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On the Sample Continuity of Random Mappings*

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Abstract. In this paper the sample continuity of random mappings between a separable metric space and a separable Banach space is considered. It is shown that the well-known Kolmogorov criterion does not hold if the domain of the random mapping is a bounded set in an infinite-dimensional Hilbert space.

1. Introduction

Let (X, d) be a separable metric space and Y a separable Banach space. By a random mapping Φ from X to Y (or a Y-valued random mapping), we mean a family $\Phi = \{\Phi x, x \in X\}$ of Y-valued random variables (r.v.'s) indexed by the parameter set X. If X is an interval of the real line $\mathbf{R}^1$, we say that Φ is a Y-valued stochastic process, and if $Y = \mathbf{R}^1$, we say that Φ is a random function on X.

An important result on the existence of sample continuous modification of the stochastic process on an interval $[0, T]$ is provided by a well-known Kolmogorov criterion (see [5]). This criterion was extended by Totoki [7] to the case of a Y-valued random mapping on a bounded set of a finite-dimensional Euclidean space. Namely, if $\Phi = (\Phi x)$ is a Y-valued random mapping on a bounded set $X \subset \mathbf{R}^k$ such that, for some $p > 0, \alpha > 0$ and all x_1, x_2 in X

$$E\|\Phi x_1 - \Phi x_2\|^p \leq C\|x_1 - x_1\|^{k+\alpha},$$

then Φ is sample continuous (i.e., there exists a modification of Φ whose sample paths are continuous). By applying this result, it is not difficult to show that, if Φ is a Y-valued Gaussian random mapping with mean 0 defined on a bounded set X of a finite-dimensional Euclidean space, then the condition

$$E\|\Phi x_1 - \Phi x_2\|^2 \leq C\|x_1 - x_2\|^r \tag{1.1}$$

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some $C > 0$, $r > 0$ and all $x_1, x_2 \in X$ is sufficient for the sample continuity
eorem 2.4). However, Theorem 2.5 shows that the above assertion does not hold if X
bounded set in an infinite-dimensional Hilbert space. A sufficient condition, which
ures the sample continuity of Y-valued random mappings satisfying (1.1), is found. In
. 3, we restrict ourselves to random operators and obtain some new results about their
nple continuity. Many other properties of random operators have been considered in
8–10].

Sample Continuity of Random Mapping

$(\Omega, \mathcal{F}, P)$ be a complete probability space, (X, d) a separable metric space, and Y
eparable Banach space.

finition 2.1. *A family $\Phi = \{\Phi x, x \in X\}$ of Y-valued r.v.'s Φx indexed by the ameter set X is called a random mapping from X into Y or a Y-valued random pping on X. We set*

$$\Phi(x, \omega) = \Phi x(\omega) \quad \textit{for all} \quad x \in X \textit{ and } \omega \in \Omega.$$

X is an interval $[0, T]$ of the real line, then Φ is said to be a Y-valued stochastic cess on $[0, T]$.

For each $\omega \in \Omega$, the mapping $\Phi(., \omega) : x \longrightarrow \Phi(x, \omega)$ is called a sample path of Φ.
other random mapping Ψ from X into Y is said to be a modification of Φ if

$$\forall x \in X \quad \Phi x(\omega) = \Psi x(\omega) \quad \text{almost surely (a.s.).}$$

hould be noted that the set of ω in which the above equality holds depends on x.

finition 2.2.

A random mapping Φ from X into Y is said to be stochastically continuous at $x_0 \in X$ if

$$\forall \varepsilon > 0 \quad \lim_{x_n \to x_0} P\big\{\|\Phi x_n - \Phi x_0\| > \varepsilon\big\} = 0.$$

Φ is stochastically continuous on X if it is stochastically continuous at every point of X.

Φ is said to be sample continuous if there exists a modification Ψ of Φ such that all sample paths of Ψ are continuous.

finition 2.3. *A random mapping Φ from X into Y is called a Gaussian random pping (with mean 0, resp.) if the stochastic process $\big\{(\Phi x, a), (x, a) \in X \times Y'\big\}$ is a ussian stochastic process (with mean 0, resp.).*

eorem 2.4. *Let X be a bounded set in a finite-dimensional Euclidean space and Φ -valued Gaussian random mapping with mean 0 on X. Suppose for some $C > 0$, 0 and all $x_1, x_2 \in X$, we have*

$$E\|\Phi x_1 - \Phi x_2\|^2 \le C d^{\delta}(x_1, x_2), \tag{2.1}$$

n Φ is sample continuous.

Proof. Let $X \subset \mathbf{R}^k$ and without loss of generality, assume $d(x_1, x_2) = \|x_1 - x_2\|$.

By the crucial property of Gaussian random variables with values in Banach spaces (see [3]), for each $p > 0$, there exists a positive number C_p such that

$$E\|\Phi x_1 - \Phi x_2\|^p \leq C_p \left\{ E\|\Phi x_1 - \Phi x_2\|^2 \right\}^{\frac{p}{2}}.$$

From (2.1), we obtain

$$E\|\Phi x_1 - \Phi x_2\|^p \leq D_p \|x_1 - x_2\|^{\frac{\delta p}{2}}$$

for all $x_1, x_2 \in X$, where $D_p = C_p C^{\frac{\delta p}{2}}$. Let p be sufficiently large such that

$$\frac{\delta p}{2} > k + 1$$

and M the diameter of X. We have the following estimation:

$$\begin{aligned} E\|\Phi x_1 - \Phi x_2\|^p &\leq D_p M^{\frac{\delta p}{2}} \left\| \frac{x_1 - x_2}{M} \right\|^{\frac{\delta p}{2}} \\ &\leq D_p M^{\frac{\delta p}{2}} \left\| \frac{x_1 - x_2}{M} \right\|^{k+1} = L\|x_1 - x_2\|^{k+1}, \end{aligned}$$

where $L = D_p M^{\frac{\delta p}{2} - k - 1}$.

By the extended Kolmogorov's criterion due to Totoki [7] (which holds for Y-valued random mappings on a bounded set in $\mathbf{R}^k$), we conclude that Φ is sample continuous. ∎

Next, we shall show that condition (2.1) is not sufficient for sample continuity of Gaussian random mappings on X with mean 0 if X is a bounded set of an infinite-dimensional Hilbert space H. To this end, let (α_n) be the sequence of independent identical distributive (i.i.d. for short) $N(0, 1)$ random variables and (y_n) a bounded sequence in Y. Suppose X is the unit ball of a Hilbert space H with the orthonormal basis (e_n). We have the following theorem.

Theorem 2.5. *Assume Y is a Banach space of type 2. Then*

(i) *for each $x \in X$, the series*

$$\Phi x = \sum_{n=1}^{\infty} \alpha_n y_n (x, e_n)$$

converges a.s. in Y and define a Gaussian random mapping Φ with mean 0 satisfying the condition

$$E\|\Phi x_1 - \Phi x_2\|^2 \leq C\|x_1 - x_2\|^2 \tag{2.2}$$

for all $x_1, x_2 \in X$ and some $C > 0$.

(ii) *if* $\sum_{n=1}^{\infty} \|y_n\|^2 < \infty$, *then* Φ *is sample continuous.*

(iii) *a necessary condition for sample continuity of* Φ *is that there exists a positive number* $K > 0$ *such that*

$$\sum_{n=1}^{\infty} \exp\left\{-\frac{K}{\|y_n\|^2}\right\} < \infty. \tag{2.3}$$

In particular, if $y_n = y$, $y \neq 0$ *for all n, then* Φ *is not sample continuous.*

Proof. (i) Because Y is of type 2, there exists $C > 0$ such that, for all independent Y-valued r.v.'s $X_1, X_2, \dots, X_n$ with mean 0 and finite second moment, we have

$$E\|\sum_{i=1}^{n} X_i\|^2 \leq C \sum_{i=1}^{n} E\|X_i\|^2. \tag{2.4}$$

So we find (with $A = \sup \|y_n\|^2$)

$$E\| \sum_{k=m}^{n} \alpha_k y_k(x, e_k)\|^2 \leq CA \sum_{k=m}^{n} |(x, e_k)|^2,$$

which proves that the series $\sum_{n=1}^{\infty} \alpha_n y_n(x, e_n)$ converges in probability, and hence, converges a.s. by Ito–Nisio theorem. It is easy to check that Φ is a Gaussian random mapping with mean 0. Moreover, using (2.4), we obtain

$$E\left\| \sum_{i=1}^{n} \alpha_i y_i(x_1, e_i) - \sum_{k=1}^{n} \alpha_i y_i(x_2, e_i)\right\|^2 \leq CA\|x_1 - x_2\|^2$$

for all n which proves (2.2).

(ii) We have

$$E\Big(\sum_{n=1}^{\infty} \|\alpha_n y_n\|^2\Big) = E|\alpha_1|^2 \sum_{n=1}^{\infty} \|y_n\|^2 < \infty,$$

so $\sum_{n=1}^{\infty} \|\alpha_n y_n\|^2 < \infty$ a.s. Put

$$\Omega_0 = \left\{\omega \in \Omega : \sum_{n=1}^{\infty} \|\alpha_n(\omega) y_n\|^2 < \infty\right\}.$$

For each $\omega \in \Omega_0$, $x \in X$, we have

$$\left\| \sum_{i=m}^{n} \alpha_i(\omega) y_i(x, e_i)\right\| \leq \sum_{i=m}^{n} \|\alpha_i(\omega) y_i\| \, |(x, e_i)| \leq \Big(\sum_{i=m}^{n} \|\alpha_i(\omega) y_i\|^2\Big)^{\frac{1}{2}} \|x\|.$$

From this it follows that the series $\sum_{n=1}^{\infty} \alpha_n(\omega) y_n(x, e_n)$ converges in Y for each $\omega \in \Omega_0$ and $x \in X$.

Define a mapping $\Psi : X \times \Omega \to Y$ by

$$\Psi(x, \omega) = \begin{cases} \sum_{n=1}^{\infty} \alpha_n(\omega) y_n(x, e_n), & \text{if } \omega \in \Omega_0 \\ 0, & \text{otherwise.} \end{cases}$$

For each $x \in X$, by definition, the series $\sum_{n=1}^{\infty} \alpha_n y_n(x, e_n)$ converges a.s. to $\Psi(x, \omega)$. Consequently,

$$P\Big\{\omega \in \Omega : \Psi(x, \omega) = \Phi(x, \omega)\Big\} = 1.$$

It remains to show that all sample paths of Ψ are continuous. Indeed, it is easy to see that the mapping

$$x \longrightarrow \sum_{i=1}^{n} \alpha_i(\omega) y_n(x, e_i)$$

is linear and continuous. By the Banach–Steinhaus theorem, the mapping $x \to \Psi(x, \omega)$ is continuous as desired.

(iii) Suppose there exists a modification Ψ with continuous sample paths. Because $\Phi(x, \omega) = \Psi(x, \omega)$ a.s., we can find a set Ω_0 of probability 1 such that

$$\Phi(re_n, \omega) = \Psi(re_n, \omega)$$

for all rational number $r \in Q$ $|r| \leq 1$ and all e_n. Clearly, for each $r \in Q$, $|r| \leq 1$ and each e_n

$$\Phi(re_n) = r\Phi(e_n) \quad \text{a.s.}$$

Hence, we can find a set Ω_0 of probability 1 such that

$$\Phi(re_n, \omega) = r\Phi(e_n, \omega)$$

for all $\omega \in \Omega_0$, all $r \in Q, |r| \leq 1$, and all e_n.

Now, fix $\omega \in \Omega_1 \cap \Omega_0$. For each rational number $r \in [0, 1]$ and each n, we have

$$\Psi(re_n; \omega) = \Phi(re_n, \omega) = r\Phi(e_n, \omega) = r\Psi(e_n, \omega).$$

Since the mapping $x \to \Psi(x, \omega)$ is continuous at O and $\Psi(0, \omega) = 0$, there exists $r \in Q, 0 < r \leq 1$ such that $\|\Psi(x, \omega)\| < 1$ whenever $\|x\| < r$. Consequently,

$$\frac{r}{2}\|\Psi(e_n, \omega)\| = \|\frac{r}{2}\Psi(e_n, \omega)\| = \|\Psi(\frac{re_n}{2}, \omega)\| < 1 \quad \text{for all } n.$$

From this, we obtain

$$\|\Phi(e_n, \omega)\| = \|\Psi(e_n, \omega)\| < \frac{2}{r} < \infty,$$

for all n and all $\omega \in \Omega_1 \cap \Omega_0$.

Since $P(\Omega_1 \cap \Omega_0) = 1$, this means that

$$\sup_n \|\Phi(e_n, \omega)\| < \infty \quad \text{a.s.},$$

i.e.,

$$\sup_n |\alpha_n| \, \|y_n\| < \infty. \tag{2.5}$$

By Vakhania's theorem [12], (2.5) implies (2.3) as desired.

Now, let Φ be a random mapping from X into Y (Gaussian or not) satisfying the following conditions:

(i) $\forall x \in X \quad E\|\Phi x\|^2 < \infty$,

(ii) $\exists C > 0, \ \exists \delta > 0$ such that

$$\left(E\|\Phi x_1 - \Phi x_2\|^2\right)^{\frac{1}{2}} \le C d^{\delta}(x_1, x_2) \tag{2.6}$$

for all $x_1, x_2 \in X$.

Without loss of generality, we can assume

$$d(x_1, x_2) \le M \quad \forall x_1, x_2 \in X.$$

The problem considered here is to determine sufficient conditions for Φ to be sample continuous.

Denote by $L_2(\Omega)$ the Hilbert space of real-valued random variable ξ with

$$\|\xi\| = \left(E|\xi|^2\right)^{\frac{1}{2}} < \infty.$$

If $C(X, Y)$ stands for the set of all bounded continuous mappings from X into Y, then $C(X, Y)$ becomes a Banach space under the norm

$$\|f\|_C = \sup_{x \in X} \|f(x)\|.$$

It should be noted that $C(X, Y)$ is not necessarily separable.

We associate to Φ a mapping T from $L_2(\Omega)$ into the set of all mappings from X into Y defined by

$$(T\xi)x = \int_{\Omega} \xi(\omega)\Phi x(\omega) dP(\omega). \tag{2.7}$$

Here, the Bochner integral (2.7) exists since $E\|\Phi x\|^2 < \infty$. ∎

Lemma 2.6. *T is a linear continuous mapping from $L_2(\Omega)$ into $C(X, Y)$.*

Proof. We have to show that $T\xi \in C(X,Y)$ for $x_1, x_2 \in X$ we have

$$\begin{aligned}\|(T\xi)x_1 - (T\xi)x_2\| &\le \int_\Omega |\xi|\,\|\Phi x_1 - \Phi x_2\| dP \\ &\le \left(E\|\xi\|^2\right)^{\frac{1}{2}} \left(E\|\Phi x_1 - \Phi x_2)\|^2\right)^{\frac{1}{2}} \le C\|\xi\| d^\delta(x_1,x_2)\end{aligned}$$

showing that $T\xi$ is continuous.

Fix $x_0 \in X$. For all $x \in X$, we have

$$\begin{aligned}\|(T\xi)x\| &\le \|(T\xi)x_0\| + \|(T\xi)x - (T\xi)x_0\| \\ &\le \|\xi\|\left\{\left(E\|\Phi x_0\|^2\right)^{\frac{1}{2}} + Cd^\delta(x,x_0)\right\} \\ &\le \|\xi\|\left\{\left(E\|\Phi x_0\|^2\right)^{\frac{1}{2}} + CM^\delta\right\} \\ &= \|\xi\|K, \end{aligned} \tag{2.8}$$

which proves that $T\xi$ is bounded. The linearity of T is obvious. Hence, T is a linear mapping from X into $C(X,Y)$.

Moreover, from (2.8), we obtain

$$\|T\xi\|_C \le K\|\xi\|,$$

which proves the continuity of T. ■

Lemma 2.7. *Suppose there exists a Radon measure μ on $C(X,Y)$ such that, for all $(x_1, x_2, \dots, x_n)$ in X, all $(a_1, a_2, \dots, a_n)$ in Y' and all Borel set B in $\mathbf{R}^n$,*

$$P\left\{(\Phi x_i, a_i)_{i=1}^n \in B\right\} = \mu\left\{f : (fx_i, a_i)_{i=1}^n \in B\right\}. \tag{2.9}$$

Then Φ is sample continuous.

Proof. For each pair $v = (x,a) \in X \times Y'$, the mapping $v : C(X,Y) \to R$ given by $v(f) = (fx, a)$ is linear and continuous. Suppose (x_m) is the countable set dense in X and $(a_n) \subset Y'$ is a sequence in Y' such that

$$\|y\| = \sup_n |(y, a_n)| \quad \forall y \in Y.$$

Let

$$M = \left\{(x_n, a_m)_{n,m=1}^\infty\right\} \subset X \times Y'.$$

M is a countable set in $X \times Y'$, so it can be written as a sequence $M = \{v_1, v_2, \dots\}$.

For brevity, if $v = (x,a) \in X \times Y'$, then $v(\Phi)$ denotes the r.v. $(\Phi x, a)$, i.e.,

$$v(\Phi)(\omega) = \left(\Phi x(\omega), a\right).$$

Define the mapping $A : C(X,Y) \to R^\infty$ by

$$A(f) = \left\{v_i(f)\right\}_{i=1}^\infty$$

It is easy to verify that A is one-to-one and continuous. Since μ is a Radon measure, there exists a sequence of compact sets $(K_n) \subset C(X, Y)$ such that $\lim_n \mu(K_n) = 1$. Put $K = \bigcup_{n=1}^{\infty} K_n$, then K is a Borel set and $\mu(K) = 1$ and $A(K) = \bigcup_{n=1}^{\infty} A(K_n)$ is a Borel set in R^{∞}. The restriction of A on K has an inverse from $A(K)$ into $C(X, Y)$ denoted by B. Put

$$\Omega_0 = \Big\{\omega \in \Omega : \{v_i(\Phi)\}_{i=1}^{\infty} \in A(K)\Big\}.$$

Then by (2.9),

$$\begin{aligned} P(\Omega_0) &= P\Big\{\omega : \{v_i(\Phi)\}_{i=1}^{\infty} \in A(K)\Big\} \\ &= \mu\Big\{f \in (X, Y) : \{v_i(f)\}_{i=1}^{\infty} \in A(K)\Big\} \\ &= \mu\Big\{f : A(f) \in A(K)\Big\} \geq \mu(K) = 1. \end{aligned}$$

Consider the mapping $G : \Omega_0 \to A(K)$ defined by

$$G(\omega) = \{v_i(\Phi)\}_{i=1}^{\infty}$$

and the mapping $\Psi : \Omega_0 \to C(X, Y)$ by

$$Y(\omega) = B(G(\omega)).$$

By (2.9), the distribution of A is the same as the distribution of G.

Fix an element $v \in M$. We have

$$\begin{aligned} &P\Big\{\omega : v(\Phi)(\omega) = v(\Psi(\omega))\Big\} \\ &= P\Big\{\omega : v(\Phi)(\omega) = v\big[B(G(\omega))\big]\Big\} \\ &= \mu\Big\{f \in C(X, Y) : v(f) = v\big[B(A(f))\big]\Big\} \\ &\geq \mu\Big\{f : f = BA(f)\Big\} \\ &\geq \mu(K) = 1. \end{aligned} \tag{2.10}$$

Consider the random mapping Ψ given by

$$\Psi(x, \omega) = \Psi(\omega)x.$$

It is obvious that all sample paths of Ψ are elements of $C(X, Y)$. We shall show that Ψ is a modification of Φ. From (2.10), we have

$$(\Phi x_n, a_m) = (\Psi x_n, a_m) \quad \text{a.s.}$$

which implies that $\forall n = 1, 2, \ldots$

$$\Phi x_n(\omega) = \Psi x_n(\omega) \quad \text{a.s.}$$

Now, if x is an arbitrary element in X, then we can choose a subsequence (x_{n_k}) bending to x. By (2.6), Φx_{n_k} converges to Φx in probability.

On the other hand, $\Psi_{n_k}(\omega)$ converges to $\Psi x(\omega)$ for all $\omega \in \Omega$. Consequently,

$$\Phi x(\omega) = \Psi x(\omega) \quad \text{a.s.}$$

The lemma is proved. ■

We are now ready to prove the following:

Theorem 2.8. *A sufficient condition for Φ to be sample continuous is that the linear continuous mapping $T : L_2(\Omega) \to C(X, Y)$ given by (2.7) is 2-summing.*

Proof. Let γ be a cylindrical measure mapping $I_d : L_2(\Omega) \to L_2(\Omega)$. In other words, γ is defined by the formula

$$(g_1, g_2, \ldots, g_n) \in L_2(\Omega) : \gamma_{g_1,\ldots,g_n} = \mathcal{L}(g_1, \ldots, g_n),$$

or

$$\gamma\Big[\xi \in L_2 : (\xi, g_i)_{i=1}^n \in B\Big] = P\Big\{\omega : \big(g_i(\omega)\big)_{i=1}^n \in B\Big\}. \tag{2.11}$$

γ is a cylindrical measure of type 2 because

$$\sup_{\|g\|\le 1} \int |t|^2 d\gamma_g(t) = \sup_{\|g\|\le 1} E\|g\|^2 \le 1.$$

Since T is 2-summing, by Schwartz's Radonification Theorem (see [4]), the image measure $\mu = T(\gamma)$ is a Radon measure on $C(X, Y)$. Now, for $(x_1, x_2, \ldots, x_n) \subset X$, $(a_1, \ldots, a_n) \subset Y'$ and for all Borel sets $B \subset R^n$ by (2.11), we obtain

$$\begin{aligned}
&\mu\Big\{f \in C(X, Y) : \big(f x_i, a_i\big)_{i=1}^n \in B\Big\}\\
&= \gamma\Big\{\xi \in L_2(\Omega) : \big((T\xi)x_i, a_i\big)_{i=1}^n \in B\Big\}\\
&= \gamma\Big\{\xi \in L_2(\Omega) : \big(\xi, (\Phi x_i, a_i)\big)_{i=1}^n \in B\Big\}\\
&= P\Big\{\omega : \big(\Phi x_i, a_i\big)_{i=1}^n \in B\Big\}.
\end{aligned}$$

According to Lemma 2.7, we conclude that Φ is sample continuous. ∎

3. Sample Continuity of Random Operators

Throughout this section, X is assumed to be a separable Banach space.

Definition 3.1.

(a) *A random mapping Φ from X into Y is said to be stochastically linear if, for each $\lambda_1, \lambda_2 \in R$ and each $x_1, x_2 \in X$,*

$$\Phi(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \Phi x_1 + \lambda_2 \Phi x_2 \quad a.s.$$

It is important that the set of ω in which the above equality holds depends on λ_1, λ_2, x_1 and x_2.

(b) *If a random mapping Φ is stochastically linear and stochastically continuous, then Φ is called a* random operator *from X into Y.*

Theorem 3.2. *Let $X = \ell_p$ $(p > 1)$ and Φ be a random operator from X into Y.*

(i) *If*

$$\sum_{n=1}^{\infty} \|\Phi e_n\|^{p'} < \infty \quad a.s.,$$

then Φ is sample continuous, where (e_n) is the standard basis in ℓ_p and p' is the conjugate number of p $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$.

(ii) *A necessary condition for the sample continuity of Φ is that*

$$\sum_{n=1}^{\infty} \left|(\Phi e_n, a)\right|^{p'} < \infty \quad a.s. \quad for\ all\ a \in Y'$$

and

$$\sup_{n \in N} \|\Phi e_n\| < \infty \quad a.s.$$

Proof. (i) Put

$$\Omega_0 = \left\{\omega : \sum_{n=1}^{\infty} \|\Phi e_n(\omega)\|^{p'} < \infty\right\}.$$

If $\omega \in \Omega_0$, then

$$\sum_{n=1}^{\infty} \|(x, e_n)\Phi e_n(\omega)\| < \infty,$$

which implies that the series $\sum_{n=1}^{\infty} (x, e_n)\Phi e_n(\omega)$ is convergent in Y. Define a random mapping Ψ by

$$\Psi(x, \omega) = \begin{cases} \sum_{n=1}^{\infty} (x, e_n)\Phi e_n(\omega), & \text{if } \omega \in \Omega_0, \\ 0 & \text{otherwise.} \end{cases}$$

By the Banach–Steinhaus theorem, all sample paths of Ψ are continuous. Since Φ is stochastically continuous, linear and $x = \sum_{n=1}^{\infty} (x, e_n)e_n$, we obtain that

$$\Phi x = \sum_{n=1}^{\infty} (x, e_n)\Phi e_n,$$

where the series converges in probability. Clearly, the series $\sum_{n=1}^{\infty} (x, e_n)\Phi e_n$ converges a.s. to Ψx. Consequently, $\Phi x = \Psi x$ a.s., i.e., Ψ is a modification of Φ.

(ii) In order to prove (ii), we need the following lemma.

Lemma 3.3. *Suppose Ψ is a modification of Φ with continuous sample paths. Then for almost all ω, sample paths $\Psi(., \omega)$ are linear.*

Proof. Let Z be a countable set dense in X and $[Z]$ a linear space spanned over the field Q of rational number of Z. From the stochastic linearity of Φ and the countability of $[Z]$, it follows that there exists a set Ω_0 with $P(\Omega_0) = 1$ such that

$$\forall \omega \in \Omega_0, \forall x_1, x_2 \in [Z], \ \forall r_1, r_2 \in Q$$
$$\Phi(r_1 x_1 + r_2 x_2, \omega) = r_1 \Phi(x_1, \omega) + r_2 \Phi(x_2, \omega).$$

Since Ψ is a modification of Φ and $[Z]$ is countable, we can find a set Ω_1 with $P(\Omega_1) = 1$ such that

$$\Phi(x, \omega) = \Psi(x, \omega)$$

for all $x \in [Z]$ and all $\omega \in \Omega_1$.

Now, we claim that, for each $\omega \in \Omega_0 \cap \Omega_1$, the sample path $\Psi(., \omega)$ is linear. Indeed, for $x_1, x_2 \in [Z]$ and $r \in Q$, we have

$$\Psi(x_1 + x_2, \omega) = \Phi(x_1 + x_2, \omega) = \Phi(x_1, \omega) + \Phi(x_2, \omega) = \Psi(x_1, \omega) + \Psi(x_2, \omega)$$
$$\Psi(r x_1, \omega) = \Phi(r x_1, \omega) = r \Phi(x_1, \omega) = r \Psi(x_1, \omega).$$

For $x \in X$, $x' \in X$ and $\lambda \in R$, we can choose sequences $(x_n) \subset [Z]$, $(x_n') \subset [Z]$ and $(r_n) \subset Q$ such that $x_n \to x$, $x_n' \to x'$ and $r_n \to \lambda$. Using the continuity of the mapping $x \to \Psi(x, \omega)$, we obtain

$$\begin{aligned} \Psi(x, \omega) + \Psi(x', \omega) &= \lim \Psi(x_n, \omega) + \lim \Psi(x_n', \omega) \\ &= \lim \Psi(x_n + x_n', \omega) = \Psi(x + x', \omega), \\ \lambda \Psi(x, \omega) &= \lim r_n \Psi(x_n, \omega) = \lim \Psi(r_n x_n, \omega) = \Psi(\lambda x, \omega). \end{aligned}$$

The lemma is proved. ■

Proof of part (ii). By Lemma 3.3, there exists a set Ω_0 with $P(\Omega_0) = 1$ such that $\forall \omega \in \Omega_0$ and the mapping

$$T(\omega) : x \to \Psi(x, \omega)$$

is a linear continuous operator. Moreover, we can find a set Ω with $P(\Omega_1) = 1$ such that

$$\Phi(e_n, \omega) = \Psi(e_n, \omega) = T(\omega) e_n$$

for all $\omega \in \Omega_1$, all $n = 1, 2, \ldots$.

Consequently, for each $\omega \in \Omega_0 \cap \Omega_1$,

$$\sup_n \|\Phi(e_n, \omega)\| = \sup_n \|T(\omega) e_n\| \le \|T(\omega)\| < \infty,$$

i.e.,

$$\sup_n \|\Phi e_n\| < \infty \quad \text{a.s.}$$

and for each $a \in Y'$, $n = 1, 2, \ldots$,

$$\begin{aligned} \sum_{n=1}^{\infty} |\langle \Phi e_n(\omega), a \rangle|^{p'} &= \sum_{n=1}^{\infty} |\langle T(\omega) e_n, a \rangle|^{p'} \\ &= \sum_{n=1}^{\infty} |\langle e_n, T^*(\omega) a \rangle|^{p'} = \|T(\omega) a\|^{p'} < \infty, \end{aligned}$$

i.e.,

$$\sum_{n=1}^{\infty} |\langle \Phi e_n, a \rangle|^{p'} < \infty \quad \text{a.s.}$$

This completes the proof of the theorem. ■

Proof. (i) We have
$$\sum \|y_n(x, e_n)\|^p \leq \sup \|y_n\| \, \|x\|^p.$$

Hence, because Y is of stable type p, the series $\sum_{n=1}^{\infty} \gamma_n y_n(x, e_n)$ converges a.s. By the properties of stable measures on Banach spaces [4], there exists a constant $K > 0$ such that

$$P\Big\{\|\Phi x\| > \varepsilon\Big\} = P\Big\{\|\sum_{n=1}^{\infty} \gamma_n y_n(x, e_n)\| > \varepsilon\Big\}$$

$$\leq \frac{K}{\varepsilon^p} \sum_{n=1}^{\infty} \|(x, e_n) y_n\|^p \leq \frac{K \sup \|y_n\|^p \, \|x\|^p}{\varepsilon^p}.$$

This shows that Φ is a random operator.

(ii) If Φ is sample continuous, then by Theorem 3.2, we have
$$\sup_n \|y_n \gamma_n\| = \sup \|\Phi e_n\| < \infty \quad \text{a.s.} \tag{3.3}$$

By the Borel–Cantelli lemma, condition (3.3) holds if and only if
$$\sum_{n=1}^{\infty} P\Big\{\|y_n \gamma_n\| > t\Big\} < \infty$$

for some $t > 0$.

Since $P\Big\{\|y_n \gamma_n\| > t\Big\} = P\Big\{|\gamma_n| > \dfrac{t}{\|y_n\|}\Big\} \sim \dfrac{\|y_n\|^p}{t^p}$, it follows that $\sum_{n=1}^{\infty} \|y_n\|^p < \infty$.

In order to prove the converse, by Theorem 3.2, it suffices to show that
$$\sum_{n=1}^{\infty} \|\Phi e_n\|^{p'} < \infty \quad \text{a.s.}$$

Indeed, since $p' > p$, the space $\ell_{p'}$ is of stable type p. Therefore, condition (3.2) implies that the series $\sum_{n=1}^{\infty} \|y_n\| e_n \gamma_n$ converges a.s. in $\ell_{p'}$. Hence, we obtain

$$\sum_{n=1}^{\infty} \|\Phi e_n\|^{p'} = \sum_{n=1}^{\infty} \|y_n \gamma_n\|^{p'} < \infty \quad \text{a.s.}$$

as desired. ■

The following corollary shows that the necessary condition stated in Theorem 3.2(ii) is sufficient only if the case Y is finite-dimensional.

Corollary 3.5. *Let Y be a separable Banach space of stable type p $(1 < p < 2)$. Then the following assertions are equivalent:*

(i) *Y is finite-dimensional;*

(ii) *for every random operator Φ from ℓ_p into Y, the condition*
$$\sum_{n=1}^{\infty} |(\Phi e_n, a)|^{p'} < \infty \quad a.s.$$

for all $a \in Y'$ is sufficient for the sample continuity of Φ.

Proof. (i) → (ii) Let $Y = R^k$ and $h_1, h_2, \ldots, h_k$ be the standard basis in R^k. By the assumption, for each h_j,

$$\sum_{n=1}^{\infty} |(\Phi e_n, h_j)|^{p'} < \infty \quad \text{a.s.}$$

It is obvious that we can find a constant $C > 0$ such that

$$\|y\|^{p'} \leq C \sum_{j=1}^{k} |(y, h_j)|^{p'}$$

for all $y \in R^k$.

Consequently,

$$\sum_{n=1}^{\infty} \|\Phi e_n\|^{p'} \leq C \sum_{n=1}^{\infty} \sum_{j=1}^{k} |(\Phi e_n, h_j)|^{p'}$$

$$= C \sum_{j=1}^{k} \sum_{n=1}^{\infty} |(\Phi e_n, h_j)|^{p'} < \infty \quad \text{a.s.}$$

By Theorem 3.2, Φ is sample continuous.

(ii) → (i) Suppose Y is infinite-dimensional. By the weak Dvoretzsky–Rogers theorem, there exists a sequence $(y_n) \subset Y$ such that

$$\text{for all } a \in Y', \sum_{n=1}^{\infty} |(y_n, a)|^p < \infty \tag{3.4}$$

but

$$\sum_{n=1}^{\infty} \|y_n\|^p = \infty. \tag{3.5}$$

Taking into account that [2, p.32]

$$\|y_n\| = \sup_{\|a\| \leq 1} |(y_n, a)| \leq \sup_{\|a\| \leq 1} \left(\sum_{n=1}^{\infty} |(y_n, a)|^p \right)^{\frac{1}{p}} < \infty,$$

we see that (y_n) is a bounded sequence in Y. Define a random mapping Φ from ℓ_p into Y by

$$\Phi x = \sum_{n=1}^{\infty} \gamma_n y_n (x, e_n),$$

Φ is a random operator (Corollary 3.4). By the same argument as shown in the proof of Corollary 3.4, condition (3.4) implies that

$$\sum_{n=1}^{\infty} |(\Phi e_n, a)|^{p'} = \sum_{n=1}^{\infty} |\gamma_n (y_n, a)|^{p'} < \infty \quad \text{a.s.}$$

Using assumption (ii), we conclude that Φ is sample continuous. Thanks to Corollary 3.4, this fact implies that

$$\sum_{n=1}^{\infty} \|y_n\|^p < \infty$$

which contradicts (3.5). Therefore, Y is finite-dimensional. ■

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