Vietnam Journal of Mathematics **27**:1 (1999) 85–87

Vietnam Journal
of
MATHEMATICS

Short Communication

Structure of Space of Germs of Holomorphic Functions and Hartogs Type Theorem

Nguyen Dinh Lan

Department of Mathematics, Teachers Training College of Ho Chi Minh City
280 An Duong Vuong, 5 Distr., Ho Chi Minh City, Vietnam

Received May 22, 1997
Revised October 6, 1997

1. Introduction

In this section we introduce some notions used throughout the paper.

1.1. Some Linear Topological Invariants

Let E be a Frechet space with a fundamental system of semi-norms $\{\|\cdot\|\}$. For a subset B of E, put $\|u\|_B^* = \sup\{|u(x)| : x \in B\}$ for $u \in E'$.

Write $\|\cdot\|_k^*$ for $B = U_k = \{x \in E : \|x\|_k < 1\}$.

Using this notation, we say that E has the property

(Ω) If $\forall p\ \exists q\ \forall k\ \exists C, d > 0$ $\ \|\cdot\|_q^{*\,1+d} \le C\|\cdot\|_k^*\|\cdot\|_p^{*\,d}$.

$(\overline{\Omega})$ If $\forall p, d > 0\ \exists q\ \forall k > 0\ \exists C > 0$ $\ \|\cdot\|_q^{*\,1+d} \le C\|\cdot\|_k^*\|\cdot\|_p^{*\,d}$.

$(\widetilde{\Omega})$ If $\forall p\ \exists q, d > 0\ \forall k\ \exists C > 0$ $\ \|\cdot\|_q^{*\,1+d} \le C\|\cdot\|_k^*\|\cdot\|_p^{*\,d}$.

(LB^∞) If $\forall \rho_n \uparrow \infty\ \forall p\ \exists q\ \forall k\ \exists n_k, C > 0\ \forall u \in E'\ \exists n_u \in [k; n_k]$

$$\|u\|_q^{*\,1+\rho_{n_u}} \le C\|u\|_{n_u}^*\|u\|_p^{*\,\rho_{n_u}}.$$

1.2. Holomorphic Functions

Let E, F be locally convex spaces and D an open set in E. A function $f : D \longrightarrow F$ is called holomorphic if it is continuous and $u \circ f$ is Gateaux holomorphic for each $u \in F'$. By $\mathcal{H}(D, F)$, we denote the space of F-valued holomorphic functions on D equipped with the compact-open topology. When F is omitted, it is understood to be the scalar field $\mathbf{C}$, e.g., $\mathcal{H}(D) = \mathcal{H}(D, \mathbf{C})$.

Finally, for each compact set K in E, by $\mathcal{H}(K)$ we denote the space of germs of holomorphic functions on K equipped with the inductive topology. In other words,

$$\mathcal{H}(K) := \liminf_{U \supset K} \mathcal{H}^{\infty}(U),$$

where U ranges over all neighborhoods of K and $\mathcal{H}^{\infty}(U)$ denotes the Banach space of bounded holomorphic functions on U.

Some authors are interested in the problem of the structure of spaces of germs of holomorphic functions on a compact set in a Frechet space.

- In [3], Meise and Vogt have proved that a nuclear Frechet space E has the property (Ω) if and only if $[\mathcal{H}(K)]'$, the strong dual of the space $\mathcal{H}(K)$ of germs of holomorphic functions, has the property (Ω).

  Later, in [4], when E is nuclear with a basis, they also showed that E has the property $(\widetilde{\Omega})$ if and only if $\mathcal{H}(\mathbf{D})$ has the property $(\widetilde{\Omega})$ for some open polydisc $\mathbf{D}$ in E'.
- Recently, Khue and Danh [2] have extended the results of type (Ω) to the Frechet case.

Our main aim is to consider this structure problem for spaces having $(\overline{\Omega}, \widetilde{\Omega})$. An application to separately holomorphic functions is given.

2. The Theorems

The main results of the paper are the following:

Theorem 1. *Let E be a nuclear Frechet space and B a balanced convex compact set on E. Assume E has the property $(\widetilde{\Omega}_B)$:*

$$(\widetilde{\Omega}_B) \ \forall p \, \exists q, d, C > 0 \ \ \| \cdot \|_q^{*1+d} \le C \| \cdot \|_B^* \, \| \cdot \|_p^{*d} .$$

Then $[\mathcal{H}(B)]' \in (LB^{\infty})$.

Theorem 2. *Let E be a nuclear Frechet space with approximation property and B a balanced convex compact set in E. Then the following assertions are equivalent:*
(a) *E has the property $(\widetilde{\Omega}_B)$;*
(b) *$[\mathcal{H}(B)]'$ has the property (LB^{∞});*
(c) *B is not pluripolar.*

Theorem 3. *Let E be a nuclear Frechet space with a basis and B a balanced compact set in E. Then E has the property $(\overline{\Omega}_B, \widetilde{\Omega}_B)$ if and only if $[\mathcal{H}(B)]'$ also has the property $(\overline{\Omega}, \widetilde{\Omega})$.*

Corollary 1. [4] *A nuclear Frechet space $\Lambda(A)$ has the property $(\widetilde{\Omega})$ if and only if there exists $a \in \Lambda(A)$ such that $\mathcal{H}(\mathbf{D}_a)$ has property $(\widetilde{\Omega})$.*

Corollary 2. [4] *Let $\Lambda_1(\alpha)$ be nuclear. For $a \in \Lambda_1(\alpha)$, $\mathcal{H}(\mathbf{D}_a) \in (\widetilde{\Omega})$ if and only if* $\liminf a_j^{\frac{1}{\alpha_j}} > 0$.

3. An Application

The well-known Hartogs theorem on holomorphicity of separately holomorphic functions in $\mathbf{C}^n$ was extended to the infinite-dimensional case by some authors. In particular, this theorem is true for the classes of Frechet spaces and dual Frechet–Schwartz space. However, the problem is complicated in the mixed case. By applying Theorem 3, we obtain a Hartogs type theorem in this case.

Theorem 4. *Let E and F be two Frechet spaces having $(\widetilde{\Omega})$ and (DN), respectively. Assume E is nuclear having a basis and F is Schwartz. Then every separately holomorphic function on $X \times Y$, an open set in $E \times F'$, is holomorphic.*

Acknowledgement. The author would like to thank Professor Nguyen Van Khue for helpful suggestions in the process of completing the paper.

References

1. S. Dineen, *Complex Analysis in Locally Convex Spaces*, North-Holland Mathematical Studies, Vol. 57, North-Holland 1981.
2. N.V. Khue and Ph.Th. Danh, Structure of spaces of germs of holomorphic functions *Publications Matematiques* (to appear).
3. R. Meise and D. Vogt, Structure of spaces of holomorphic functions on infinite-dimensional polydiscs, *Studia Math.* **75** (1983) 235–253.
4. R. Meise and D. Vogt, Holomorphic functions of uniformly bounded type on nuclear Frechet spaces, *Studia Math.* **83** (1986) 147–166.
5. V. Nguyen and A. Zeriahi, Familles de polynômes presque partout bornées, *Bull. Sci. Math.* **2** (1982) 81–91.
6. A. Pietsch, *Nuclear Locally Convex Spaces*, Ergeb. Math., Vol. 66, Springer-Verlag, 1972.