -T]$ and $\delta \leq x^0(t) \leq \Delta$ for $-\infty < t \leq -T$. Let

$$\widetilde{\Delta} = \max\left\{\sup_{|t|\leq T} \frac{a(t)}{b(t)} + 1,\ \Delta\right\},$$

then $0 < \widetilde{\Delta} < +\infty$.

By the same argument given before and by the Comparison Lemma, it is clear that $x^0(t)$ is defined on $[-T, +\infty)$ and $x^0(t) < \widetilde{\Delta}$ for $-T \leq t < +\infty$. Since $(0, +\infty)$ is positively invariant with respect to (2.1), it follows that $x^0(t) > 0$ for $-T \leq t \leq T$. Therefore, $\overline{\delta} := \min_{|t|\leq T} x^0(t) > 0$. Let $\widetilde{\delta} = \min\{\delta, \overline{\delta}\}$. Then $x^0(t) > \delta$ for all $t > T$. Hence, $\widetilde{\delta} \leq x^0(t) < \widetilde{\Delta}$ for all $t \in R$.

Uniqueness. Suppose $x^1(t)$ is another solution to (2.1) defined on $(-\infty, +\infty)$ such that $0 < \inf_{t\in R} x^1(t) \leq \sup_{t\in R} x^1(t) < +\infty$. Let $t_0 \in (-\infty, +\infty)$ be such that $x^0(t_0) \neq x^1(t_0)$. Without loss of generality, we suppose $x^1(t_0) < x^0(t_0)$. Since (2.1) is a scalar equation, we can assume, by the uniqueness of solutions of Cauchy problems for Eq. (2.1), that

$$0 < \gamma_1 := \inf_{t\in R} x^1(t) \leq x^1(t) < x^0(t) \leq \sup_{t\in R} x^0(t) =: \gamma_2 < +\infty \ (t \in R).$$

We have

$$\frac{d}{dt} \ln \frac{x^1(t)}{x^0(t)} = b(t)\big[x^0(t) - x^1(t)\big].$$

Then, for any $M > 0$,

$$0 < \int_{-M}^{M} b(t)\big[x^0(t) - x^1(t)\big]dt = \ln\frac{x^1(M)}{x^0(M)} - \ln\frac{x^1(-M)}{x^0(-M)} \leq 2\ln\frac{\gamma_2}{\gamma_1}$$

Hence, $\int_{-\infty}^{+\infty} b(t)\left[x^0(t)-x^1(t)\right]dt < +\infty$. Consequently, $\lim_{t\to\pm\infty}\left[x^0(t)-x^1(t)\right]=0$, and this leads to

$$\int_{-\infty}^{+\infty} b(t)\left[x^0(t)-x^1(t)\right]dt = \lim_{M\to\infty}\left[\ln\frac{x^1(M)}{x^0(M)} - \ln\frac{x^1(-M)}{x^0(-M)}\right] = 0.$$

Thus, $x^0(t) = x^1(t)$, $t \in R$, and in particular, $x^0(t_0) = x^1(t_0)$. This contradiction implies the uniqueness.

Asymptoticity. Let $x(t)$ be a solution to (2.1) with $x(t_0) > 0$, $t_0 \in R$. It can be shown that

$$x(t) \le \max\left\{\sup_{t\in R}\frac{a(t)}{b(t)},\ x(t_0)\right\} =: \alpha_1\,, \quad t \ge t_0\,.$$

Let t_1 satisfy $t_1 > \max\{t_0,\ T\}$. By the Comparison Lemma, we have

$$x(t) \ge \alpha_2 := \min\left\{x(t_0),\ \min_{t_0\le t\le t_1} x(t),\ \inf_{t\ge T}\frac{a(t)}{b(t)}\right\} > 0, \quad t \ge t_0\,.$$

Since (2.1) is scalar, it follows that either

(a) $x(t) > x^0(t)$, $t \ge t_0$ or
(b) $x(t) < x^0(t)$, $t \ge t_0$.

If (a) holds, from

$$\frac{d}{dt}\ln\frac{x(t)}{x^0(t)} = b(t)\left[x^0(t) - x(t)\right],$$

it can be shown that

$$0 < \int_{t_0}^{M} -b(t)\left[x^0(t) - x(t)\right]dt = \ln\frac{x(t_0)}{x^0(t_0)} - \ln\frac{x(M)}{x^0(M)} \le 2\ln\frac{\alpha_1}{\alpha_2} \quad \text{for any } M > t_0.$$

Thus, $\int_{t_0}^{+\infty} -b(t)\left[x^0(t)-x(t)\right]dt < +\infty$ and this leads to $\lim_{t\to+\infty}\left[x^0(t)-x(t)\right] = 0$.

Similarly, we prove that $\lim_{t\to+\infty}\left[x^0(t) - x(t)\right] = 0$ if (b) holds. Therefore, the lemma is proved. ∎

Remark 1. It follows from Lemma 1 that, if (1.5) holds, then (1.3j) has a unique solution $U_j^0(t)$ defined on $(-\infty,\ +\infty)$ which is bounded above and below by positive constants.

Remark 2. It is not hard to see that (1.5)–(1.7) are equivalent to the following:

There exist positive numbers $T,\ \varepsilon,\ b_{iL},\ a_{ijL}\ (1 \le i,\ j \le n)$ such that

$$b_i(t) > b_{iL}\,,\quad a_{ij}(t) > a_{ijL} \quad \text{for } |t| \ge T, \tag{2.2}$$

$$b_i(t) - \sum_{j\in J_i} a_{ij}(t)\,U_j^0(t) \ge \varepsilon, \quad 1 \le i \le n, \quad |t| \ge T, \tag{2.3}$$

$$\alpha_i^- a_{ii}(t) - \sum_{j\in J_i} a_{ji}(t)\,\alpha_j^- \ge \varepsilon, \quad 1 \le i \le n, \quad t < -T, \tag{2.4'}$$

and

$$\alpha_i^+ a_{ii}(t) - \sum_{j\in J_i} a_{ji}(t)\,\alpha_j^+ \ge \varepsilon, \quad 1 \le i \le n, \quad t > T. \tag{2.4''}$$

Lemma 2. *Let (1.5) and (1.6) (or (2.2) and (2.3)) hold. Let $u(t)$ be a solution to (1.1) with $u(t_0) \in \operatorname{int}(R^n_+)$, for some $t_0 \in R$. Then its right maximal interval of existence is $[t_0, +\infty)$ and there exist positive numbers $t_1, \eta_1, \ldots, \eta_n, \Delta_1, \ldots, \Delta_n$ ($t_1 > T$) such that $\eta_i < u_i(t) < \Delta_i$ ($t \ge t_1$, $1 \le i \le n$).*

Proof. Since $\inf_{t\in R} a_{ii}(t) > 0$ $(1 \le i \le n)$, it follows that

$$0 < u_i(t) \le \max\left\{u_i(t_0), \sup \frac{b_i(t)}{a_{ii}(t)}\right\} := \Delta_i, \quad t > t_0.$$

Let $t_2 = \max\{T, t_0\}$. From (2.3), it follows that there exists a $\gamma > 0$ (for example, $\gamma = \min_{1\le i\le n}\{\epsilon/2\left[\sum_{j=1}^n \sup_{t\ge T} a_{ij}(t)\right]^{-1}\}$) such that

$$b_i(t) - \gamma\, a_{ii}(t) - \sum_{j\in J_i} a_{ij}(t)\left[U_j^0(t) + \gamma\right] > 0 \quad (1 \le i \le n, \ t \ge t_2). \tag{2.5}$$

Let us denote by $U_i(t)$ the solution to (1.3j) given by $U_i(t_2) = u_i(t_2)$. From (1.1) and (1.3j), it is easy to see that

$$u_i(t) < U_i(t), \quad t > t_2. \tag{2.6}$$

By Lemma 1, $U_i(t) - U_i^0(t) \to 0$ as $t \to +\infty$ $(1 \le i \le n)$. Consequently, there is $t_3 > t_2$ such that

$$U_i(t) \le U_i^0(t) + \gamma \quad (t \ge t_3, \ 1 \le i \le n). \tag{2.7}$$

We claim that

$$u_i(t) \ge \eta_i := \min\{u_i(t_3), \gamma\}, \quad t \ge t_3, \quad 1 \le i \le n. \tag{2.8}$$

Suppose that it is false. For each $i = 1, 2, \ldots, n$, let us define $g_i(t) = \eta_i - u_i(t)$. Then there exist i and $t_4 > t_3$ such that $g_i(t_4) > 0$. Since $g_i(t_3) \le 0$, there exists $t_5 > t_3$ such that $g_i(t_5) > 0$ and $\dot g_i(t_5) > 0$. It follows that

$$0 < -b_i(t_5) + a_{ii}(t_5)\,u_i(t_5) + \sum_{j\in J_i} a_{ij}(t_5)\,u_j(t_5).$$

Hence,

$$0 < -b_i(t_5) + a_{ii}(t_5)\,\gamma + \sum_{j\in J_i} a_{ij}(t_5)\,u_j(t_5). \tag{2.9}$$

From (2.6), (2.7), and (2.9), we have

$$0 < -b_i(t_5) + a_{ii}(t_5)\,\gamma + \sum_{j\in J_i} a_{ij}(t_5)\left[U_j^0(t_5) + \gamma\right],$$

which contradicts (2.5). The claim is proved. Therefore, the lemma is proved. ∎

3. Proof of the Main Result

Proof of Theorem 1. By Remark 2, we assume (2.2), (2.3), (2.4′) and (2.4″) instead of (1.5)–(1.7).

Existence. Let us define, for each $1 \le i,\ j \le n$,

$$\overline{a}_{ij}(t) = \begin{cases} a_{ij}(t), & t < -T, \\ a_{ij}(-T), & t \ge -T, \end{cases} \qquad \overline{b}_i(t) = \begin{cases} b_i(t), & t < -T, \\ b_i(-T), & t \ge -T. \end{cases}$$

Consider

$$\dot{\overline{u}}_i = \overline{u}_i \left[\overline{b}_i(t) - \sum_{j=1}^{n} \overline{a}_{ij}(t)\, \overline{u}_j \right], \quad 1 \le i \le n. \tag{3.1}$$

Clearly, (3.1) satisfies all conditions in Theorem 2.3 in [7]. Therefore, (3.1) has a unique solution $\overline{u}^0(t)$ defined on $(-\infty,\ +\infty)$ whose components are all bounded above and below by positive constants. Let $u(t)$ be the solution to (1.1) with $u(-T) = \overline{u}^0(-T)$. By Lemma 2, the right maximal interval of existence of $u(t)$ is $[-T,\ +\infty)$. It is easy to see that

$$u^0(t) := \begin{cases} \overline{u}^0(t), & t \le -T, \\ u(t), & t > -T, \end{cases}$$

is a solution to (1.1). From Lemma 2, $u_i^0(t)$ $(1 \le i \le n)$ is bounded above and below by positive constants.

Uniqueness. Suppose $u^1(t)$ is a solution to (1.1) defined on $(-\infty,\ +\infty)$ whose components are bounded above and below by positive constants. Let $\overline{u}^1(t)$ be the solution to (3.1) with $\overline{u}^1(-T) = u(-T)$. By Lemma 2 applying to (3.1), the right maximal interval of existence of $\overline{u}^1(t)$ is $[-T,\ +\infty)$.

Define

$$\widetilde{u}^1(t) = \begin{cases} \overline{u}^1(t), & t \ge -T, \\ u^1(t), & t < -T, \end{cases}$$

then $\widetilde{u}^1(t)$ is a solution to (3.1). From Lemma 2, $\widetilde{u}_i^1(t)$ $(1 \le i \le n)$ is bounded above and below by positive constants. By Theorem 2.3 in [7] applying to (3.1), $\widetilde{u}^1(t) = \overline{u}^0(t)$, $t \in R$.

Thus, $u^1(t) = u^0(t)$, $t \le -T$, and this leads to $u^1 \equiv u^0$. The uniqueness is proved.

Asymptoticity. Let $u^k(t)$ $(k = 1,\ 2)$ be solutions to (1.1) with $u^k(t_0) \in \operatorname{int}(R_+^n)$, $t_0 \in R$. It suffices to show that $u_i^1(t) - u_i^2(t) \to 0$ as $t \to +\infty$.

Let us define $t_1 := \max\{T,\ t_0\}$ and

$$\widetilde{a}_{ij}(t) = \begin{cases} a_{ij}(t), & t > t_1, \\ a_{ij}(t_1), & t \le t_1, \end{cases} \qquad \widetilde{b}_i(t) = \begin{cases} b_i(t), & t > t_1, \\ b_i(t_1), & t \le t_1, \end{cases} \qquad 1 \le i,\ j \le n.$$

Consider

$$\dot{\widetilde{u}}_i = \widetilde{u}_i \left[\widetilde{b}_i(t) - \sum_{j=1}^{n} \widetilde{a}_{ij}(t)\, \widetilde{u}_j \right], \quad 1 \le i \le n. \tag{3.2}$$

Clearly, $u^1(t)$ and $u^2(t)$ are solutions to (3.2) for $t \ge t_1$.

By Theorem 2.3 in [7] applying to (3.2), we have $u_i^1(t) - u_i^2(t) \to 0$ as $t \to +\infty$ $(1 \le i \le n)$. The theorem is proved. ■

Corollary. *Let* $\liminf_{t\to\pm\infty} b_i(t) = b_{iL}^{\pm} > 0$, $\liminf_{t\to\pm\infty} a_{ij}(t) = a_{ijL}^{\pm} > 0$. *If*

$$b_{iL}^{\pm} - \sum_{j\in J_i} a_{ijM}^{\pm} \frac{b_{jM}^{\pm}}{a_{jjL}^{\pm}} > 0, \quad 1 \le i \le n, \tag{3.3}$$

where $a_{ijM}^{\pm} = \limsup_{t\to\pm\infty} a_{ij}(t)$, $b_{iM}^{\pm} = \limsup_{t\to\pm\infty} b_i(t)$, *hold. Then the assertion in Theorem 1 is valid.*

Proof. It is not hard to see that $\limsup_{t\to\pm\infty} U_j^0(t) \le b_{jM}^{\pm}/a_{jjL}^{\pm}$, $1 \le i \le n$. Thus, (3.3) implies (1.6).

It is suffices to show that (3.3) implies (1.7). Let $B = (b_{ij})$ be the real $n \times n$ matrix defined by

$$b_{ij} = \begin{cases} 0, & i = j, \\ a_{ijM}^{+}/a_{jjL}^{+}, & i \ne j. \end{cases}$$

It follows from (3.3) that $B\beta < \beta$, where $\beta = (b_{1L}^{+}, ..., b_{nL}^{+})^T$. Let $\bar\varepsilon > 0$ be such that $B_{\bar\varepsilon}\beta < \beta$, where $B_{\bar\varepsilon} = B + \bar\varepsilon I$ (I is the identity matrix). By Perron's theorem, there exists a real positive eigenvalue λ of $B_{\bar\varepsilon}$ such that $\lambda < 1$ and $|\mu| \le \lambda$ for all eigenvalue μ of $B_{\bar\varepsilon}$. Once again, from Perron's theorem, we have $B_{\bar\varepsilon}^{*}\alpha^{+} = \lambda\alpha^{+}$ for some vector $\alpha^{+} > 0$, where $B_{\bar\varepsilon}^{*}$ is the adjoint matrix of $B_{\bar\varepsilon}$. Therefore, $B^{*}\alpha^{+} = (\lambda - \bar\varepsilon)\alpha^{+} < \alpha^{+}$ which implies

$$\liminf_{t\to+\infty}\Big[\alpha_i^{+} a_{ii}(t) - \sum_{j\in J_i} a_{ji}(t)\alpha_j^{+}\Big] > 0 \quad (1 \le i \le n).$$

Similarly, we can prove that there exists a vector $\alpha^{-} > 0$ such that

$$\liminf_{t\to-\infty}\Big[\alpha_i^{-} a_{ii}(t) - \sum_{j\in J_i} a_{ji}(t)\alpha_j^{-}\Big] > 0 \quad (1 \le i \le n).$$

Therefore, (3.3) implies (1.7). The corollary is proved. ■

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