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Short Communication

On the First Occurrence of Irreducible Modular Representations of Semigroups of all Matrices

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1. Introduction

Let $F_p[x_1, ..., x_n]$ be the commutative polynomial algebra in n indeterminants, $x_1, ..., x_n$ say, over the field F_p of p elements, and S^d the subspace of $F_p[x_1, ..., x_n]$ consisting of all homogeneous polynomials of degree d. M_n acts on $F_p[x_1, ..., x_n]$ in the usual way, therefore, M_n acts on S^d and S^d becomes M_n-module.

Mitchell showed that every irreducible M_n-module occurs as a composition factor in M_n-module S^d for some $d \le \sum_{i=1}^{n}(p^i - 1)$ and then, Doty and Walker [1] showed that this module can be embedded in S^d as an M_n-submodule for at least one value of d in this range. However, it is not known whether the degree d given above is the minimum possible degree of an embedding of the irreducible M_n-module. Independently, by using Dickson invariants, a complete set of distinct irreducible modules H_β was constructed by Tri [3]; every module H_β is a submodule of S^d for some d in the above range. The aim of this paper is to show that the occurrence of this module is the first occurrence of this module as a submodule in $F_p[x_1, ..., x_n]$.

To state our results we recall that the Dickson invariant $L_n = L_n(x_1, ..., x_n)$ is defined as follows:

$$L_n = \begin{vmatrix} x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_1^{p^{n-1}} & \cdots & x_n^{p^{n-1}} \end{vmatrix}$$

Then $\sigma.L_n = \det\sigma \; L_n$ for $\sigma \in M_n$.

Let $\beta = (\beta_1, ..., \beta_n)$ and $L^\beta = \prod_{i=1}^{n} L_i^{\beta_i} \in F_p[x_1, ..., x_n]$.

We denote by H_β the M_n-module generated by L^β, it means that H_β is an F_p-vector space generated by the set $\{\sigma.L^\beta \,:\, \sigma \in M_n\}$.

Theorem 1.1. [3, 1.1]

$$\{H_\beta : \beta = (\beta_1, \dots, \beta_n),\ 0 \le \beta_i \le p-1,\ 1 \le i \le n\}$$

is a complete set of p^n distinct irreducible modules for the algebra $F_p[M_n]$.

Let $n = 2$ and i, j be integers such that $0 \le i \le p-1$. Let W_{ij} be the M_n-module generated by $x_1^{ip^j}$. Then W_{ij} is irregular and isomorphic to $H_{(i,0)}$. W_{ij} occurs in $F_p[x_1, x_2]$ in dimension ip^j whereas $H_{(i,0)}$ occurs in $F_p[x_1, x_2]$ in dimension i. Let V_{ij} be the M_n-module generated by $L_2^{ip^j}$. Then V_{ij} is irregular and isomorphic to $H_{(0,i)}$. V_{ij} occurs in $F_p[x_1, x_2]$ in dimension $(1+p)ip^j$ whereas $H_{(0,i)}$ occurs in $F_p[x_1, x_2]$ in dimension $(1+p)i$. Generally, we have

Theorem 1.2. *H_β mentioned in Theorem 1.1 is the representative of lowest degree for its class of isomorphic irreducible M_n-submodules of $F_p[x_1, \dots, x_n]$.*

2. Proof of Theorem 1.2

We need the following results.

Proposition 2.1. [3, 3.5] *Let $\beta = (\beta_1, \dots, \beta_n)$ be such that $0 \le \beta_i \le p-1$ for $i = 1, \dots, n$.*

(i) *If $\beta_n \ge 1$, then $H_{(\beta_1,\dots,\beta_n)} \cong H_{(\beta_1,\dots,\beta_{n-1},\beta_n-1)} \otimes \det$.*

(ii) *Let $e_{n-1} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$. Then $e_{n-1}.H_{(\beta_1,\dots,\beta_{n-1},0)} = H_{(\beta_1,\dots,\beta_{n-1})}$ as M_{n-1}-modules.*

Lemma 2.2. *Let $\beta_1, \dots, \beta_n$ be such that $0 \le \beta_i \le p-1$, $1 \le i \le n-1$, $0 \le \beta_n \le p-2$ and W an M_n-module.*

(i) *If $1 \le \beta_n$ and $\det \otimes W \cong \det \otimes H_{(\beta_1,\dots,\beta_{n-1},\beta_n)}$, then $W \cong H_{(\beta_1,\dots,\beta_{n-1},\beta_n)}$.*

(ii) *If $\det \otimes W \cong \det \otimes H_{(\beta_1,\dots,\beta_{n-1},0)}$, then*

$$W \cong H_{(\beta_1,\dots,\beta_{n-1},0)} \text{ or } W \cong H_{(\beta_1,\dots,\beta_{n-1},p-1)}.$$

Proof. From the hypothesis $\det \otimes W \cong \det \otimes H_{(\beta_1,\dots,\beta_{n-1},\beta_n)}$, it implies that $\det \otimes W \cong H_{(\beta_1,\dots,\beta_{n-1},\beta_n+1)}$ by Proposition 2.1. $H_{(\beta_1,\dots,\beta_{n-1},\beta_n+1)}$ is irreducible since W is too. The modules H_β for $\beta = (\beta_1, \dots, \beta_n)$, $0 \le \beta_i \le p-1$, $1 \le i \le n$ form a complete set of p^n distinct irreducible modules for the algebra $F_p[M_n]$, so $W \cong H_{(\alpha_1,\dots,\alpha_{n-1},\alpha_n)}$ for some $\alpha_1, \dots, \alpha_{n-1}, \alpha_n$, $0 \le \alpha_i \le p-1$, $1 \le i \le n$.

For $1 \le \beta_n \le p-2$, if $\alpha_n = p-1$, from the definition of $H_{(\alpha_1,\dots,\alpha_{n-1},\alpha_n)}$, it is not hard to show that $\det \otimes H_{(\alpha_1,\dots,\alpha_{n-1},\alpha_n)}$ is isomorphic to $H_{(\alpha_1,\dots,\alpha_{n-1},1)}$. Then, from $H_{(\alpha_1,\dots,\alpha_{n-1},1)} \cong H_{(\beta_1,\dots,\beta_{n-1},\beta_n+1)}$, it implies that $\beta_n = 0$. It is impossible. Thus, $\alpha_n < p-1$ and $\det \otimes W \cong H_{(\alpha_1,\dots,\alpha_{n-1},\alpha_n+1)} \cong H_{(\beta_1,\dots,\beta_{n-1},\beta_n+1)}$. From this, we have $\alpha_i = \beta_i$, $1 \le i \le n$ and the assertion (i) is proved.

To prove the assertion (ii), we note

$$\det \otimes W \cong \begin{cases} H_{(\alpha_1,\dots,\alpha_{n-1},\alpha_n+1)} & \text{for } \alpha_n < p-1, \\ H_{(\alpha_1,\dots,\alpha_{n-1},1)} & \text{for } \alpha_n = p-1. \end{cases}$$

From the hypothesis of (ii) and Proposition 2.1, we have

$$\det \otimes W \cong H_{(\beta_1,\dots,\beta_{n-1},1)},$$

therefore, $\alpha_i = \beta_i, 1 \le i \le n-1$, $\alpha_n = 0$ or $\alpha_n = p-1$ and the assertion (ii) is proved. ∎

Proof of Theorem 1.2. Let T_n be the group consisting of upper triangular matrices with 1 on the diagonal. In [2], Mui showed that

$$F_p[x_1,\dots,x_n]^{T_n} = F_p[V_1,\dots,V_n],$$

where $V_i = V_i(x_1,\dots,x_i) = \prod(\alpha_1 x_1 + \cdots + \alpha_{i-1}x_{i-1} + x_i)$ and $L_n = V_1 \cdots V_n$.

The theorem is proved by induction on n.

For $n = 1$, let W be an irreducible M_1-module as a submodule of homogeneous polynomials in $F_p[x_1]$, isomorphic to H_{β_1} for some β_1, $0 \le \beta_1 \le p-1$. H_{β_1} is an M_1-module generated by $x_1^{\beta_1}$ and the modules H_i, $0 \le \subset \le p-1$, form a complete set of p distinct irreducible modules for $F_p[M_1]$. Therefore, W contains nonzero homogeneous polynomials with their degree not less than β_1 and the theorem is proved.

For $n > 1$, suppose the theorem is true for every integer less than n.

Let $\beta = (\beta_1,\dots,\beta_n)$, $0 \le \beta_i \le p-1$, $1 \le i \le n$ and H_β an irreducible module generated by L^β. Let W be an irreducible module as a submodule of homogeneous polynomials in $F_p[x_1,\dots,x_n]$ and η an isomorphism from W onto H_β. Set $f(x_1,\dots,x_n) = \eta^{-1}(L^\beta)$, then according to the proof of [3, 3.2], $f(x_1,\dots,x_n)$ is a unique T_n-invariant up to constant $a \in F_p, a \ne 0$. For $1 \le i \le n$, take $\sigma_i = (a_{jk}) \in M_n$, $a_{jj} = 1, j \ne i$ and $a_{jk} = 0$ at other positions. We have $\eta(\sigma_i.f(x_1,\dots,x_n)) = \sigma_i.\eta(f(x_1,\dots,x_n)) = \sigma_i.L^\beta = 0$, therefore, $\sigma_i.f(x_1,\dots,x_n) = 0$ and then $f(x_1,\dots,x_n)$ has x_i as a factor. $f(x_1,\dots,x_n)$ is an T_n-invariant, so $f(x_1,\dots,x_n)$ has $\alpha_1 x_1 + \cdots + \alpha_{i-1}x_{i-1} + x_i$ as a factor and V_i as a factor for $1 \le i \le n$. Thus, $f(x_1,\dots,x_n) = L_n f_1(x_1,\dots,x_n)$ for some T_n-invariant $f_1(x_1,\dots,x_n)$. Let W_1 be the M_n-module generated by $f_1(x_1,\dots,x_n)$, then $W = L_n W_1 \cong \det \otimes W_1$ as M_n-modules.

Note that $H_{(\beta_1,\dots,\beta_{n-1},\beta_n)} \cong \det \otimes H_{(\beta_1,\dots,\beta_{n-1},\beta_n-1)}$ if $\beta_n \ge 1$. Lemma 2.2 implies $W_1 \cong H_{(\beta_1,\dots,\beta_{n-1},\beta_n-1)}$. By repeating this procedure for $\beta_n - 1$ times and also by Lemma 2.2, we have $f(x_1,\dots,x_n) = L_n^{\beta_n} f_{\beta_n}(x_1,\dots,x_n)$ for some T_n-invariant $f_{\beta_n}(x_1,\dots,x_n)$ and $W_{\beta_n} \cong H_{(\beta_1,\dots,\beta_{n-1},0)}$ or $H_{(\beta_1,\dots,\beta_{n-1},p-1)}$, where W_{β_n} is the M_n-module generated by $f_{\beta_n}(x_1,\dots,x_n)$. If $W_{\beta_n} \cong H_{(\beta_1,\dots,\beta_{n-1},p-1)}$, by the above method, there finally exists k such that

$$f(x_1,\dots,x_n) = L_n^{\beta_n+k(p-1)} f_{\beta_n+k(p-1)}(x_1,\dots,x_n)$$

for some T_n-invariant $f_{\beta_n+k(p-1)}(x_1,\dots,x_n)$. If we denote by $W_{\beta_n+k(p-1)}$ the M_n-module generated by $f_{\beta_n+k(p-1)}(x_1,\dots,x_n)$, then $W_{\beta_n+k(p-1)} \cong H_{(\beta_1,\dots,\beta_{n-1},0)}$. Take $e_{n-1} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix} \in M_n$, then $\eta|_{e_{n-1}W} : e_{n-1}.W_{\beta_n+k(p-1)} \cong e_{n-1}.H_{(\beta_1,\dots,\beta_{n-1},0)} = H_{(\beta_1,\dots,\beta_{n-1})}$ as M_{n-1}-modules by Proposition 2.1. Note that $e_{n-1}.W_{\beta_n+k(p-1)}$ is the M_{n-1}-module generated by $e_{n-1}.f_{\beta_n+k(p-1)}(x_1,\dots,x_n)$, so $e_{n-1}.f_{\beta_n+k(p-1)}(x_1,\dots,x_n) \ne 0$ and $\deg f_{\beta_n+k(p-1)}(x_1,\dots,x_n) = \deg e_{n-1}.f_{\beta_n+k(p-1)}(x_1,\dots,x_n)$. By the induction assumption, we have $\deg e_{n-1}.f_{\beta_n+k(p-1)}(x_1,\dots,x_n) \ge \deg L^{(\beta_1,\dots,\beta_{n-1})}$. Thus,

$$\deg f(x_1,\dots,x_n) \ge \deg L^{(\beta_1,\dots,\beta_{n-1},\beta_n)}$$

and the theorem follows.

Now, let $r \geq 1$ and $n_1, \ldots, n_r$ positive integers such that $n_1 + \cdots + n_r = n$. Denoted by $M_{n_1,\ldots,n_r}$, the parabolic subsemigroup of semigroup M_n, $M_{n_1,\ldots,n_r}$ is defined as follows:

$$M_{n_1,\ldots,n_r} = \left\{ \begin{pmatrix} B_1 & & * \\ & \ddots & \\ 0 & & B_r \end{pmatrix} \in M_n \ : \ B_i \in M_{n_i}\,,\ 1 \leq i \leq r \right\}.$$

Denote by $H_\beta(M_{n_1,\ldots,n_r})$ the $M_{n_1,\ldots,n_r}$ module generated by L^β; it means that $H_\beta(M_{n_1,\ldots,n_r})$ is an F_p-vector space generated by the set $\{\sigma.L^\beta \ : \ \sigma \in M_{n_1,\ldots,n_r}\}$.

In the same way as in [3], we also have

Proposition 2.3.

$$\{H_\beta(M_{n_1,\ldots,n_r}) : \beta = (\beta_1, \ldots, \beta_n),\ 0 \leq \beta_i \leq p-1,\ 1 \leq i \leq n\}$$

is a complete set of p^n *distinct irreducible modules for* $F_p[M_{n_1,\ldots,n_r}]$.

Then the proof of Theorem 1.2 is also valid for the following proposition.

Proposition 2.4. $H_\beta(M_{n_1,\ldots,n_r})$ *mentioned in Proposition 2.3 is the representative of lowest degree for its class of isomorphic irreducible* $M_{n_1,\ldots,n_r}$ *submodules of* $F_p[x_1, \ldots, x_n]$.

References

1. S. Doty and G. Walker, Truncated symmetric powers and modular representations of GL_n, *Math. Proc. Cambridge Philos. Soc.* **119** (1996) 231–242.
2. Huynh Mui, Modular invariant theory and the cohomology algebras of symmetric groups, *J. Fac. Sci. Univ. Tokyo Sec. IA Math.* **22** (1975) 319–369.
3. Ton That Tri, The irreducible modular representations of semigroups of all matrices, *Acta Math. Vietnam* **20**(1) (1995) 43–53.