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Stable Integral Currents in Riemannian Manifolds with Positive Sectional Curvature*

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Abstract. In this paper, we study the nonexistence of stable integral currents in compact Riemannian manifolds of positive sectional curvature.

1. Introduction

The existence theorem due to Federer and Fleming [3] states that, for any compact Riemannian manifold M^m, each non-trivial integral homology class in $H_p(M, Z)$ corresponds to a stable integral current. By applying the Federer Fleming theorem and the techniques from the calculus of variations in the geometric measure theory, Lawson and Simons [4] investigated the topology and geometry of submanifolds of the sphere, and showed that approriate assumptions on the extrinsic geometry of the submanifolds imply the vanishing of a given homology group. In [4], they conjectured that there are no stable integral currents in any compact, simply-connected Riemannian manifold which is $\frac{1}{4}$-pinched. As a "version" of the above famous conjecture, there have been many results on stable minimal submanifolds and stable harmonic maps. But the problem offered by Lawson and Simons is still open.

Recently, Cheng [2] showed that if M^m can be immersed as a hypersurface in the Euclidean space E^{m+1}, and the sectional curvature $K_M > A\lambda^2$ (where $A \geq \dfrac{1}{3+2\sqrt{2}}$ is constant, λ^2 is the maximum of the squared of principal curvatures of M^m). Then there are no stable integral currents in M^m.

In this paper, we shall give a partial answer of the Lawson–Simons conjecture. The obtained results show that if a complete δ-pinched Riemannian manifold can be immersed as a submanifold with parallel mean curvature vector and flat normal connection or as a

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hypersurface with constant mean curvature, in a space form of non-negative curvature, then the conjecture holds true.

2. Preliminaries

The same notations as in [6] will be used throughout this paper. In this section, we only list some main formulas employed in [6].

Let M^m be an m-dimensional compact Riemannian manifold with Riemannian metric $\langle , \rangle$ and the Levi–Civita connection ∇. Denote by (S, ξ) the oriented, p-rectifiable set in M^m. The set of rectifiable p-currents is (see [3,4])

$$\mathcal{R}_p(M) = \Big\{\mathcal{S} = \sum_{n=1}^{\infty} n\mathcal{S}_n\,;\ \mathcal{S}_n = (S_n, \xi_n),\ M(\mathcal{S}) = \sum_{n=1}^{\infty} n\,\mathcal{H}^p(S_n) < \infty\Big\}.$$

$\mathcal{S} \in \mathcal{R}_p(M)$ is called an *integral p-current* if $\mathcal{S}$ and $\partial\mathcal{S}$ are both rectifiable currents.

For a smooth vector field $X \in C(TM)$, let $\phi_t : M^m \to M^m$ be the 1-parameter group of local diffeomorphism generated by X, and we define [4]

$$Q_\xi(X) = \frac{d^2}{dt^2}\|\phi_{t*}\xi\|\Big|_{t=0}.$$

Then

$$\frac{d^2}{dt^2}\, M(\phi_{t*}\,\mathcal{S})\Big|_{t=0} = \sum_n n \int_{S_n} Q_{\xi_n}(X)\, d\mathcal{H}^p(x). \tag{2.1}$$

If $X = \nabla f$ for some $f \in C^3(M)$ and $\{e_i, e_\alpha\}$ $(i = 1, \ldots, p;\ \alpha = p+1, \ldots, m)$ is an orthonormal basis of T_xM with $\xi = e_1 \wedge e_2 \wedge \cdots \wedge e_p$, then (see [4, p. 436])

$$\begin{aligned} Q_\xi(X) &= -\langle a^X(\xi), \xi\rangle^2 + 2\|a^X(\xi)\|^2 + \langle \nabla_{X,\xi} X, \xi\rangle \\ &= \Big[\sum_j \langle a^X(e_j), e_j\rangle\Big]^2 + 2\sum_{j,\alpha}\langle a^X(e_j), e_\alpha\rangle^2 + \sum_j \langle \nabla_{X,e_j} X, e_j\rangle. \end{aligned} \tag{2.2}$$

Let $\phi : M^m \to N^n$ be an isometric immersion of M^m into a Riemannian manifold N^n. The Levi–Civita connection of N^n is $\overline{\nabla}$. Denote by $V(N, M)$ the normal bundle of M^m in N^n. For a smooth section $\nu \in C(V(N, M))$ and $X \in C(TM)$, we have

$$\overline{\nabla}_{X^\nu} = -A_\nu X + \nabla^{\perp}_{X^\nu},$$

where A_ν is the so-called shape operator determined by ν associated with the immersion ϕ.

For a given integer $p \in (0, m)$, let V be a p-dimensional subspace in T_xM. Define a map $B_\nu :\ V \to V$ associated with A_ν by [6].

$$B_\nu X = \text{orthogonal projection of } A_\nu X \text{ onto } V, \tag{2.3}$$

where $X \in V$. If $\{e_i\}$ is an orthonormal basis of V, we have

$$B_\nu X = \sum_i \langle A_\nu X, e_i\rangle e_i\,.$$

Let $\{\nu_\lambda\}$ be an orthonormal basis of the normal space $V_x(N, M)$ and $A_\lambda = A_{\nu_\lambda}$. Define a self-adjoint linear map $Q^A : V \to V$ associated with the immersion ϕ by [6].

$$Q^A X = \sum_\lambda \big[2\big(\sum_i \langle A_\lambda^2 X, e_i\rangle e_i - B_\lambda^2 X\big) - (\operatorname{tr} A_\lambda - \operatorname{tr} B_\lambda) B_\lambda X\big], \tag{2.4}$$

where $X \in V$ and $\{e_i\}$ is an orthonormal basis of V. Let $\{e_\alpha\}$ be an orthonormal basis of $V^\perp$ which is the orthogonal complement of V in $T_x M$. Then $\{e_i, e_\alpha\}$ is an orthonormal basis of $T_x M$ and

$$\operatorname{tr} Q^A = \sum_i \langle Q^A e_i, e_i\rangle = \sum_\lambda \big[2\sum_{i,\alpha} \langle A_\lambda e_i, e_\alpha\rangle^2 - (\operatorname{tr} A_\lambda - \operatorname{tr} B_\lambda) \operatorname{tr} B_\lambda\big]. \tag{2.5}$$

Now, assume $\psi : N^n \to E^l$ is an isometric immersion of the Riemannian manifold N^n in the Euclidean space E^l with the Levi–Civita connection D. The shape operator, associated with the isometric immersion $x = \psi \circ \phi : M^m \to E^l$, A'_ν determined by $\nu \in C(V(E^l, M))$, is given by

$$A'_\nu Y = -(D_Y \nu)^T,$$

where $Y \in C(TM)$.

Let (S, ξ) be an oriented, p-rectifiable set. At $x \in S$, we consider the tangent p-space $V = T_x S \subset T_x M$. Choose an orthonormal basis $\{e_i, e_\alpha\}$ of $T_x M$ such that $\{e_i\}$ is a basis of V and $\xi = e_1 \wedge e_2 \wedge \cdots \wedge e_p$. At $x \in M^m$, let $\{\nu_\sigma\}$ be an orthonormal basis of $V_x(E^l, M)$ and $A'_\sigma = A'_{\nu_\sigma}$. Consider Q_ξ, given by (2.2), as a quadratic form (see [4]) on the set

$$\theta = \{v^T;\ v \in E^l,\ v^T = \text{orthogonal projection of } v \text{ onto } T_x M\}. \tag{2.6}$$

Note that at the given point $x \in M^m$, $\{e_i, e_\alpha, \nu_\sigma\}$ is an orthonormal basis of E^l, hence, we have

$$\operatorname{tr} Q_\xi = \sum_i Q_\xi(e_i) + \sum_\alpha Q_\xi(e_\alpha) + \sum_\sigma Q_\xi(\nu_\sigma).$$

By making use of the proof given in [6], we have

Lemma. [6] $\operatorname{tr} Q = \operatorname{tr} Q^{A'}$, *where $Q^{A'}$ is the self-adjoint linear operator on the p-subspace $T_x S \subset T_x M$ associated with the immersion $\psi \circ \phi : M^m \to E^l$ and*

$$\operatorname{tr} Q^{A'} = \sum_\sigma \big[2\sum_{i,\alpha} \langle A'_\sigma e_i, e_\alpha\rangle^2 - (\operatorname{tr} A'_\sigma - \operatorname{tr} B'_\sigma) \operatorname{tr} B'_\sigma\big]. \tag{2.7}$$

At a point $x \in M^m$, we take an orthonormal basis $\{\nu_\lambda, \eta_\alpha\}$ of $V_x(E^l, M)$ so that $\{\nu_\lambda\}$ and $\{\eta_\alpha\}$ are bases of $V_x(N, M)$ and $V_x(E^l, N)$, respectively. Let $\bar{A}_\alpha = \bar{A}_{\eta_\alpha}$ be the shape operator associated with the immersion $\psi : N^n \to E^l$. Then

$$\operatorname{tr} Q^{A'} = \operatorname{tr} Q^A + \bar{A}(V), \tag{2.8}$$

where $\operatorname{tr} Q^A$ is given by (2.5) and

$$\bar{A}(V) = \sum_{a,i,\alpha} \big[2\langle \bar{A}_\alpha e_i, e_\alpha\rangle^2 - \langle \bar{A}_\alpha e_\alpha, e_\alpha\rangle \langle \bar{A}_\alpha e_i, e_i\rangle\big]. \tag{2.9}$$

3. Main Results

Theorem 1. *Let $\phi : M^m \to N^n(c)$ be an isometric immersion, where M^m is a complete Riemannian manifold with the sectional curvature $K_M \geq \delta > 0$ and $N^n(c)$ is a simply connected space form with $c \geq 0$. If the normal connection of the immersion ϕ is flat and the mean curvature vector of M^m is parallel, then there are no stable integral currents in M^m, and for any $p \in (0, m)$,*

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

Proof. According to the assumption, $N^n(c)$ can be considered as a totally umbilical hypersurface of E^{n+1} [1, p. 41]. In this case, (2.9) becomes

$$\bar{A}(V) = -p(m-p)c.$$

From (2.8), we obtain that

$$\operatorname{tr} Q^{A'} = \operatorname{tr} Q^A - p(m-p)c. \tag{3.1}$$

Because the normal connection of the submanifold M^m is flat, we know that all $A_\lambda = A_{\eta_\lambda}$ are simultaneously diagonalizable. Hence, there is an orthonormal basis $\{E_i\}$ of T_xM such that $A_\lambda E_i = k_{\lambda i} E_i$. Set $s = \sum_\lambda \operatorname{tr} A_\lambda^2$, $H = \sum_\lambda (\operatorname{tr} A_\lambda)\eta_\lambda / m$, that is, s is the square length of the second fundamental form and H is the mean curvature bundle, and the Laplacian of s is [5, p. 105]

$$\frac{1}{2}\Delta s = \sum_\lambda \sum_{i<j} (k_{\lambda i} - k_{\lambda j})^2 K(E_i \wedge E_j) + \sum_\lambda \|\nabla A_\lambda\|^2 \tag{3.2}$$

where $K(E_i \wedge E_j)$ is the sectional curvature of the plane $E_i \wedge E_j \subset T_xM$.

Since the sectional curvatures $K_M \geq \delta > 0$ and M^m is complete, $\operatorname{Ric}(v, v) \geq (m-1)\delta$ for a unit vector field v in T_xM. By Myer's theorem, M^m is compact. From (3.2), we have $\Delta s = 0$ and thus, $A_\lambda = k_\lambda I$, $\operatorname{tr} A_\lambda = m k_\lambda$, $H = \sum_\lambda k_\lambda \eta_\lambda$.

Let (S, ξ) be a p-rectifiable set and the p-space $V = T_xS \subset T_xM$. For the operator B_λ on V defined by (2.3), we have $\operatorname{tr} B_\lambda = p k_\lambda$. From (2.5), we obtain

$$\operatorname{tr} Q^A = -\sum_\lambda p(m-p)\, k_\lambda^2 = -p(m-p)\, \|H\|^2.$$

So (3.1) becomes

$$\operatorname{tr} Q^{A'} = -p(m-p)\,(c + \|H\|^2). \tag{3.3}$$

If $k_\lambda = 0$ for $\lambda = 1, 2, \ldots, n-m$, then $A_\lambda = 0$ for any λ and thus, M^m is totally geodesic. So $K_M = 0$ when $c = 0$. Therefore, $c + \|H\|^2 > 0$ when $c \geq 0$ and $K_M > 0$. By the Lemma, (3.3) implies $\operatorname{tr} Q_\xi = \operatorname{tr} Q^{A'} < 0$.

Let θ be the set given by (2.6). If $v^T \in \theta$, then v^T is the gradient ∇f of the function $f(x) = \langle v, x\rangle$ on M^m. To each $S \in \mathcal{R}_p(M)$ we associate a quadratic form Q_S on θ as follows. For $X \in \theta$, let ϕ_t be the flow generated by X and set

$$Q_S(X) = \frac{d^2}{dt^2} M(\phi_{t^*}S)\Big|_{t=0}.$$

Then from (2.1), (3.3) and the Lemma, we obtain

$$\operatorname{tr} Q_S = \sum_n n \int_{S_n} \operatorname{tr} Q_{\xi_n} \, d\mathcal{H}^p(x) < 0.$$

This implies that there is no stable integral p-current in M^m. Using the Federer–Fleming theorem, we have

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

Corollary 1. *Let M^m be a complete hypersurface with the sectional curvature $K_M \geq \delta > 0$ and constant mean curvature in E^{m+1} or S^{m+1}. Then there are no stable integral currents in M^m and hence, for any $p \in (0, m)$,*

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

Remark. When M^m is the unit sphere S^m, Corollary 1 is due to Lawson and Simons [4].

The assumption on M^m in Theorem 1 is stronger than that in Theorems 3.2 and 4.2 in [5]. The obtained result shows that M^m is topologically a sphere, and the Lawson–Simons conjecture is true in this case.

Let N^n be a complete submanifold with sectional curvature $K_N \geq \delta > 0$ immersed in E^l. Denote by $\bar{A}_\lambda$ the shape operator associated with the immersion $N^n \to E^l$. Suppose the normal connection of N^n is flat and the mean curvature vector $\bar{H}$ is parallel. Then for $\bar{A}(V)$ given by (2.9), a calculation similar to that given for (3.3) will give

$$\bar{A}(V) = -p(m-p)\,\|\bar{H}\|^2. \tag{3.4}$$

Let M^m be a compact submanifold immersed in N^n. Denote by A_λ the shape operator associated with the immersion $M^m \to N^n$, and by A'_λ the shape operator associated with the immersion $M^m \to E^l$. From (2.8) and (3.4), we have

$$\operatorname{tr} Q^{A'} = \operatorname{tr} Q^A - p(m-p)\,\|\bar{H}\|^2.$$

Hence, a similar proof as in Theorem 1 gives the following:

Theorem 2. *Suppose M^m is a compact submanifold immersed in N^n, where N^n is a complete submanifold in E^l with flat normal connection and parallel mean curvature vector $\bar{H}$. Denote by A_λ the shape operator associated with the immersion $M^m \to N^n$. Let $K_N \geq \delta > 0$ and p be a given integer $p \in (0, m)$. If, for any $x \in M^m$ and any p-subspace V in $T_x M$,*

$$\operatorname{tr} Q^A < p(m-p)\|\bar{H}\|^2,$$

then there is no stable integral p-currents in M^m and

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

Corollary 2. [4] *Let M^m be a compact submanifold of S^n and p a given integer $p \in (0, m)$. If, for any $x \in M^m$ and any p-subspace V in $T_x M$,*

$$\operatorname{tr} Q^A < p(m-p),$$

then there is no stable integral p-current in M^m and

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

Corollary 3. *Let M^m and N^n be as in Theorem 2. If the immersion $M^m \longrightarrow N^n$ is minimal and the square length of the second fundamental form of M^m satisfies $s < 2\min\{p, m-p\}\,\|\bar{H}\|^2$, then*

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

Proof. Because the immersion $M^m \to N^n$ is minimal, we have $\operatorname{tr} A_\lambda = 0$. So, from (2.5), we have

$$\begin{aligned}\operatorname{tr} Q^A &= \sum_\lambda \big[2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2 + (\operatorname{tr} B_\lambda)^2\big]\\ &= \sum_\lambda \big[2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2 + \big(\sum_i\langle A_\lambda e_i, e_i\rangle\big)^2\big].\end{aligned}$$

Since $\sum_i \langle A_\lambda e_i, e_i\rangle + \sum_\alpha \langle A_\alpha e_\alpha, e_\alpha\rangle = \operatorname{tr} A_\lambda = 0$, we obtain

$$\begin{aligned}\big(\sum_i\langle A_\lambda e_i, e_i\rangle\big)^2 &= \frac{1}{2}\big(\sum_i\langle A_\lambda e_i, e_i\rangle\big)^2 + \frac{1}{2}\big(\sum_\alpha\langle A_\lambda e_\alpha, e_\alpha\rangle\big)^2\\ &\le \frac{p}{2}\sum_i\langle A_\lambda e_i, e_i\rangle^2 + \frac{m-p}{2}\sum_\alpha\langle A_\lambda e_\alpha, e_\alpha\rangle^2.\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{tr} Q^A &\le \sum_\lambda \big(2\sum_{i,\alpha}\langle A_\lambda e_i, e_\alpha\rangle^2 + \frac{p}{2}\sum_i\langle A_\lambda e_i, e_i\rangle^2 + \frac{m-p}{2}\sum_\alpha\langle A_\lambda e_\alpha, e_\alpha\rangle^2\big)\\ &\le \frac{1}{2}\max\{p, m-p\}s.\end{aligned}$$

By the conditions, we obtain $\operatorname{tr} Q^A < p(m-p)\,\|\bar{H}\|^2$. By applying Theorem 2, we complete the proof. ■

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