Vietnam Journal of Mathematics **27**:1 (1999) 53–59

Vietnam Journal
of
MATHEMATICS

On Zahariuta's Extremal Function for Harmonic Functions

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Received January 5, 1998
Revised September 1, 1998

Abstract. In the study of spaces of harmonic functions, Zahariuta [11] introduced an extremal function and the associated regularity (Lh-regularity). Our purpose is first to study the relationship between the Lh-regularity and the H-regularity [8] and to give some properties of this function.

1. Preliminary and Definitions

We denote $Ha(\Omega)$ the set of harmonic functions on the open set Ω.

Definition 1. [11] *Let Ω be an open in $\mathbf{R}^N$ and K a compact in Ω. We pose*

$$\chi_0(\Omega, K, x) := \lim_{\varepsilon\to 0} \chi_\varepsilon(\Omega, K, x)$$

where $\chi_\varepsilon(\Omega, K, x) := \overline{\lim_{y\to x}} \sup\{\alpha \ln|u(y)|,\ u \in Ha(\Omega),\ 0 < \alpha < \varepsilon,\ \|u\|_K \le 1,$ $\|u\|_\Omega^\alpha \le e\}$.

Definition 2. [11] *Let $(\Omega_s)_{s\in\mathbf{N}}$ be a sequence of open subsets of $\mathbf{R}^N$ such that $\Omega_s \subset\subset \Omega_{s+1}$, $\cup_{s\in\mathbf{N}}\Omega_s = \Omega$ and $(K_r)_{r\in\mathbf{N}}$ a sequence of compact subsets of Ω_1 such that $K_{r+1} \subset\subset Int(K_r)$, $\cap_{r\in\mathbf{N}} K_r = K$. We define the Zahariuta extremal function $h(\Omega, K, .)$ associated with (Ω, K) by the formula:*

$$h(\Omega, K, x) := \overline{\lim_{y\to x}} \lim_{r\to\infty} \chi(\Omega, K_r, x),\ x \in \Omega,$$

where

$$\chi(\Omega, K, x) := \lim_{s\to\infty} \chi_0(\Omega_s, K, x).$$

Remark. In the case $N = 2$, Zahariuta proved that $h(\Omega, K, .)$ is the usual harmonic measure $\omega(\Omega, K.)$.

It is easy to see that $\chi(\Omega, K, .) \geq \chi_0(\Omega, K, .)$ and $\chi(\Omega, K, .) \geq h(\Omega, K, .)$.

Definition 3. [11] *An open subset of $\Omega \subset \mathbf{R}^N$ is called Lh-regular if, for every K compact subset of Ω, we have $h(K^*, \Omega^*, x) = 0$ for all $x \in \Omega^*$ (where $E^* = \mathbf{R}^N \setminus E$).*

A compact set $K \subset \mathbf{R}^N$ is called Lh-regular if K^* is Lh-regular.

A compact set K is called Lh_0-regular if, for every open neighborhood Ω of K, we have the following identity $\chi_0(\Omega, K, .) \equiv 0$ on K.

Zahariuta [11] proved that if a compact is Lh_0-regular, then it is Lh-regular (but the inverse conclusion is not true). The next theorem shows the utility of the Lh-regularity.

Theorem 1. [11] *Let Ω be a connected open in $\mathbf{R}^N$.*

(i) *The space $Ha(\Omega)$ is isomorphic to $Ha(B(0, 1))$ if and only if Ω is Lh-regular.*
(ii) *If Ω is Lh-regular and $K \subset \Omega$ is a Lh-regular compact such that K^* is connected, then there exists a common basis for the spaces $Ha(\Omega)$ and $Ha(K)$.*

We refer to [11] for more details on the Lh-regularity.

Definition 4. *We say that a compact set $K \subset \mathbf{R}^N$ is H-regular at a if, for every $b > 1$ there exists $M > 0$ and an open neighborhood V of a such that*

$$\|p\|_V \leq Mb^n \|p\|_K, \ \forall p \in \mathcal{P}_n(\mathbf{R}^N), \forall n \in \mathbf{N}$$

where $\mathcal{P}_n(\mathbf{R}^N)$ denotes the vector space of all harmonic polynomials of degree $\leq n$. K is H-regular if, for every $a \in K$, K is H-regular at a.

The H-regularity takes a very important place in the harmonic polynomial approximation theory [6, 8–10].

Definition 5. [6] *We say that a compact $K \subset \mathbf{R}^N$ is of a positive capacity if $c_H(K) > 0$, where*

$$c_H(K) := \liminf_{k \to \infty} (c_k(K))^{\frac{1}{k}},$$

and $c_k(K) := \inf\{\|p\|_K / \|p\|_B, \ p \in PH_k(\mathbf{R}^N)\}$, $B := \{x \in \mathbf{R}^N, \ |x| \leq 1\}$.

Remark. A H-regular compact is of positive capacity.

2. Relation Between H-Regularity and Lh-Regularity

Subsequently, we need the following lemma:

Lemma 1. *Let K be a H-regular compact subset of $\mathbf{R}^N$ at a and Ω an open set containing $K \cup \{a\}$ and the bounded connected components of K^*. Let $(\lambda_k)_{k \in \mathbf{N}}$ be a sequence of positive reals and $(f_k)_{k \in \mathbf{N}}$ a sequence of harmonic functions on Ω. If the following conditions are fulfilled for a constant $M > 0$:*

(i) $|f_k(x)| \leq e^{M\lambda_k}$, $\forall k \in \mathbf{N}, \forall x \in \Omega$;
(ii) $|f_k(x)| \leq 1$, $\forall x \in K$, $\forall k \in \mathbf{N}$. *Then for all* $\varepsilon > 0$, *there exists a positive constant* C *and an open neighborhood* U *of* a *such that*

$$|f_k(x)| \leq Ce^{\varepsilon\lambda_k}, \ \forall k \in \mathbf{N}, \forall x \in U.$$

Proof. Since Ω contains the bounded connected components of K^*, we can choose a compact E in Ω such that $K \cup \{a\}$ is in the interior of E and that E^* is connected.

According to Bagby and Levenberg [2], there exists $r \in]0, 1[$ (depending on Ω and E) such that, for all harmonic function f on Ω,

$$\limsup_{n\to\infty}(\inf\{\|f - p\|_E, \ p \in \mathcal{P}_n(\mathbf{R}^N)\})^{\frac{1}{n}} \leq r.$$

Using this result, we can repeat the proof of Lemma 2.1 in [7] in the harmonic case. So there exists a positive constant c and a constant $r \in]0, 1[$ such that, for all harmonic function f,

$$\inf\{\|f - p\|_E, \ p \in \mathcal{P}_n(\mathbf{R}^N)\} \leq \|f\|_\Omega cr^n, \ \forall n \in \mathbf{N}.$$

Also, we can easily obtain the result, using the last inequality and repeating the proof of Theorem 2 in [5]. □

Theorem 2. *Let* K *be a compact in* $\mathbf{R}^N$ *of positive capacity such that* K^* *is connected. Then the following conditions are equivalent:*

(i) K *is* Lh_0*-regular;*
(ii) K *is* H*-regular.*

Proof. (ii) ⇒ (i) Let Ω be an open neighborhood of K and $a \in K$. According to Klimek [3, Lemma 2.3.2], there exists $(\varepsilon_n)_{n\in\mathbf{N}}$ a sequence of numbers such that $\varepsilon_n \to 0$ and

$$\chi_0(\Omega, K,) = \lim_{n\to\infty} \chi_{\varepsilon_n}(\Omega, K,).$$

Now, by the Choquet lemma (see, for example, [3, Lemma 2.3.4]), we have, for all $n \in \mathbf{N}$, the existence of $(\alpha_k^n)_{k\in\mathbf{N}}$, a sequence of positive numbers with $\alpha_k^n < \varepsilon_n$, $\forall k \in \mathbf{N}$, and $(f_k^n)_{k\in\mathbf{N}}$, a sequence of harmonic functions on Ω with $\alpha_k^n \ln \|f_k^n\|_K \leq 0$ and $\alpha_k^n \ln \|f_k^n\|_\Omega \leq 1$ such that

$$\chi_{\varepsilon_n}(\Omega, K, x) = \limsup_{y\to x} \sup_k \{\alpha_k^n \ln |f_k^n(y)|\}, \ \forall x \in \Omega.$$

Therefore, we obtain a countable family $(f_k^n, \alpha_k^n)_{k,n}$ that we can write $(F_l, \beta_l)_{l\in\mathbf{N}}$ and so we have

$$\beta_l \ln |F_l(x)| \leq 1, \ \forall l \in \mathbf{N}, \ \forall x \in \Omega$$
$$\beta_l \ln |F_l(x)| \leq 0, \ \forall x \in K.$$

It follows from the previous lemma that, for every $\eta > 0$, there exists a positive constant C and U, an open neighborhood of a such that

$$|F_l(x)| \leq Ce^{\eta\frac{1}{\beta_l}}, \ \forall l \in \mathbf{N}, \ \forall x \in K$$

and so, for all $\eta > 0$ and $(n, k) \in \mathbf{N}^2$,

$$\alpha_k^n \ln |f_k^n(x)| \leq \mathcal{C}\alpha_k^n + \eta, \ \forall x \in U.$$

If we take the supremum, we have

$$\sup_k \{\alpha_k^n \ln |f_k^n(x)|\} \leq \mathcal{C}\varepsilon_n + \eta, \ \forall x \in U.$$

Using this inequality, we easily obtain for all $n \in \mathbf{N}$ the following estimate:

$$\chi_{\varepsilon_n}(\Omega, K, a) \leq \mathcal{C}\varepsilon_n + \eta.$$

Now, let $n \to \infty$ and $\eta \to 0$ to obtain $\chi_0(\Omega, K, a) = 0$. Ω and a are arbitrary, so we obtain the result.

(i)$\Rightarrow$(ii) Let $x \in K$ and Ω be a bounded open neighborhood of K. We have $\chi_0(\Omega, K, x) = 0$. Then we obtain (from the definition of χ_0) $\forall\, \varepsilon > 0$, $\exists\, \theta_0 > 0$, $\forall\, \theta \in]0, \theta_0[$, $\exists\, r_0 > 0$, $\forall\, r \in]0, r_0[$:

$$|f(y)| \leq e^{\frac{\varepsilon}{\theta'}}, \ \forall y \in B(x, r), \ \forall \theta' \in]0, \theta[, \ \forall f \in Ha(\Omega), \tag{1}$$

with $\|f\|_K \leq 1$ and $\|f\|_\Omega \leq e^{\frac{1}{\theta'}}$.

Since $c_H(K) > 0$, there exists a constant $A > 1$ (see [9, 10]) such that

$$|p(y)| \leq A^n \|p\|_K (1 + |y|)^n, \ \forall y \in \mathbf{R}^N, \ \forall p \in \mathcal{P}_n(\mathbf{R}^N), \ \forall n \geq 0.$$

Let $p \in \mathcal{P}_n(\mathbf{R}^N)$ and denote $q(x) := p(x)/\|p\|_K$, then $\|q\|_K \leq 1$ and

$$\|q\|_\Omega \leq A^n (1 + \sup_{x \in \Omega} \|x\|)^n = C^n.$$

There exists n_0 such that, for every $n > n_0$, we have $\frac{1}{n \ln C} < \theta$. Also, if we take $\theta' = \frac{1}{n \ln C}$, then we have $\|q\|_\Omega \leq e^{n \ln C} = e^{\frac{1}{\theta'}}$. By (1), for all $\varepsilon > 0$, there exists n_0 and a neighborhood V of x such that

$$\|p\|_V \leq \|p\|_K e^{\varepsilon n \ln C}, \ \forall n \geq n_0, \ p \in \mathcal{P}_n(\mathbf{R}^N).$$

Using a compactness argument, the last inequality remains true if V is a neighborhood of K. Moreover, $c_H(K) > 0$, then there exists $M > 0$ such that, for every $n \leq n_0$, we have $\|p\|_V \leq M\|p\|_K$. Now, the H-regularity of K follows from the last two inequalities. ■

From the proof of Theorem 2, it is not difficult to prove the following corollary.

Corollary 1. *With the hypothesis of Theorem 2 on K, let Ω be an open neighborhood of K. Then for $a \in \Omega$, the following conditions are equivalent:*

(i) $\chi_0(\Omega, K, a) = 0$;
(ii) *K is H-regular at a.*

Remark. In Theorem 2, we cannot replace the Lh_0-regularity by Lh-regularity; indeed, if $K \subset \mathbf{R}^2$ is a piece of the unit circle, then K is Lh-regular and $c_H(K) > 0$ but K is not H-regular. We only have the following:

Corollary 2. *If $K \subset \mathbf{R}^N$ is a H-regular compact such that K^* is connected, then K is Lh-regular.*

3. Some Properties of the Function χ_0

Proposition 1. *Let $K \subset \mathbf{R}^N$ be a H-regular compact such that K^* is connected. Then for every open neighborhood Ω of K, we have*

$$K = \{x \in \Omega,\ \chi_0(\Omega, K, x) = 0\}.$$

Proof. Let Ω be an open neighborhood of K. From Theorem 2, it follows that $K \subset \{x \in \Omega,\ \chi_0(\Omega, K, x) = 0\} = F$.

Assume there is a point $a \in F \backslash K$ and denote $E = K \cup \{a\}$. We know that, for every open neighborhood U of E, there exists a positive constant $c = c(U, E)$ and a constant $r = r(U, E) \in]0, 1[$ such that, for all $f \in Ha(U)$,

$$\inf\{\|f - p\|_E,\ p \in \mathcal{P}_n(\mathbf{R}^N)\} \le \|f\|_U c r^n,\ \ \forall n \in \mathbf{N}.$$

Let $\alpha \in]0, 1 - r[$. For $n \in \mathbf{N}$, the functions $f_n \equiv \frac{1}{2}$ on K and $f_n(a) = \left(\frac{1}{r+\alpha}\right)^n$ are harmonic on a small neighborood (not connected) of E. It follows from the remark above that, for all $n \in N$, there exists $q_n \in \mathcal{P}_n(\mathbf{R}^N)$ such that

$$\|f_n - q_n\|_E \le \|f\|_U c r^n \le c\left(\frac{r}{r+\alpha}\right)^n.$$

Consequently, there exists $n_0 \in \mathbf{N}$, such that, for all $n \ge n_0$,

$$\left(\frac{1}{r+\alpha}\right)^n (1 - cr^n) \le |q_n(a)| \text{ and } \|q_n\|_K \le 1,$$

so

$$0 < \ln\frac{1}{r+\alpha} \le \phi_K(a) := \overline{\lim_{\xi \to a}}\ \overline{\lim_{n \to \infty}} \sup\{\frac{1}{n} \ln|p(\xi)|,\ p \in \mathcal{P}_n(\mathbf{R}^N),\ \|p\|_K \le 1\}.$$

From the results of Siciak [10], it follows that K is not H-regular at a and then $\chi_0(\Omega, K, a) \ne 0$ which contradicts $a \in F$. □

Proposition 2. *Let Ω be an open subset in $\mathbf{R}^N$ and $E \subset \Omega$ a compact. Then for any $\alpha \in]0, 1[$, $\varepsilon \in]0, 1 - \alpha[$ and K compact subset of Ω_α, there exists a positive constant $c = c(\alpha, \varepsilon, K, \Omega)$ such that, for every harmonic function f on Ω, we have*

$$\|f\|_K \le c\|f\|_E^{1-\alpha-\varepsilon}\|f\|_\Omega^{\alpha+\varepsilon},$$

where $\Omega_\alpha := \{x \in \Omega,\ \chi_0(\Omega, E, x) < \alpha\}$.

Proof. Let K be a compact subset in Ω_α. By Dini's theorem for all $\varepsilon \in]0, 1-\alpha[$, there exists $\theta_0 > 0$ such that

$$\chi_\theta(\Omega, E, x) \leq \alpha + \varepsilon, \ \forall x \in K, \ \forall \theta < \theta_0.$$

Let $f \in Ha(\Omega)$. If $(\ln \|f\|_\Omega - \ln \|f\|_E)^{-1} < \theta_0$, then

$$\frac{\ln |f(x)| - \ln \|f\|_E}{\ln \|f\|_\Omega - \ln \|f\|_E} \leq \alpha + \varepsilon, \ \forall x \in K.$$

Otherwise, $(\|f\|_\Omega)/(\|f\|_E) < e^{\frac{1}{\theta_0}}$, and since $\|f\|_\Omega / \|f\|_E \geq 1$, we have

$$\frac{|f(x)|}{\|f\|_E} \leq e^{\frac{1}{\theta_0}} \left(\frac{\|f\|_\Omega}{\|f\|_E} \right)^{\alpha+\varepsilon}, \ \forall x \in K.$$

By the arbitrary character of K in Ω_α, we obtain the result. ■

Remark. It is clear that if $\chi_0(\Omega, E, .) \not\equiv 1$, then $= E$ is determining for the harmonic functions on Ω. It is impossible to replace the set Ω_α by $\{x \in \Omega, \ h(\Omega, E, x) \leq \alpha\}$ because there exists compacts E such that $h(\Omega, E, .) \not\equiv 1$ and E is not determining for the harmonic functions on Ω.

We denote $B_r = \{x \in \mathbf{R}^N, \ \|x\| \leq r\}$.

Proposition 3.

$$\chi_0(B_R, \overline{B}_r, x) \leq \begin{cases} \dfrac{\ln |x| - \ln r}{\ln R - \ln r} & \textit{if} \ \ |x| > r, \\ 0 & \textit{if} \ \ |x| \leq r. \end{cases}$$

Proof. Let $f \in Ha(B_R)$. There exists a holomorphic function $\tilde{f}$ on BL_R such that $\tilde{f}_{|B_R} \equiv f$ (see [1]), where $BL_R = BL(0, R)$ is the Lie ball in $\mathbf{C}^n$ and where the Lie norm is $L(z) = \sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |\sum_{j=1}^n z_j^2|^2}}$. We know that

$$\omega(BL_R, \overline{BL_r}, z) = \begin{cases} \dfrac{\ln L(z) - \ln r}{\ln R - \ln r} & \text{si} \ \ L(z) > r, \\ 0 & \text{si} \ \ L(z) \leq r, \end{cases}$$

where $\omega(BL_R, \overline{BL_r}, .)$ denotes the extremal plurisubharmonic function associated with $(BL_R, \overline{BL_r})$. Therefore, for all $R > t > r$ and $\varepsilon > 0$,

$$\|\tilde{f}\|_{BL_t} \leq \|\tilde{f}\|_{BL_{r(1-\varepsilon)}}^{1-\alpha_\varepsilon} \|\tilde{f}\|_{BL_{R(1-\varepsilon)}}^{\alpha_\varepsilon},$$

where $\alpha_\varepsilon = (\ln t - \ln r(1-\varepsilon)) / (\ln R - \ln r)$. By [1], there exist two constants $c_1 = c_1(\varepsilon, r)$ and $c_2 = c_2(\varepsilon, R)$ such that

$$\|\tilde{f}\|_{BL_{r(1-\varepsilon)}} \leq (1 + c_1)\|f\|_{B_r} \ \text{and} \ \|\tilde{f}\|_{BL_{R(1-\varepsilon)}} \leq (1 + c_2)\|f\|_{B_R}.$$

Then there exists a constant $c(\varepsilon, R, r) > 0$ such that

$$\|f\|_{BL_t} \leq C\|f\|_{B_r}^{1-\alpha_\varepsilon}\|f\|_{B_R}^{\alpha_\varepsilon}.$$

Now, let $f \in Ha(B_R)$, $\theta > 0$ such that $\ln\|f\|_{B_r} \leq 0$ and $\theta \ln\|f\|_{B_R} \leq 1$. For all $\varepsilon > 0$, there exists (using the last estimate) $c = c(\varepsilon, R, r)$ such that

$$\theta \ln|f(x)| \leq \theta \ln c + \alpha_\varepsilon, \forall x \in B_t.$$

So for every $x \in B_t \backslash B_r$,

$$\chi_0(B_R, B_r, x) \leq \alpha_\varepsilon, \forall \varepsilon.$$

We may now let $\varepsilon \to 0$ to obtain the required inequality. ∎

Using the previous proposition, we can say that Proposition 2 improves the "three-balls theorem" for harmonic functions (see [4] for more information about the "three-balls theorem").

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