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On Matheron Theorem for Non-Locally Compact Metric Spaces

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Abstract. In this paper we investigate the Matheron theorem in the case of non-locally compact setting, i.e., without the hypothesis of the local compactness for the topology on a metric space E. Specifically, we show that the miss-and-hit topology on the space $\mathcal{F}$ of all closed subsets of E is still separable and compact. However, the space $\mathcal{F}$ is no longer Hausdorff if the Polish space E contains at least one non-locally compact point.

1. Introduction

Matheron [2] has shown that, for a locally compact Polish space E, the miss-and-hit topology on the space $\mathcal{F}$ of all closed subsets of E is compact, separable, and Hausdorff. The Matheron theorem is essential for the proof of the Choquet theorem in the case of locally compact spaces. Note that in his proof, Matheron relied heavily on the hypothesis of the local compactness. Since the natural domain of probability is that of non-locally compact Polish spaces (or, more generally, metric spaces), it is necessary to know whether or not the Matheron theorem holds for non-locally compact metric spaces.

It is shown in [5] that, if E is not locally compact at any point, then the miss-and-hit topology on $\mathcal{F}$ is no longer Hausdorff.

Also, the miss-and-hit topology has been studied in two particular cases: the metric space E being locally compact at every point and E being *not* locally compact at any point.

We investigate the Matheron theorem in the non-locally compact setting, i.e., without the hypothesis of the local compactness for the topology on E. Specifically, we show that, with the miss-and-hit topology, $\mathcal{F}$ is still separable and compact. As far as the Hausdorff property is concerned, we show that $\mathcal{F}$ is no longer Hausdorff if the Polish space E contains at least one non-locally compact point.

2. The Miss-and-Hit Topology

Let E be a metric space. Let $\mathcal{K}$, $\mathcal{F}$, and $\mathcal{G}$ denote the classes of all compact, closed, and open subsets of E, respectively. Following Matheron [2], we topologize $\mathcal{F}$ as follows. For every $A \subset E$, we denote

$$\mathcal{F}_A = \{F \in \mathcal{F} : F \bigcap A \neq \emptyset\} \quad \text{and} \quad \mathcal{F}^A = \{F \in \mathcal{F} : F \bigcap A = \emptyset\}.$$

The *miss-and-hit topology* on $\mathcal{F}$ is the topology with the base

$$\{\mathcal{F}^K_{G_1,\ldots,G_n} : K \in \mathcal{K} \quad \text{and} \quad G_1, \ldots, G_n \in \mathcal{G}\},$$

where

$$\mathcal{F}^K_{G_1,\ldots,G_n} = \mathcal{F}^K \bigcap \mathcal{F}_{G_1} \bigcap \ldots \mathcal{F}_{G_n}, \quad n \in \mathbf{N}.$$

3. The Results

By a *Polish space*, we mean a complete, separable metric space. Theorems 1 and 2 are generalizations of the Matheron theorem [2].

Theorem 1. *If the space E is separable, then so is the space $\mathcal{F}$. In particular, if E is a Polish space, then the miss-and-hit topology on $\mathcal{F}$ is separable.*

Proof. Let M be a dense countable set in E. By $\mathcal{M}$, we denote the family of all finite subsets of M. Then $\mathcal{M}$ is countable. We will show that $\mathcal{M}$ is dense in $\mathcal{F}$ in the miss-and-hit topology.

Suppose $F \in \mathcal{F}$ and $\mathcal{U}(F)$ is a neighborhood of F in the miss-and-hit topology. We may assume

$$\mathcal{U}(F) = \mathcal{F}^K_{G_1,\ldots,G_n},$$

where $K \in \mathcal{K}$, $G_i \in \mathcal{G}$ for $i = 1, 2, \ldots, n$ such that $F \cap K = \emptyset$ and $F \cap G_i \neq \emptyset$ for every $i = 1, 2, \ldots, n$.

For each $i = 1, 2, \ldots, n$, let $x_i \in F \cap G_i$ and $U(x_i)$ be a neighborhood of x_i in E such that $U(x_i) \subset G_i$ and $U(x_i) \cap K = \emptyset$. Since M is dense in E, for each $i = 1, 2, \ldots, n$, there exists a point $r_i \in M \cap U(x_i)$. Then

$$T = \{r_1, \ldots, r_n\} \in \mathcal{M} \quad \text{and} \quad T \in \mathcal{U}(F) = \mathcal{F}^K_{G_1,\ldots,G_n}.$$

Thus, the theorem is proved. ∎

Theorem 2 will be proved by a similar argument as in [2] in a locally compact setting. We include it here for the reader's convenience.

Theorem 2. *The miss-and-hit topology on $\mathcal{F}$ is compact.*

Proof. By the Alexandroff Theorem (see, e.g., [1]), in order to prove that $\mathcal{F}$ is compact, it is sufficient to show that, if

$$\{\mathcal{F}^{K_i},\ K_i \in \mathcal{K},\ i \in I\} \bigcup \{\mathcal{F}_{G_j},\ G_j \in \mathcal{G},\ j \in J\}$$

is a covering consisting of elements of a sub-base for the miss-and-hit topology, then it has a finite cover consisting of elements of this sub-base.

We have

$$\mathcal{F} = \Big(\bigcup_{i\in I} \mathcal{F}^{K_i},\ K_i \in \mathcal{K}\Big) \bigcup \Big(\bigcup_{j\in J} \mathcal{F}_{G_j},\ G_j \in \mathcal{G}\Big).$$

Hence,

$$\begin{aligned}\emptyset &= \Big[\bigcap_{i\in I} (\mathcal{F}\backslash\mathcal{F}^{K_i},\ K_i \in \mathcal{K})\Big] \bigcap \Big[\bigcap_{j\in J} (\mathcal{F}\backslash\mathcal{F}_{G_j},\ G_j \in \mathcal{G})\Big] \\ &= \Big[\bigcap_{i\in I} (\mathcal{F}_{K_i},\ K_i \in \mathcal{K})\Big] \bigcap \Big(\bigcap_{j\in J} \mathcal{F}^{G_j},\ G_j \in \mathcal{G}\Big).\end{aligned}$$

Let us put

$$\Omega = \bigcup_{j\in J} G_j.$$

Then the set Ω is open in E and

$$\begin{aligned}\Big(\bigcap_{i\in I} \mathcal{F}_{K_i}\Big) \bigcap \Big(\bigcap_{j\in J} \mathcal{F}^{G_j}\Big) &= \Big(\bigcap_{i\in I} \mathcal{F}_{K_i}\Big) \bigcap \big(\mathcal{F}^{\bigcup_{j\in J} G_j}\big) \\ &= \Big(\bigcap_{i\in I} \mathcal{F}_{K_i}\Big) \bigcap \mathcal{F}^{\Omega} = \bigcap_{i\in I} \mathcal{F}^{\Omega}_{K_i} = \emptyset. \qquad (1)\end{aligned}$$

From the last equality, it follows that there exists an index $i_0 \in I$ with $K_{i_0} \subset \Omega$. Otherwise, we have $\Omega^c \cap K_i \neq \emptyset$ for every $i \in I$, where $\Omega^c = E\backslash\Omega$. It implies that $\emptyset \neq \Omega^c \in \cap_{i\in I} \mathcal{F}^{\Omega}_{K_i}$, which is a contradiction to (1).

Thus, let $i_0 \in I$ be an index such that $K_{i_0} \subset \Omega = \cup_{j\in J} G_j$. Since K_{i_0} is compact, there exists a finite cover $\{G_{j_1}, \ldots, G_{j_n}\}$ of the compact set K_{i_0} for indices $j_1, \ldots, j_n \in J$. Since $\{G_{j_1}, \ldots, G_{j_n}\}$ is a cover of K_{i_0}, we have $\mathcal{F}_{K_{i_0}} \cap \mathcal{F}^{G_{j_1}} \cap \cdots \cap \mathcal{F}^{G_{j_n}} = \emptyset$. It implies that

$$\mathcal{F}\backslash\Big(\mathcal{F}^{K_{i_0}} \bigcup \mathcal{F}_{G_{j_1}} \bigcup \cdots \bigcup \mathcal{F}_{G_{j_n}}\Big) = \emptyset.$$

Therefore,

$$\mathcal{F} = \mathcal{F}^{K_{i_0}} \bigcup \mathcal{F}_{G_{j_1}} \bigcup \cdots \bigcup \mathcal{F}_{G_{j_n}}.$$

The theorem is proved. ■

A point $x \in E$ is said to be *locally compact* if it has a compact neighborhood in E. It is shown in [2] that if E is locally compact at every point, then the miss-and-hit topology on $\mathcal{F}$ is Hausdorff. However, there is an another extreme case: If the space E is *not* locally compact at any point, then the miss-and-hit topology on $\mathcal{F}$ is not Hausdorff [5].

We prove here the following theorem.

Theorem 3. *If the separable metric space E contains at least one non-locally compact point, then the miss-and-hit topology on $\mathcal{F}$ is no longer Hausdorff.*

Proof. Assume E is a non-locally compact metric space at a point $x_0 \in E$, i.e., no neighborhood of x_0 is compact. Let $x_1 \in E$ be a point such that $x_1 \neq x_0$.

Putting

$$F = \{x_0, x_1\} \quad \text{and} \quad F' = \{x_1\},$$

we will show that $\mathcal{U}(F) \cap \mathcal{U}(F') \neq \emptyset$ for every neighborhood $\mathcal{U}(F)$ of F and $\mathcal{U}(F')$ of F' in the miss-and-hit topology.

In fact, we may assume

$$\mathcal{U}(F) = \mathcal{F}^{K}_{G_1,\dots,G_n} \text{ and } \mathcal{U}(F') = \mathcal{F}^{K'}_{G'_1,\dots,G'_m},$$

where $K, K' \in \mathcal{K}$ and $G_i, G'_j \in \mathcal{G}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$ such that $F \cap K = \emptyset$, $F \cap G_i \neq \emptyset$ for every $i = 1, \dots, n$ and $F' \cap K' = \emptyset$, $F' \cap G'_j \neq \emptyset$ for every $j = 1, \dots, m$.

Assume $G_1, \dots, G_{n_0}$ $(n_0 \leq n)$ contain x_0, and $G_{n_0+1}, \dots, G_n$ contain x_1 but do not contain x_0. Putting

$$G(x_0) = \bigcap_{i=1}^{n_0} G_i,\ G(x_1) = \bigcap_{i=n_0+1}^{n} G_i, \quad \text{and} \quad G'(x_1) = \bigcap_{j=1}^{m} G'_j,$$

we have

$$\mathcal{F}^{K}_{G(x_0),G(x_1)} \subset \mathcal{F}^{K}_{G_1,\dots,G_n} \quad \text{and} \quad \mathcal{F}^{K'}_{G'(x_1)} \subset \mathcal{F}^{K'}_{G'_1,\dots,G'_m}.$$

Let $U(x_0) \subset G(x_0)$ be a neighborhood of x_0 in E such that $U(x_0) \cap K = \emptyset$. Since x_0 has no compact neighborhood and K' is compact, there exists a point $y_0 \in U(x_0)$ such that $y_0 \notin K'$. Putting $T = \{y_0, x_1\}$, we obtain

$$T \in \mathcal{F}^{K}_{U(x_0),G(x_1)} \bigcap \mathcal{F}^{K'}_{G'(x_1)} \subset \mathcal{F}^{K}_{G(x_0),G(x_1)} \bigcap \mathcal{F}^{K'}_{G'(x_1)} \subset \mathcal{F}^{K}_{G_1,\dots,G_n} \bigcap \mathcal{F}^{K'}_{G'_1,\dots,G'_m}.$$

The theorem is proved. ■

4. Remark

By Theorem 1, if the metric space is separable, so is the space $\mathcal{F}$ equipped with the miss-and-hit topology. Here, we will show by an example that the space $\mathcal{F}$ may not be separable if it is equipped with an another topology.

Let $C[0, 1]$ be the space of continuous functions on the unit interval $[0, 1]$. Then $C[0, 1]$ is an infinite-dimensional Polish space, so it is not locally compact. Let E be the closed unit ball of the space $C[0, 1]$, that is,

$$E = \{x(t) \in C[0, 1] : \|x\| = \max_{t \in [0,1]} \|x(t)\| \leq 1\}.$$

By Theorem 1, the space $\mathcal{F}$ of all closed subsets of E equipped with the miss-and-hit topology is separable, but by Theorem 3, it is no longer Hausdorff. Thus, if we insist on having $\mathcal{F}$ as a Hausdorff topological space, then we need to use another topology. The most popular topology for the space of all closed subsets of a metric space is, perhaps, the topology induced by the Hausdorff metric, that is,

$$d(A, B) = \begin{cases} \max\{\sup_{x \in A} \|x - B\|, \sup_{x \in B} \|x - A\|\}, & \text{if } A \neq \emptyset,\ B \neq \emptyset, \\ 0 & \text{if } A = B = \emptyset, \\ 2 & \text{otherwise,} \end{cases}$$

where $\|x - A\| = \inf\{\|x - y\| : y \in A\}$.

We will show that, although the space $\mathcal{F}$ with the miss-and-hit topology is still separable, but with the topology induced by the Hausdorff metric, it is not separable.

In fact, let $\{e_n\}$ be a sequence of continuous functions on the unit interval [0, 1] such that

$$e_1(t) = \begin{cases} 1 & \text{for all } t \in [\frac{1}{2}, 1], \\ 0 & \text{for all } t \in [0, 1] \backslash [\frac{5}{12}, 1], \end{cases}$$

and for $n \geq 2$

$$e_n(t) = \begin{cases} 1 & \text{for all } t \in [\frac{1}{2n}, \frac{1}{2n-1}], \\ 0 & \text{for all } t \in [0, 1] \backslash (z_n, z_{n-1}), \end{cases}$$

where $z_n = 1/2((1/2n) + 1/(2n + 1))$. Such functions e_n, $n = 1, 2, \ldots$, exist by the Urysohn–Tietze Theorem. Then the sequence $\{e_n : n \in \mathrm{N}\} \subset E$ satisfies $\|e_n - e_m\| = 1$ for every $n \neq m$.

By 2^{N}, we denote the family of all non-empty subsets of N. For every $S \in 2^{\mathrm{N}}$, let $A_S = \{e_n : n \in S\}$. It is easy to see that $d(A_S, A_T) = 1$ for every $S \neq T$. Since $\{A_S : S \in 2^{\mathrm{N}}\}$ is uncountable, it follows that $\mathcal{F}$ is not separable.

5. Examples

There is evidence of the existence of metric spaces being locally compact at every point and of metric spaces being not locally compact at any point. The following examples show that there are metric spaces which contain only one non-locally compact point.

Example 1. For each $n \in \mathrm{N}$, let e_n denote the nth standard unit vector of the Hilbert space

$$l_2 = \Big\{x = (x_n) : \|x\| = \sum_{n=1}^{\infty} (x_n^2)^{1/2} < \infty\Big\},$$

that is, e_n has 1's at its nth position and 0's elsewhere. The zero element of the space l_2 is denoted by θ. Let us put

$$E = \{\theta\} \bigcup \Big(\bigcup_{k=1}^{\infty} \frac{1}{k} S \Big),$$

where

$$S = \{e_n : n = 1, 2, \ldots\} \subset l_2.$$

Obviously, every point $x \in E \setminus \{\theta\}$ is locally compact. Let us show that the point θ is not locally compact.

To obtain a contradiction, we assume the contrary that θ is a locally compact point and U is a compact neighborhood of θ in E. Then there exists a neighborhood U_0 such that $\theta \subset U_0 \subset U$ and U_0 is of the form

$$U_0 = B\Big(\theta, \frac{2}{n_0}\Big) \bigcap E \qquad (n_0 \in \mathrm{N}),$$

where $B(x, r) = \{y \in l_2 : \|x - y\| < r\}$ $(r > 0)$. Let ϵ be a number such that $0 < \epsilon < 1/n_0$.

For each $x_n \in E$, we put

$$U_n = B(x_n, \epsilon) \bigcap E.$$

Then the family $\{U_n\}_{n=1}^{\infty}$ is an open covering of E. Because of the compactness of the set U and the relation $U_0 \subset U \subset E$, there exists a set consisting of finitely many elements $x_1, \ldots, x_k$ of E such that

$$U_0 \subset U \subset \bigcup_{i=1}^{k} U_i.$$

We may assume without loss of generality that $x_i = (1/n_i)e_i$, $i = 1, 2, \ldots, k$. Let $x_0 = (1/n_0)e_{k+1}$, then $x_0 \in U_0$. Since $\{U_i\}_{i=1}^{k}$ is a covering of U_0, there exists an index $i_0 \in \{1, 2, \ldots, k\}$ such that $x_0 \in U_{i_0}$. Then we have

$$||x_0 - x_{i_0}|| < \epsilon. \tag{2}$$

Assume x_{i_0} is of the form $x_{i_0} = (0, \ldots, 0, 1/n_{i_0}, 0, \ldots)$, that is, x_{i_0} is the vector with $1/n_{i_0}$ in the i_0th slot and 0's elsewhere. It is then straightforward to check that

$$||x_0 - x_{i_0}|| = \left(\frac{1}{n_0^2} + \frac{1}{n_{i_0}^2}\right)^{1/2} > \frac{1}{n_0} > \epsilon.$$

The latter is a contradiction to (2), and therefore the assertion is proved.

Example 2. Let $E = [0, 1] \setminus \{1/n, n \in \mathrm{N}\}$. Then the point 0 is the unique non-locally compact point of the space E. In fact, let $\{f_n(x)\}$ be a sequence of functions defined by $f_n(x) = \cos(\pi/x - (1/n))$ for $n \in \mathrm{N}$ and $x \in E$.

Then the function $f_n : E \to R^1$ is continuous. Putting

$$f(x) = \sum_{i=1}^{\infty} a^n f_n(x) \ (0 < a < 1),$$

we obtain a continuous function $f : E \to R^1$. By a similar argument as in the proof of Theorem 1 in [4], one readily verifies that there is no neighborhood of 0 in E on which f is uniformly continuous. Therefore, the point $0 \in E$ is not locally compact. Moreover, it is easy to see that every point $x \in E \setminus \{0\}$ is locally compact.

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References

1. J. Kelley, *General Topology*, Van Nostrand Rheinhold, New York, 1955.
2. G. Matheron, *Random Sets and Integral Geometry*, John Wiley & Sons, New York, 1975.
3. M. Marinacci, Choquet theorem for the Hausdorff metric, 1998 (preprint).
4. Nguyen Nhuy and Dau The Cap, The gluing theorem for locally uniform retracts, *Vietnam J. Math.* **9**(4) (1981) 6–10.
5. H. T. Nguyen and N. T. Nguyen, On Choquet theorem for random sets on Polish space, *East West Journal of Mathematics* (to appear).
6. H. T. Nguyen, N. T. Nguyen, and T. Wang, On capacity functionals in interval probabilities, *Inter. J. Uncertainty, Fuzziness and Knowledge-Based Systems* (to appear).

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References

1. [illegible]
2. [illegible]
3. [illegible]
4. [illegible]
5. [illegible] Nguyen and [illegible] Nguyen, [illegible] (to appear).
6. [illegible]

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Density and the Space of Approximately Continuous Mappings

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Abstract. Starting from the idea of density of sets in a topological space X fitted with certain axioms, we generate a density topology in the space making certain critical observations on this topology. As far as it is known, the idea of the space of approximately continuous functions considered in this paper is not available in the literature. We construct the space of such mappings from X to another topological space and show that this space is metrizable under some conditions.

1. Introduction

The notion of density of sets and approximately continuous mappings has been widely studied in various spaces such as in real number space, measure space, Romanovski space, and topological group by many authors (see, for example, [2-5, 7-11]). In this paper we treat density of sets in a topological space fitted with certain axioms. These axioms together with some others make a topological space into a Romanovski space [12]. However, we observe that to build up the said theory in a topological space, we need only some of the several axioms for a Romanovski space.

As far as it is known, no attempt has been made to study the space of approximately continuous mappings from a topological space into another. This we do in the last section where we show ultimately that this space is metrizable under some general situations.

2. The Space X and Density of Sets

Let (X, τ) be a locally compact topological space. Let $\mathbf{B}$ denote the class of all Borel sets in (X, τ). Let μ be a measure defined on $\mathbf{B}$ such that $\mu(X)$ is finite. We assume μ to be non-zero for all non-void open sets. Let μ^* be the outer measure on $\mathbf{P}(X)$, the power et of X generated by μ. Let S denote the class of all μ^*-measurable sets and let **A** be a ubfamily of **B**, each member of which is open.

Definition 1. *By a decomposition S_U of $U \in$ **B**, we mean a finite disjoint family $A_1, \ldots, A_n$ from **A** such that*

(i) $\bigcup_{i=1}^n A_i \subset U$ *and*
(ii) $\mu(U - \bigcup_{i=1}^n A_i) = 0$.

The members of **A** need to satisfy the following axioms.

Axiom I. **A** form a base of τ.

Axiom II. For arbitrary $\varepsilon > 0$ and $A \in$ **A**, there exists a decomposition S_A of A such that $B \in S_A$ implies $\mu(B) < \varepsilon$.

Axiom III. For each point $x \in X$ and for each open set U containing x, there exists $\varepsilon > 0$ such that, if $\mu(A) < \varepsilon$, $x \in \overline{A}$, and $A \in$ **A**, then $A \subset U$.

Axiom IV. Given $A \in$ **A** and $\varepsilon > 0$, there is a $B \in$ **A** such that $\overline{A} \subset B$ and $\mu(B - A) < \varepsilon$, where the bar over a set denotes the closure.

In this situation, the members of **A** are called *fundamental sets* (cf. [12]). We shall assume throughout that (X, τ) is fitted with the above axioms.

Note 1. If $X = R$, the real number space with the usual topology, μ is the Lebesgue measure, and **A** is the collection of all open intervals, then it is clear that the above axioms hold for the class **A**.

Let S_U be a decomposition of an open set U, consisting of a finite disjoint family $A_1, \ldots, A_n$ from **A**. Then

$$\mu\Big(U - \bigcup_{i=1}^{n} \overline{A_i}\Big) \leq \mu\Big(U - \bigcup_{i=1}^{n} A_i\Big) = 0.$$

Now, $U - \bigcup_{i=1}^n \overline{A_i}$ is open. Since μ is positive for all non-void open sets, $U - \bigcup_{i=1}^n \overline{A_i} = \emptyset$. Also, from Axiom I, it follows that, for $x \in X$, there is $A \in$ **A** such that $x \in A$. These facts together with Axiom II imply the following:

Remark 1. For each $x \in X$, there exists a sequence of fundamental sets $\{A_{n,x}\}$ such that $x \in \overline{A}_{n,x}$, $\mu(A_{n,x}) < 1/n\ \forall n$.

The definition of density of sets as in [10, 11] is reproduced here for easy reference.

Definition 2. *For $E \subset X$, the upper and lower outer density of E at x denoted, respectively, by $\overline{D}^*(E, x)$ and $\underline{D}^*(E, x)$ are defined by $\overline{D}^*(E, x) = \lim_{n\to\infty} \overline{D}_n^*(E, x)$ and $\underline{D}^*(E, x) = \lim_{n\to\infty} \underline{D}_n^*(E, x)$, where*

$$\overline{D}_n^*(E, x) = \sup\{m^*(E, A);\ x \in \overline{A},\ \mu(A) < \frac{1}{n}, A \in \mathbf{A}\},$$
$$\underline{D}_n^*(E, x) = \inf\{m^*(E, A);\ x \in \overline{A},\ \mu(A) < \frac{1}{n}, A \in \mathbf{A}\}$$

and $m^*(E, A) = \mu^*(E \cap A) / \mu(A)$.

Since both $\overline{D}_n^*$ and $\underline{D}_n^*$ are monotone, the limits exist, and moreover,

$$0 \leq \underline{D}^*(E,x) \leq \overline{D}^*(E,x) \leq 1.$$

If they are equal, their common value is denoted by $D^*(E,x)$ and we say that the outer density of E exists at x. If $E \in S$, we write $\overline{D}^*(E,x) = \overline{D}(E,x)$ and $\underline{D}^*(E,x) = \underline{D}(E,x)$. If they are equal, we write $\overline{D}(E,x) = \underline{D}(E,x) = D(E,x)$. We call x an *outer density point* or *an outer dispersion point* of E as $\underline{D}^*(E,x) = 1$ or $\overline{D}^*(E,x) = 0$.

The density function is monotone, non-decreasing, and subadditive and further:

Theorem 1. *If $E, F \in S$ and $D(E,x)$ and $D(F,x)$ exist for $x \in X$, where $E \subset F$, then $D(F - E, x)$ exists and $D(F - E, x) = D(F,x) - D(E,x)$.*

The proof is omitted.

3. Density Topology in X

We give the definition of density topology as in [2, 10] and make some critical observations.

Definition 3. [10] *Let $\mathbf{D} = \{U : U \subset X$ and $\overline{D}^*(X - U, x) = 0\ \forall x \in U\}$. Then $\mathbf{D}$ is a topology on X which is called the density topology (d-topology for short) on X. The sets belonging to $\mathbf{D}$ are called d-open in X.*

Thus, $(X, \mathbf{D})$ is a topological space.

Theorem 2. *If G is open, then $D^*(G,x) = 1$ for each $x \in G$.*

Proof. Let $x \in G$. Then, by Axiom III, there is an $\varepsilon > 0$ such that, if $x \in \overline{A}, \mu(A) < \varepsilon$, and $A \in \mathbf{A}$, then $A \subset G$. Let $n_0 \in N$ be such that $1/n_0 < \infty$. If $n \geq n_0$ and $x \in \overline{A}$, $\mu(A) < 1/n$ and $A \in \mathbf{A}$, then $A \subset G$. From the definition of $\underline{D}_n^*(G,x)$, it follows that $\forall n \geq n_0\ \underline{D}_n^*(G,x) = 1$. Hence, $\underline{D}^*(G,x) = 1$. Therefore, $D^*(G,x) = 1$.

Theorem 3. *The d-topology $\mathbf{D}$ is finer than τ.*

Proof. Let $G \in \tau$. Then $G \in \mathbf{B}$ and so $G \in S$. By Theorems 1 and 2, $D(X - G, x) = 0$ $\forall x \in G$. Therefore, $G \in \mathbf{D}$. ∎

The following example shows that the d-topology is strictly finer than τ.

Example 1. Let $X = R$, the real number space with the usual topology, and let be the Lebesgue measure on R. Let $\mathbf{A}$ denote the set of all open intervals of R. By Note 1, members of $\mathbf{A}$ are fundamental sets. Let B be the set of all irrational numbers in [0, 1]. Then B is not open in R. But we shall show that B is d-open. Let $\alpha \in B$. Then $\alpha \in (0,1)$. Let $\delta = \min\{\alpha, (1 - \alpha)\}$. Choose $n_0 \in N$ such that $1/n_0 < \delta$. Then for all $A \in \mathbf{A}$ with $\alpha \in \overline{A}, \mu(A) < 1/n$, $n \geq n_0$, we must have $A \subset [0, 1]$. For these fundamental sets A, $A \cap (R - B)$ is a subset of the set of the rational numbers and so $m^*(R - B, A) = \mu\{(R - B) \cap A\}/\mu(A) = 0\ \forall A \in \mathbf{A}$ with the above property. Thus,

$\overline{D}_n^*(R - B, \alpha) = 0\ \forall n \geq n_0$, and hence, $\overline{D}^*(R - B, \alpha) = 0$. Since $\alpha \in B$ is arbitrary, B is d-open.

By similar arguments, we can show that if B contains, in addition to the irrational numbers in [0, 1], a finite number of rational numbers of (0, 1), then B is also d-open. This fact will be required in Example 3.

Note 2. In Example 1, the closed interval [0, 1] may clearly be replaced by any bounded closed interval.

We observe below that Axioms I–IV have an interesting implication on the basic structure of the original topological space X. We prove this in the following theorem which will also be needed to show that $(X, \mathbf{D})$ is Hausdorff (Corollary 1).

Theorem 4. *(X, τ) is regular.*

Proof. Let E be τ-closed and $x \notin E$. Then $x \in X - E = G$, say. By Remark 1, we associate with each $x \in X$, a sequence of fundamental sets $\{A_{n,x}\}$ such that $x \in \overline{A}_{n,x}$, $\mu(A_{n,x}) < 1/n\ \forall n$. By Axiom IV, we can find a $B_{n,x} \in \mathbf{A}$ such that $x \in \overline{A}_{2n,x} \subset B_{n,x}$ and $\mu(B_{n,x} - A_{2n,x}) < 1/2n$. Then $\mu(B_{n,x}) \leq \mu(B_{n,x} - A_{2n,x}) + \mu(A_{2nx}) < 1/2n + 1/2n = 1/n$. Thus, we have a sequence $\{B_{n,x}\}$ from $\mathbf{A}$ such that $x \in B_{n,x}$ and $\mu(B_{n,x}) < 1/n\ \forall n$. Again proceeding as above, we obtain a sequence $\{C_{n,x}\}$ from $\mathbf{A}$ satisfying $x \in B_{2n,x} \subset \overline{B}_{2n,x} \subset C_{n,x}$ and $\mu(C_{n,x}) < 1/n\ \forall n$. Since $x \in G$, by Axiom III, there exists $\varepsilon > 0$ such that, if $x \in \overline{A}$, $\mu(A) < \varepsilon$, $A \in \mathbf{A}$, then $A \subset G$. Choose $m \in N$ such that $1/m < \varepsilon$. Then $C_{m,x} \subset G$ and $x \in B_{2m,x} = U$, say, $\subset \overline{B}_{2m,x} = F$, say, $\subset C_{m,x} \subset G$. Thus, we have $U \in \tau$ and $V = X - F \in \tau$ satisfying $x \in U$, $E \subset V$ and $U \cap V = \emptyset$.

Corollary 1. *If (X, τ) is T_1, then $(X, \mathbf{D})$ is Hausdorff.*

Corollary 1 follows from the fact that (X, τ) is regular T_1 and so Hausdorff, and $\mathbf{D}$ is finer than τ.

Theorem 5. (cf. [10, Theorem 2.1]) *$(X, \mathbf{D})$ is regular.*

The proof is omitted.

4. Density Topology Through Closure

For $E \subset X$ and $x \in X$, if $\overline{D}^*(E, x) > 0$, then x is called a *d-limit point* of E and the collection of all d-limit points of E is denoted by E'. For each subset A, we may consider $d\text{-}cl(A) = A \cup A'$ called the *d-closure* of A. Assuming further that X is second countable and we have the following axiom.

Axiom V. If $A \in \mathbf{A}$, then $\overline{A}$ is compact.

One can verify that the operator d-closure satisfies all the axioms of Kuratovski for being a topological space. Calling a set E to be *d-closed* if $E = d - cl(E)$ and denoting by $\mathbf{C}$ the collection of all d-closed sets in X, let $\mathbf{U} = \{A : A \in X$ and $X - A \in \mathbf{C}\}$. One can verify as in [2, 10] that $\mathbf{U} = \mathbf{D}$.

5. Approximately Continuous Mappings

In this section, we consider the idea of approximately continuous mapping in topological spaces and state only those basic properties which do not seem to be available in the literature for other spaces.

Definition 4. [10] *A mapping $f : (X, \tau) \to (Y, \sigma)$, where (Y, σ) is an arbitrary topological space is called approximately continuous in X if, for each σ-open set V in Y, $f^{-1}(V)$ is d-open in X.*

By Theorem 3, a continuous mapping is approximately continuous but the converse is not true as the following example shows.

Example 2. Let X and μ be as in Example 1. Let $a, b \in R$, $a \neq b$ and let Y be the topological space consisting of the open sets $\phi, \{a, b\}$ and $\{a\}$. Let $f : R \to Y$ be defined by

$$f(x) = \begin{cases} a & \text{if } x \text{ is irrational,} \\ b & \text{if } x \text{ is rational.} \end{cases}$$

Clearly, f is approximately continuous.

Note 3. The class of all continuous but not continuous mapping is a proper subclass of the class of all approximately continuous mappings.

In this section, we shall assume further that X is second countable and Axiom V also holds. The proofs of the following theorems are omitted.

Theorem 6. *Let $f : X \to Y$ be a mapping. Then the following are equivalent.*
(i) *f is approximately continuous;*
(ii) *for each σ-closed set F is Y, $f^{-1}(F)$ is d-closed in X;*
(iii) *for $A \subset X$, f (d-cl $(A)) \subset \overline{f(A)}$;*
(iv) *for $B \subset Y$, $f^{-1}(\overline{B}) \supset$ d-cl $(f^{-1}(B))$;*
where bar here denotes the σ-closure.

Theorem 7. *If X is the union of two d-closed sets A, B and $f : A \to Y$ and let $g : B \to Y$ be continuous mappings in the subspace topology $\mathbf{D}_A$ and $\mathbf{D}_B$, respectively, such that $f(x) = g(x)$, $\forall x \in A \cap B$, then $h : X \to Y$ defined by $h(x) = f(x)$, $\forall x \in A$, and $h(x) = g(x)$, $\forall x \in B$, is approximately continuous.*

Theorem 8. *Let (Y, ρ) be a metric space. Then the uniform limit of a sequence of approximately continuous mappings is also approximately continuous.*

6. K_d-Topology

The following definition and lemma will be needed in this section.

Definition 5. *$A \subset X$ is said to be d-compact if every d-open cover of A has a finite subcover.*

But the converse is not true as the following example shows:

Clearly, every d-compact set is compact because any open set is d-open.

Example 3. With reference to Example 1, let us consider the closed interval $[0, 1]$ which is compact in R with the usual topology. We shall show that $[0, 1]$ is not d-compact. Let $B = [1, 2]$. Write all rational numbers in $(-1, 2)$ as a sequence $\{x_n\}$ and consider a sequence of sets $\{G_n\}$, where G_n contains all irrational numbers in $[-1, 2]$ and x_n. By Example 1, the sets G_n are d-open $\forall n$. Clearly, $\{G_n\}$ forms a d-open cover of $[0, 1]$. But it cannot contain a finite subcover $\{G_{r_1}, \ldots, G_{r_n}\}$ of $[0, 1]$ for this will imply that there are only a finite number of rational numbers in $[0, 1]$. Hence, $[0, 1]$ is not d-compact.

Lemma 1. *The approximately continuous image of a d-compact set is compact.*

The proof is omitted.

If X and Y are topological spaces, Arens [1] introduced a topology on the set of all continuous mappings from X to Y as follows. If $K \subset X$ is compact and $W \subset Y$ is open, then the collection of all continuous mappings f such that $f(K) \subset W$ is denoted by the symbol (K, W). The collection of all such (K, W) forms a sub-base of a topology which Arens calls a *K-topology.*

Now, if (X, τ) is equipped with the above d-topology and we include the set of all approximately continuous mappings from X to (Y, σ) into our consideration, then we observe that following the above method, we can generate several topologies as follows:

(i) $K \subset X$ d-compact, $W \subset Y$ open, and f continuous. The corresponding topology is called K_1-topology.
(ii) $K \subset X$ compact, $W \subset Y$ open, and f approximately continuous. The corresponding topology is called K_2-topology.
(iii) $K \subset X$ d-compact, $W \subset Y$ open, and f approximately continuous. The corresponding topology is called K_d-topology.

To obtain various topological properties of these topologies (including K-topology), we feel that the property "the image of a compact set is compact" is frequently needed. This is true if the mapping is continuous (and so is true in the case of K-topology [1]), but may fail to be true if the mapping is approximately continuous, because d-open sets need not be open (see Example 1). Clearly, K_1-topology is a subspace topology of the K_d-topology. As such in our present discussion, we dispense with K_1 and K_2 topologies and adhere to the K_d-topology.

We denote by AC the collection of all approximately continuous mappings from X to Y. From the definition of K_d-topology as given in (iii) and considering the fact that sets (K, W) form a sub-base of the K_d-topology, we observe that given any $f \in AC$, there exists a basis of K_d-open neighborhoods of f of the form $U(f) = (K_1, W_1) \cap \cdots \cap (K_n, W_n)$, where K_i is d-compact, W_i open, and $f(K_i) \subset W_i$ for $i = 1, 2, \ldots, n$. The set $U(f)$ will be denoted by $(K_1, \ldots, K_n; W_1, \ldots, W_n)$.

Note 4. Clearly, $f : (X, \tau) \to (Y, \sigma)$ is approximately continuous if and only if $f : (X, \mathbf{D}) \to (Y, \sigma)$ is continuous. So it appears that the study of K_d-topology should run analogously to the study of K-topology [1]. In Arens' paper [1], the question of metrizability of the K-topology has been dealt with under several conditions including the assumption that the range space is a metric space. The primary object of this section is to show that the K_d-topology is metrizable under certain conditions which do not go in the line of Arens' treatment. To do this, we need to present the proofs of two theorems where we use only the properties of d-compact sets and d-open sets as formulated in this paper.

Theorem 9. *The K_d-topology in AC is T_0, T_1, T_2 and regular if Y is so.*

Proof. First let Y be a T_2-space. Let $f, g \in AC$ and $f \neq g$. So there is $x \in X$ such that $f(x) \neq g(x)$. Further, there exists $U, V \in \sigma$ such that $f(x) \in U$, $g(x) \in V$, and $U \cap V = \emptyset$. Then we have $f \in (\{x\}, U)$ and $g \in (\{x\}, V)$ and clearly, these two K_d-open sets are disjoint. So K_d-topology is T_2. The proof is similar when Y is T_0 or T_1.

Let Y now be regular. Let $f \in AC$ and U be a K_d-open set containing f. Then there is a set of the form $U(f) = (K_1, \ldots, K_n; W_1, \ldots, W_n)$ such that $f \in U(f) \subset U$. Since K_i is d-compact, $f(K_i)$ is compact in Y (by Lemma 1) and $f(K_i) \subset W_i$.

Now, for each $y \in f(K_i)$, there is a $V_y \in \sigma$ such that

$$y \in V_y \subset \overline{V}_y \subset W_i \text{ (since } Y \text{ is regular).}$$

The collection of sets $\{V_y : y \in f(K_i)\}$ form an open cover of $f(K_i)$ and so there exist $y_1, \ldots, y_m \in f(K_i)$ such that $f(K_i) \subset \bigcup_{j=1}^{m} V_{y_j} = G_i$, say. Then $f(K_i) \subset G_i \subset \bigcup_{j=1}^{m} \overline{V}_{y_j} = \overline{G}_i \subset W_i$ and this is true for $i = 1, 2, \ldots, n$. Let $V(f) = (K_1, \ldots, K_n; G_1, \ldots, G_n)$. Then $f \in V(f) \subset U(f)$. We shall show that the K_d-closure of $V(f)$ is a subset of $U(f)$. Let $g \notin U(f)$. Then for some $l = 1, 2, \ldots, n$, $g \notin (K_l, W_l)$, i.e., $g(x) \notin W_l$ for some $x \in K_l$. So $g(x) \in Y - W_l \subset Y - \overline{G}_l = H_l$, say. Clearly, the K_d-open set $(\{x\}, H_l)$ containing g is disjoint from $V(f)$. Thus, $g \notin K_d\text{-}cl(V(f))$. This shows that $K_d\text{-}cl(V(f)) \subset U(f)$ and so the K_d-topology is regular. ∎

Theorem 10. *If $(X, \mathbf{D})$ is locally compact and has a basis of cardinality c and Y has a basis of cardinality c, then the K_d-topology has a basis whose cardinality does not exceed c where c denotes the power of the continuum.*

Proof. The basis $\mathbf{B}_x$ of $(X, \mathbf{D})$ may be supposed to consist of only those d-open sets whose d-closure are d-compact. Let $\mathbf{B}_Y$ be the basis of Y.

We shall show that the members of the K_d-topology of the form

$$V_0 = (d\text{-}cl(U_1), \ldots, d\text{-}cl(U_n); W_1, \ldots, W_n),$$

where $U_i \in \mathbf{B}_x$ for $i = 1, \ldots, n$ form a basis of the K_d-topology. Let (K, W) be a sub-base member of the K_d-topology. Let $x \in K$ and $f \in (K, W)$ and so $f(x) \in W$. Then there is a $W_x \in \mathbf{B}_Y$ such that $f(x) \in W_x \subset W$. Since f is approximately continuous, there exists $U_x \in \mathbf{B}_x$ such that $x \in U_x$ and $f(U_x) \subset W_x$. Again since $(X, \mathbf{D})$ is regular (see Theorem 5), we can find a d-open set V_x such that

$$x \in V_x \subset d\text{-}cl(V_x) \subset U_x\,.$$

Finally, we can obtain $G_x \in \mathbf{B}_X$ satisfying $x \in G_x \subset V_x \subset d\text{-}cl(Vx) \subset U_x$ and this gives $d\text{-}cl(G_x) \subset U_x$ and so $f(d\text{-}cl(Gx)) \subset W_x$. It may be noted that $d\text{-}cl(G_x)$ is compact. Now, $\{G_x : x \in K\}$ form a d-open cover of K and so there exist $x_1, \ldots, x_m \in K$ such that

$$K \subset \bigcup_{j=1}^{m} G_{x_j} .$$

This shows that

$$f \in (d\text{-}cl(G_{x_1}), \ldots, d\text{-}cl(G_{x_m}); W_{x_1}, \ldots, W_{x_m}) \subset (K, W).$$

Thus, the sets of the form V_0 form an open base of the K_d-topology. Since $\mathbf{B}_x$ and $\mathbf{B}_Y$ both have cardinality c, it is clear from the above construction that the cardinality of the class V_0 does not exceed c. This proves the theorem. ■

The proof of the following theorem is similar and so omitted.

Theorem 11. *Suppose $(X, \mathbf{D})$ is locally compact and second countable. If Y is second countable, then so is the K_d topology.*

Corollary 2. *If $(X, \mathbf{D})$ is locally compact second countable and Y is regular T_1, second countable, then the K_d-topology is metrizable.*

This follows from Theorems 9 and 11.

References

1. R.F. Arens, A topology for spaces of transformations, *Annals of Math.* **47** (1946) 480–495.
2. S. Chakraborty and B.K. Lahiri, Density topology in a topological group, *Indian J. Pure Appl. Math.* **15**(7) (1984) 753–764.
3. W. Eames, Local property of measurable sets, *Canad. J. Math.* **12** (1960) 632–640.
4. C. Goffman and D. Waterman, Approximately continuous transformations, *Proc. Amer. Math. Soc.* **12**(1) (1961) 116–121.
5. C. Goffman, C.J. Neugebauer, and T. Nishiura, Density topology and approximately continuity, *Duke Math. J.* **28** (1961) 497–503.
6. P.R. Halmos, *Measure Theory*, Van Nostrand Rheinhold, 1950.
7. B.K. Lahiri, Density and approximate continuity in topological groups, *J. Indian Math. Soc.* **41** (1977) 129–141.
8. B.K. Lahiri and S. Chakraborty, Approximately continuous functions in a measure space, *Vietnam J. Math.* **25**(1) (1997) 59–64.
9. N.F.G. Martin, A topology for certain measure spaces, *Trans. Amer. Math. Soc.* **112** (1964) 1–18.
10. P.K. Saha and B.K. Lahiri, Density topology in Romanovski spaces, *J. Indian Math. Soc.* **54** (1989) 65–84.
11. D.W. Soloman, On separation in measure and metric density on Romanovski spaces, *Duke Math. J.* **36** (1969) 81–90.
12. D.W. Soloman, *Denjoy Integration in Abstract Spaces*, Memoirs of the American Math. Soc., Vol. 85, American Mathematical Society, 1969.