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Left $\aleph$-Coherent Dimension of Rings*

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Abstract. Let $\aleph$ be an infinite cardinal number. A ring R is said to be left $\aleph$-coherent if every finitely generated left ideal of R is $\aleph$-finitely presented. In this paper, we define a dimension called the *left $\aleph$-coherent dimension* for a ring R. We show that a ring R is left $\aleph$-coherent if and only if the left $\aleph$-coherent dimension of R is equal to zero. Some characterizations of this dimension are given. We also show that if a ring S is an excellent extension of a ring R, then the left $\aleph$-coherent dimension of S is equal to that of R.

1. Introduction

A ring R is called left coherent if every finitely generated left ideal is finitely presented. It is well known that R is left coherent if and only if every finitely presented left R-module M has a finite 2-presentation in the sense of Bourbaki [3], that is, there exists an exact sequence $0 \longrightarrow K_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ such that F_0 and F_1 are finitely generated free left R-modules and K_2 is finitely generated (see, for example, [7] or [9]). In [13], the concept of left $\aleph$-coherent rings was introduced and investigated for any infinite cardinal number $\aleph$. A ring R is said to be left $\aleph$-coherent if every finitely generated left ideal is $\aleph$-finitely presented.

In this paper, using the concepts of finite n-presentations, as defined in [3], and $\aleph$-finite n-presentations, we define a dimension called the *left $\aleph$-coherent dimension* for a ring R. We show that a ring R is left $\aleph$-coherent if and only if the left $\aleph$-coherent dimension of R is equal to zero. Thus, the left $\aleph$-coherent dimension can be used to measure how far a ring is from being left $\aleph$-coherent. In Sec. 2, we give the definition and show some characterizations of this dimension. In Sec. 3, we show that if S is an excellent extension of R, then the left $\aleph$-coherent dimension of S is equal to that of R. Take the special infinite cardinal number $\aleph$, we obtain some results for left coherent dimension of R.

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Throughout this paper, R denotes an associative ring with identity. For any left R-module M, we denote by M^+ the character module $Hom_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$ of M. If N_R is a submodule of M_R, the notation $N_R|M_R$ means that N_R is a summand of M_R.

2. Definition and Characterizations

Let $\aleph$ be an infinite cardinal number and M a left R-module. Following Loustaunau [13], M is said to be $\aleph$-finitely generated, denoted by $\aleph$-fg, if every subset X of M, with $|X| < \aleph$, is contained in a finitely generated submodule of M. For example, every left R-module is $\aleph_0$-fg, and every finitely generated left R-module is $\aleph$-fg for all $\aleph > \aleph_0$. If $\aleph > |M|$ and M is $\aleph$-fg, then M is finitely generated.

Let M be a finitely generated left R-module. Then M is said to be $\aleph$-finitely presented, denoted by $\aleph$-fp, if there exists an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ with F free of finite rank and K $\aleph$-fg. A ring R is said to be left $\aleph$-coherent if every finitely generated left ideal is $\aleph$-fp. For example, every ring is left $\aleph_0$-coherent. If R is left coherent, then it is left $\aleph$-coherent for all infinite cardinal number $\aleph$. If $\aleph > \aleph'$, then every left $\aleph$-coherent ring is left $\aleph'$-coherent.

Let M be a left R-module. According to [3], we will say that M is n-finitely presented (n-$\aleph$-finitely presented), denoted by n-FP ($(n, \aleph)$-FP, respectively), if there exists an exact sequence:

$$0 \longrightarrow K_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where $F_0, \ldots, F_{n-1}$ are finitely generated free modules and K_n is finitely generated ($\aleph$-fg, respectively). In this case, we also say that M has a finite n-presentation ($\aleph$-finite n-presentation, respectively).

It is easy to see that a left R-module M is finitely presented if and only if M is 1-FP, and that M is $\aleph$-FP if and only if M is $(1, \aleph)$-FP. Clearly, $(n+1)$-FP implies $(n+1, \aleph)$-FP and $(n+1, \aleph)$-FP implies n-FP, but not conversely.

From Theorem 3.3 in [9], it is clear that R is left coherent if and only if every 1-FP left R-module is 2-FP. Generalizing this result, we give the following definition.

Definition 2.1. *Let R be a ring. We define the left $\aleph$-coherent dimension of R, denoted by* $\aleph$-lc.dim$R$*, as*

$$\inf\{n|\ every\ (n+1)\text{-}FP\ left\ R\text{-}module\ is\ (n+2, \aleph)\text{-}FP\}.$$

If no such n exists, we say that $\aleph$-lc.dim$R = \infty$.

Take $\aleph > |R|^{\aleph_0}$. For every $\aleph$-fg left R-module K with $K \leq F$ for some free left R-modules F of finite rank, we have $|K| \leq |F| \leq |R|^{\aleph_0} < \aleph$. This implies that K is finitely generated. Thus, when $\aleph > |R|^{\aleph_0}$, Definition 2.1 gives a concept of left coherent dimension, denoted by lc.dim$R$, that is,

$$\text{lc.dim}R = \inf\{n|\ \text{every } (n+1)\text{-}FP \text{ left } R\text{-module is } (n+2)\text{-}FP\}.$$

If no such n exists, we say that lc.dim$R = \infty$.

The following result appeared in [13].

Lemma 2.2. *Let $M_1, \dots, M_k$ be left R-modules. Then $\oplus_{i=1}^{k} M_i$ is ℵ-fg if and only if every M_i is ℵ-fg.*

Proposition 2.3. *R is left ℵ-coherent if and only if ℵ-lc.dim$R = 0$.*

Proof. Suppose R is left ℵ-coherent. Then, by Theorem 1.6 in [13], every finitely generated submodule of every ℵ-fp left R-module is ℵ-fp. Let M be a 1-FP left R-module. Then there exists an exact sequence:

$$0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

such that F_0 is free of finite rank and K_1 is finitely generated. Since F_0 is ℵ-FP, it follows that K_1 is ℵ-fp. Thus, there exists an exact sequence:

$$0 \longrightarrow K_2 \longrightarrow F_1 \longrightarrow K_1 \longrightarrow 0$$

such that F_1 is free of finite rank and K_2 is ℵ-fg. Now, it is clear that M is (2, ℵ)-FP.

Conversely, suppose ℵ-lc.dim$R = 0$. Then every 1-FP left R-module is (2, ℵ)-FP. Let L be a finitely generated left ideal of R. Then R/L is 1-FP. Thus, R/L is (2, ℵ)-FP. So there exists an exact sequence:

$$0 \longrightarrow K_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R/L \longrightarrow 0$$

such that F_0, F_1 are free of finite rank and K_2 is ℵ-fg. Take an exact sequence $0 \longrightarrow H \longrightarrow F \longrightarrow L \longrightarrow 0$ such that F is free of finite rank. Then by the Schanuel' lemma, $H \oplus F_1 \oplus R \simeq K_2 \oplus F \oplus F_0$. Now, by Lemma 2.2, it follows that H is ℵ-fg, which implies that L is ℵ-fp. Thus, R is left ℵ-coherent.

Because of this proposition, we may regard our left ℵ-coherent dimension as a measure of how far a ring R is from being left ℵ-coherent. Since a ring R is left coherent if and only if every 1-FP left R-module is 2-FP, it follows that when $\aleph > |R|^{\aleph_0}$, we may regard the left ℵ-coherent dimension as a measure of how far a ring R is from being left coherent. ∎

Lemma 2.4. *Let M be a left R-module and ℵ an infinite cardinal number. Then M is n-FP ($(n, \aleph)$-FP) if and only if there exists an exact sequence:*

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where $P_0, \dots, P_{n-1}$ are finitely generated projective left R-modules and K_n is finitely generated (ℵ-fg, respectively).

Proof. By induction for n, it follows from the Schanuel' lemma and standard techniques. ∎

In order to establish some characterization of left ℵ-coherent dimension, we need the following lemma.

Lemma 2.5. *If $lc.\dim R = m$, then for any $n \geq m$, every $(n+1)$-FP left R-module is $(n+2, \aleph)$-FP.*

Proof. Suppose left R-module M is $(m+2)$-FP. Then there exists an exact sequence:

$$0 \longrightarrow K_{m+2} \longrightarrow F_{m+1} \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where $F_{m+1}, F_m, \ldots, F_0$ are finitely generated free modules and K_{m+2} is finitely generated. Denote $K_1 = \mathrm{Ker}\,(F_0 \longrightarrow M)$. Then K_1 is $(m+1)$-FP. Since $\mathrm{lc.dim}R = m$, it follows that K_1 is $(m+2, \aleph)$-FP. Thus, there exists an exact sequence:

$$0 \longrightarrow H_{m+2} \longrightarrow G_{m+1} \longrightarrow G_m \longrightarrow \cdots \longrightarrow G_0 \longrightarrow K_1 \longrightarrow 0,$$

where $G_{m+1}, G_m, \ldots, G_0$ are finitely generated free modules and H_{m+2} is $\aleph$-finitely generated. Therefore, M is $(m+3, \aleph)$-FP. ■

Now, the result follows by induction.

Proposition 2.6. *Let R, S be rings. Then*

$$\aleph\text{-}lc.\dim(R \oplus S) = \sup(\aleph\text{-}lc.\dim R, \aleph\text{-}lc.\dim S).$$

Proof. Suppose $\aleph$-lc.dim$(R \oplus S) = k < \infty$. Let A be a $k+1$-FP left R-module. For any left R-module X, we can regard X as an $(R \oplus S)$-module by defining $(r, s)x = rx$, for $r \in R, s \in S$, and $x \in X$. Then $(1, 0)X \simeq X$ as an R-module. It is well known that $_RX$ is projective if and only if $_{(R\oplus S)}X$ is projective. Thus, by Lemma 2.4, it is easy to see that A is a $k+1$-FP left $(R \oplus S)$-module. Hence, $_{(R\oplus S)}A$ is $(k+2, \aleph)$-FP, that is, there exists an exact sequence:

$$0 \longrightarrow K_{m+2} \longrightarrow F_{m+1} \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_0 \longrightarrow A \longrightarrow 0,$$

where $F_{m+1}, F_m, \ldots, F_0$ are finitely generated free $(R \oplus S)$-modules and K_{m+2} is $\aleph$-fg. Thus, we have the following exact sequence:

$$0 \longrightarrow (1,0)K_{m+2} \longrightarrow (1,0)F_{m+1} \longrightarrow \cdots \longrightarrow (1,0)F_0 \longrightarrow A \longrightarrow 0,$$

where $(1,0)F_{m+1}, (1,0)F_m, \ldots, (1,0)F_0$ are finitely generated projective left R-modules and $(1,0)K_{m+2}$ is $\aleph$-fg. This means that $_RA$ is $(k+2, \aleph)$-FP. Thus, $\aleph$-lc.dim$R \leq k$. Similarly, we have $\aleph$-lc.dim$S \leq k$. Thus, sup($\aleph$-lc.dim$R$, $\aleph$-lc.dim$S$) $\leq$ $\aleph$-lc.dim$(R \oplus S)$. If $\aleph$-lc.dim$(R \oplus S) = \infty$, then clearly sup($\aleph$-lc.dim$R$, $\aleph$-lc.dim$S$) $\leq$ $\aleph$-lc.dim$(R \oplus S)$.

Let $\aleph$-lc.dim$R = m < \infty$ and $\aleph$-lc.dim$S = n < \infty$, $m \geq n$. Let M be an $m+1$-FP left $(R \oplus S)$-module. Then there exists an exact sequence:

$$0 \longrightarrow K_{m+1} \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where $F_m, \ldots, F_0$ are finitely generated free left $(R \oplus S)$-modules and K_{m+1} is finitely generated left $(R \oplus S)$-module. Thus, we have exact sequences:

$$0 \longrightarrow (1,0)K_{m+1} \longrightarrow (1,0)F_m \longrightarrow \cdots \longrightarrow (1,0)F_0 \longrightarrow (1,0)M \longrightarrow 0$$

and

$$0 \longrightarrow (0,1)K_{m+1} \longrightarrow (0,1)F_m \longrightarrow \cdots \longrightarrow (0,1)F_0 \longrightarrow (0,1)M \longrightarrow 0,$$

where $(1,0)F_m, \ldots, (1,0)F_0$ are finitely generated projective left R-modules, $(1,0)K_{m+1}$ is R-finitely generated, $(0,1)F_m, \ldots, (0,1)F_0$ are finitely generated projective left S-modules, and $(0,1)K_{m+1}$ is S-finitely generated. By Lemma 2.5, every $m+1$-FP left R-(left S-) module is $(m+2, \aleph)$-FP. Thus, $(1,0)M$, $(0,1)M$ is an $(m+2,\aleph)$-FP left R-, left S-, respectively, module. Now, by Lemma 2.4, it is easy to see that M is an $(m+2,\aleph)$-FP left $R \oplus S$-module. Therefore, $\aleph\text{-lc.dim}(R \oplus S) \leq m = \sup(\aleph\text{-lc.dim}R, \aleph\text{-lc.dim}S)$. If $\sup(\aleph\text{-lc.dim}R, \aleph\text{-lc.dim}S) = \infty$, then obviously

$$\aleph\text{-lc.dim}(R \oplus S) \leq \sup(\aleph\text{-lc.dim}R, \aleph\text{-lc.dim}S).$$

Lemma 2.7. *Let X be a right R-module and M a left R-module. Then the following conditions are equivalent:*

(1) $\mathrm{Ext}_R^n(X, M^+) = 0$;
(2) $\mathrm{Tor}_n^R(X, M) = 0$;
(3) $\mathrm{Ext}_R^n(M, X^+) = 0$.

Proof. By standard techniques, we have an isomorphism:

$$\mathrm{Ext}_R^n(X, \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})) \cong \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Tor}_n^R(X, M), \mathbf{Q}/\mathbf{Z}).$$

Thus, (1) $\Longleftrightarrow$ (2) follows. The proof of (2) $\Longleftrightarrow$ (3) is similar. ■

Suppose I is a set and $\{M_i | i \in I\}$ is a family of right R-modules. Let $x = (x_i)_{i \in I} \in \prod_{i \in I} M_i$. We define the support of x as $\mathrm{supp}\,(x) = \{i \in I | x_i \neq 0\}$. For an infinite cardinal number $\aleph$, define the $\aleph$-product of the M_i's as

$$\prod_{i \in I}^{\aleph} M_i = \left\{ x \in \prod_{i \in I} M_i \mid |\mathrm{supp}(x)| < \aleph \right\}.$$

Clearly, one may view the direct sum and the direct product of a family of modules as two special cases of the same object, namely, the $\aleph$-product of the family of modules. $\aleph$-products of some families of modules have been studied by Loustaunau [13], Dauns [5, 6], Teply [20, 21], and Oyonarte and Torrecillas [15]. The following lemma appeared in [13].

Lemma 2.8. *Let $\aleph$ be an infinite cardinal number and M a right R-module. Then the following statements are equivalent:*

(1) *M is $\aleph$-fp;*
(2) *if $\{L_i | i \in I\}$ is any family of left R-modules and if $\phi : M \otimes_R \left(\prod_{i \in I}^{\aleph} L_i\right) \longrightarrow \prod_{i \in I}^{\aleph} (M \otimes L_i)$ is defined via $\phi(m \otimes (x_i)_{i \in I}) = (m \otimes x_i)_{i \in I}$, then ϕ is an isomorphism;*
(3) *if I is any index set and if $\phi : M \otimes_R \left(\prod_{i \in I}^{\aleph} R\right) \longrightarrow \prod_{i \in I}^{\aleph} M$ is defined via $\phi(m \otimes (r_i)_{i \in I}) = (mr_i)_{i \in I}$, then ϕ is an isomorphism.*

We are now ready to give our characterizations of left $\aleph$-coherent dimension of rings.

Theorem 2.9. *Let $\aleph$ be an infinite cardinal number or an integer $m \geq 0$. The following conditions on a ring R are equivalent:*

(1) $\aleph$-lc.dim$R \leq m$;

(2) *if $(L_i)_{i\in I}$ is a family of flat right R-modules, then* $\text{Tor}^R_{m+1}(\prod^{\aleph}_{i\in I} L_i, M) = 0$ *for each $(m+1)$-FP left R-module M;*

(3) $\text{Tor}^R_{m+1}(\prod^{\aleph}_I R, M) = 0$ *for each $(m+1)$-FP left R-module M and for every set I;*

(4) *for each set I, if* $\text{Tor}^R_{m+1}(\prod^{\aleph}_I R, N) = 0$ *for all $(m+2, \aleph)$-FP left R-modules N, then* $\text{Tor}^R_{m+1}(\prod^{\aleph}_I R, M) = 0$ *for all $(m+1)$-FP left R-modules M;*

(5) *if X is a right R-module such that* $\text{Ext}^{m+1}_R(X, N^+) = 0$ *for all $(m+2, \aleph)$-FP left R-modules N, then* $\text{Ext}R^{m+1}(X, M^+) = 0$ *for all $(m+1)$-FP left R-modules M.*

Proof. (1) $\Longrightarrow$ (2) Suppose M is $(m+1)$-FP. Then M is $(m+2, \aleph)$-FP by Lemma 2.5. Thus, there exists an exact sequence:

$$0 \longrightarrow K_{m+2} \longrightarrow F_{m+1} \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where $F_{m+1}, F_m, \ldots, F_0$ are finitely generated free modules and K_{m+2} is $\aleph$-finitely generated. Denote $K_{m+1} = \text{Ker}(F_m \to F_{m-1})$ and $K_m = \text{Ker}(F_{m-1} \longrightarrow F_{m-2})$. If $m = 0$, then take $K_m = F_{m-1} = M$. Consider the following exact sequence:

$$0 \longrightarrow K_{m+1} \longrightarrow F_m \longrightarrow K_m \longrightarrow 0.$$

We obtain a commutative diagram

$$\begin{array}{ccc} \text{Tor}^R_1(\prod^{\aleph} L_i, K_m,) & \xrightarrow{f} & (\prod^{\aleph} L_i) \otimes K_{m+1} \\ \alpha\downarrow & & \beta\downarrow \\ 0 & \longrightarrow & \prod^{\aleph}(L_i \otimes K_{m+1}) \end{array}$$

where f is a monomorphism. When K_{m+2} is $\aleph$-fg, K_{m+1} is $\aleph$-fp and hence β is an isomorphism by Lemma 2.8. Thus, α is an isomorphism and hence $\text{Tor}^R_1(\prod^{\aleph} L_i, K_m) = 0$. Now, it is easy to see that $\text{Tor}^R_{m+1}(\prod^{\aleph} L_i, M) = 0$.

The implications (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) are clear.

(4) $\Longrightarrow$ (1) Let N be an $(m+2, \aleph)$-FP left R-module. By analogy with the proof of (1) $\Longrightarrow$ (2), we can obtain $\text{Tor}^R_{m+1}(\prod^{\aleph}_I R, N) = 0$ for every set I. Thus, by (4), it follows that $\text{Tor}^R_{m+1}(\prod^{\aleph}_I R, M) = 0$ for all $(m+1)$-FP left R-modules M. In order to complete the proof, it is enough to show that every $(m+1)$-FP left R-module is $(m+2, \aleph)$-FP.

Let M be an $(m+1)$-FP left R-module. If $m = 0$, then the result follows from Proposition 2.3 and from the fact that R is left $\aleph$-coherent if and only if every $\aleph$-product of any family of copies of R is flat as a right R-module (see [13]) since every left R-module is a direct limit of finitely presented left R-modules. Now, suppose $m \geq 1$. Then there exists an exact sequence:

$$0 \longrightarrow K_{m+1} \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where $F_m, \ldots, F_0$ are finitely generated free left R-modules and K_{m+1} is finitely generated. Denote $K_m = \mathrm{Ker}(F_{m-1} \longrightarrow F_{m-2})$ (if $m = 1$, then set $K_m = \mathrm{Ker}(F_0 \longrightarrow M)$). Then $\mathrm{Tor}_1^R(\prod^{\aleph} R, K_m) \cong \mathrm{Tor}_{m+1}^R(\prod^{\aleph} R, M) = 0$. Thus, we obtain a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & (\prod^{\aleph} R) \otimes K_{m+1} & \longrightarrow & (\prod^{\aleph} R) \otimes F_m & \longrightarrow & (\prod^{\aleph} R) \otimes K_m \\
\downarrow & & \gamma \downarrow & & \beta \downarrow & & \alpha \downarrow \\
0 & \longrightarrow & \prod^{\aleph} K_{m+1} & \longrightarrow & \prod^{\aleph} F_m & \longrightarrow & \prod^{\aleph} K_m
\end{array}$$

with exact rows, where β and α are isomorphisms by Lemma 2.8 since F_m and K_m are finitely presented. Thus, γ is an isomorphism and, hence K_{m+1} is ℵ-fp by Lemma 2.8. Now, the result follows.

(1) $\Longrightarrow$ (5) It follows from Lemma 2.5.

(5) $\Longrightarrow$ (3) Let N be an $(m+2, \aleph)$-FP left R-module. By analogy with the proof of (1) $\Longrightarrow$ (2), we can obtain

$$\mathrm{Tor}_{m+1}^R(\prod_I^{\aleph} R, N) = 0,$$

or every set I. Thus, $\mathrm{Ext}_R^{m+1}(\prod_I^{\aleph} R, N^+) = 0$ by Lemma 2.7. From (5), it follows that $\mathrm{Ext}_R^{m+1}(\prod_I^{\aleph} R, M^+) = 0$ for every $(m+1)$-FP left R-module M. Now, we have $\mathrm{Tor}_{m+1}^R(\prod_I^{\aleph} R, M) = 0$ for every set I and every $(m+1)$-FP left R-module M by Lemma 2.7. ■

We use w.gl.dim$R$ to denote the weak global dimension of ring R. As a direct consequence of Definition 2.1 and Theorem 2.9, we have

Corollary 2.10. ℵ-lc.dim$R \leq$ lc.dim$R \leq$ w.gl.dim$R$.

Example 1. We remark that lc.dim$R$ can be much smaller than w.gl.dim$R$. Take $R = F[x]$, the polynomial ring over a field F. Then lc.dim$R = 0$ but w.gl.dim$R \neq 0$.

We also remark that ℵ-lc.dim$R$ can be much smaller than lc.dim$R$. For example, let ω_1 be the first uncountable ordinal number and $R = \mathbf{Z}_2[x_\mu | \mu \leq \omega_1]$ the commutative polynomial ring with relations $x_\alpha = x_\alpha x_\beta$ for $\alpha < \beta \leq \omega_1$ and $x_\alpha^2 = x_\alpha$ for $\alpha < \omega_1$. By [13], R is $\aleph_1$-coherent but not coherent. Thus, $\aleph_1$-lc.dim$R = 0$ but lc.dim$R > 0$.

Example 2. Couchot [5] pointed out that there exists a commutative ring R such that w.gl.dim$R \leq 1$ but R is not semi-hereditary. It is well known that R is semi-hereditary if and only if w.gl.dim$R \leq 1$ and R is left coherent. Thus, there exists a commutative ring R such that w.gl.dim$R \leq 1$ but R is not coherent. For these rings, we have lc.dim$R =$ w.gl.dim$R = 1$ by Corollary 2.10. Take $\aleph > |R|^{\aleph_0}$. Then R is not left ℵ-coherent. Therefore, ℵ-lc.dim$R = 1$ by Corollary 2.10 and Proposition 2.3.

According to [7], a left (right) R-module X is called 2-FP-injective (2-FP-flat) if $\mathrm{Ext}_R^1(M, X) = 0$ ($\mathrm{Tor}_1^R(X, M) = 0$) for each 2-FP left R-module M. We will say that a left (right) R-module X is called $(2, \aleph)$-FP-injective ($(2, \aleph)$-FP-flat) if $\mathrm{Ext}_R^1(M, X) = 0$ ($\mathrm{Tor}_1^R(X, M) = 0$) for each $(2, \aleph)$-FP left R-module M. A left R-module X is called FP-injective if $\mathrm{Ext}_R^1(M, X) = 0$ for each finitely presented left R-module M. As an immediate consequence of Theorem 2.9 when $m = 0$, we have the following result, some parts of which are well known.

Corollary 2.11. *The following are equivalent for a ring R.*

(1) *R is left $\aleph$-coherent.*
(2) *The $\aleph$-product of any family of flat right R-modules is flat.*
(3) *For every set I, $\prod_I^{\aleph} R$ is a flat right R-module.*
(4) *For every set I, if the right R-module $\prod_I^{\aleph} R$ is $(2, \aleph)$-FP-flat, then it is flat.*
(5) *For every set I, if $(\prod_I^{\aleph} R)^+$ is $(2, \aleph)$-FP-injective, then it is FP-injective.*

Proof. The result follows from Theorem 1.6 in [13], Theorem 2.9, Proposition 2.3 and Lemma 2.7, bearing in mind that each left R-module is a direct limit of finitely presented modules, and that the functor $\mathrm{Tor}_1^R(X, -)$ preserves direct limits. ∎

For special $\aleph$ (for example, $\aleph > |R|^{\aleph_0}$), we have

Corollary 2.12. *(See [9]) For an integer $m \geq 0$, the following conditions on a ring R are equivalent.*

(1) $\mathrm{lc.dim} R \leq m$.
(2) *If $(L_i)_{i \in I}$ is a family of flat right R-modules, then $\mathrm{Tor}_{m+1}^R(\prod_{i \in I} L_i, M) = 0$ for each $(m+1)$-FP left R-module M.*
(3) $\mathrm{Tor}_{m+1}^R(\prod_I R, M) = 0$ *for each $(m+1)$-FP left R-module M and for every set I.*
(4) *For each set I, if $\mathrm{Tor}_{m+1}^R(\prod_I R, N) = 0$ for all $(m+2)$-FP left R-modules N, then $\mathrm{Tor}_{m+1}^R(\prod_I R, M) = 0$ for all $(m+1)$-FP left R-modules M.*
(5) *If X is a right R-module such that $\mathrm{Ext}_R^{m+1}(X, N^+) = 0$ for all $(m+2)$-FP left R-modules N, then $\mathrm{Ext}_R^{m+1}(X, M^+) = 0$ for all $(m+1)$-FP left R-modules M.*

3. Left $\aleph$-Coherent Dimension of Excellent Extensions

Suppose R is a subring of the ring S, and R and S have the same identity.

(1) The ring S is said to be an excellent extension of R if
   (i) S is a free normalizing extension of R with a basis that includes 1, that is, there is a finite set $\{a_1, \ldots, a_n\} \subseteq S$ such that $a_1 = 1$, $S = Ra_1 + \cdots + Ra_n$, $a_i R = Ra_i$ for all $i = 1, \ldots, n$ and S is free with basis $\{a_1, \ldots, a_n\}$ as both a right and left R-module, and
   (ii) S is right R-projective, that is, if N_S is a submodule of M_S, then $N_R | M_R$ implies $N_S | M_S$.

Excellent extensions were introduced by Passman [17], named by Bonami [2], and recently studied in [10–12, 16, 22]. Examples include finite matrix rings (see [17]), and crossed product $R * G$ where G is a finite group with $|G|^{-1} \in R$ (see [18]).

The following lemma is well known.

Lemma 3.1.

(1) *Let $R \longrightarrow S$ be a homomorphism of rings and let there be given modules U_R and ${}_SV$. Assume ${}_RS$ is a flat left R-module. Then there exists an isomorphism*

$$\mathrm{Tor}_n^R(U, {}_RV) \cong \mathrm{Tor}_n^S(U \otimes_R S, V).$$

(2) *Let R, S be rings and let there be given modules U_S, ${}_SV_R$, and ${}_RW$. Assume ${}_SV$ and V_R are flat. Then there exists an isomorphism*

$$\mathrm{Tor}_n^R(U \otimes_S V, W) \cong \mathrm{Tor}_n^S(U, V \otimes_R W).$$

Theorem 3.2. *Let S be an excellent extension of R. Then*

$$\aleph\text{-lc.dim}R = \aleph\text{-lc.dim}S.$$

Proof. Suppose $\aleph$-lc.dim$R = m < \infty$. Let M be an $(m+1)$-FP left S-module. There exists an exact sequence:

$$0 \longrightarrow K_{m+1} \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_0 \longrightarrow_S M \longrightarrow 0,$$

where $F_m, \ldots, F_0$ are finitely generated free left S-modules and K_{m+1} is finitely generated. Suppose K_{m+1} is generated by $x_1, \ldots, x_p$. Since S is an excellent extension of R, it follows that ${}_R(K_{m+1})$ is generated by $y_{ij} = a_i x_j$, $1 \le i \le n$, $1 \le j \le p$. It is easy to see that ${}_RF_m, \ldots, {}_RF_0$ are finitely generated free left R-modules. Thus, ${}_RM$ is an $(m+1)$-FP left R-module. Since $\aleph$-lc.dim$R = m$ and S_R is flat right R-module, it follows that

$$\mathrm{Tor}_{m+1}^R(\prod_I^{\aleph} S, M) = 0$$

for every set I by Theorem 2.9. Thus, by Lemma 3.1, we have

$$\mathrm{Tor}_{m+1}^S((\prod_I^{\aleph} S) \otimes_R S, M) = 0.$$

Define an S-homomorphism $f : (\prod_I^{\aleph} S) \otimes_R S \longrightarrow \prod_I^{\aleph} S$ via $f(x \otimes s) = xs$, where $s \in S$ and $x \in \prod_I^{\aleph} S$. Then we have an exact sequence of right S-modules

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow (\prod_I^{\aleph} S) \otimes_R S \longrightarrow \prod_I^{\aleph} S \longrightarrow 0.$$

Define an R-homomorphism $g : \prod_I^{\aleph} S \longrightarrow (\prod_I^{\aleph} S) \otimes_R S$ via $g(x) = x \otimes 1$. Then $fg = 1$. This means that the exact sequence $0 \longrightarrow (\mathrm{Ker}(f))_R \longrightarrow ((\prod_I^{\aleph} S) \otimes_R S)_R \longrightarrow (\prod_I^{\aleph} S)_R \longrightarrow 0$ splits. Thus, $(\mathrm{Ker}(f))_R | ((\prod_I^{\aleph} S) \otimes S)_R$, which implies that $(\mathrm{Ker}(f))_S | ((\prod_I^{\aleph} S) \otimes S)_S$ by the right R-projectivity of S. Hence, $\prod_I^{\aleph} S$ is isomorphic to a direct summand of $(\prod_I^{\aleph} S) \otimes S$. It now follows that

$$\mathrm{Tor}_{m+1}^S(\prod_I^{\aleph} S, M) = 0,$$

and so $\aleph$-lc.dim$S \le m$ by Theorem 2.9.

If $\aleph$-lc.dim$R = \infty$, then obviously $\aleph$-lc.dim$S \leq \aleph$-lc.dim$R$.

Conversely, suppose $\aleph$-lc.dim$S = m < \infty$. Let M be an $(m+1)$-FP left R-module. Then there exists an exact sequence:

$$0 \longrightarrow K_{m+1} \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where $F_m, \ldots, F_0$ are finitely generated free left R-modules and K_{m+1} is finitely generated. Since S_R is flat, we obtain an exact sequence:

$$0 \longrightarrow S \otimes K_{m+1} \longrightarrow S \otimes F_m \longrightarrow \cdots \longrightarrow S \otimes F_0 \longrightarrow S \otimes M \longrightarrow 0.$$

It is clear that $S \otimes F_m, \ldots, S \otimes F_0$ are finitely generated projective left S-modules and $S \otimes K_{m+1}$ is finitely generated. This means that $S \otimes M$ is an $(m+1)$-FP left S-module by Lemma 2.4. Thus, by Theorem 2.9, we have

$$\mathrm{Tor}^S_{m+1}(\prod_I^{\aleph} S, S \otimes_R M) = 0$$

for every set I, which implies that $\mathrm{Tor}^R_{m+1}((\prod_I^{\aleph} S)_R, M) = 0$ by Lemma 3.1. Now, it follows that

$$\mathrm{Tor}^R_{m+1}(\prod_I^{\aleph} R, M) = 0,$$

since $(\prod_I^{\aleph} S)_R \cong \prod_I^{\aleph}(\oplus_{i=1}^n R) \cong \oplus_{i=1}^n(\prod_I^{\aleph} R)$, and hence, $\aleph$-lc.dim$R \leq m =$ $\aleph$-lc.dim$S$ by Theorem 2.9.

If $\aleph$-lc.dim$S = \infty$, then clearly $\aleph$-lc.dim$R \leq \aleph$-lc.dim$S$. ∎

Corollary 3.3.

(1) $\aleph$-lc.dim$R = \aleph$-lc.dim$(M_n(R))$ *for every ring* R*, where* $M_n(R)$ *is the matrix ring over* R.

(2) *If* G *is a finite group such that* $|G|^{-1} \in R$*, then* $\aleph$-lc.dim$R = \aleph$-lc.dim$(R * G)$*, where* $R * G$ *is a crossed product of* R *with* G.

Let R be graded by a finite group G. The smash product, $R\#G^*$, is a free right and left R-module with basis $\{p_a | a \in G\}$ and multiplication determined by

$$(rp_a)(sp_b) = rs_{ab^{-1}}p_a,$$

where $s_{ab^{-1}}$ is the ab^{-1} component of s.

Corollary 3.4. *Let* R *be graded by a finite group* G*, and* $|G|^{-1} \in R$*. Then* $\aleph$-lc.dim$R =$ $\aleph$-lc.dim$(R\#G^*)$.

Proof. The group G acts as automorphisms on $R\#G^*$ with

$$a(rp_a) = rp_{ba},$$

so we may form the skew group ring $(R\#G^*) * G$. By [14] or [19], it follows that $(R\#G^*) * G \cong M_n(R)$, the ring of $n \times n$ matrices over R, where $n = |G|$. Thus, the result follows from Corollary 3.3. ∎

Now, we give a generalization of Corollary 3.3(1).

Proposition 3.5. *If R and S are two equivalent rings, then* ℵ-lc.dim$R$ = ℵ-lc.dim$S$.

Proof. Let $F : R \approx S$ define an equivalence and ℵ-lc.dim$S = m < \infty$. Assume M is an $(m + 1)$-FP left R-module. Then there exists an exact sequence:

$$0 \longrightarrow K_{m+1} \longrightarrow H_m \longrightarrow \cdots \longrightarrow H_0 \longrightarrow M \longrightarrow 0,$$

such that $H_m, \ldots, H_0$ are finitely generated free left R-modules and K_{m+1} is finitely generated. By [1, pp. 254, 258], we obtain an exact sequence of left S-modules:

$$0 \longrightarrow F(K_{m+1}) \longrightarrow F(H_m) \longrightarrow \ldots \longrightarrow F(H_0) \longrightarrow F(M) \longrightarrow 0,$$

where $F(H_m), \ldots, F(H_0)$ are finitely generated projective left S-modules and $F(K_{m+1})$ is finitely generated. Thus, by Lemma 2.4, $F(M)$ is $(m+1)$-FP. Since ℵ-lc.dim$S = m$, it follows that $F(M)$ is $(m+2, \aleph)$-FP. Similar arguments will show that M is $(m+2, \aleph)$-FP by [1, p. 256] and Lemma 2.4. Thus, ℵ-lc.dim$R \leq m$. If ℵ-lc.dim$S = \infty$, then obviously ℵ-lc.dim$R \leq$ ℵ-lc.dim$S$. Now, the result follows.

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