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An Eigenvalue Problem with Singularity for Fourth Order Ordinary Differential Equation*

Ta Van Dinh

Department of Applied Mathematics
Hanoi University of Technology, Hanoi, Vietnam

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Abstract. We are dealing with some eigenvalue problems with singularity for fourth order ordinary differential equation. The existence of a countable set of real eigenvalues and eigenfunctions is established. At the same time the high smoothness of the first eigenfunction is investigated.

1. Introduction

The eigenvalue problems in which the parameter is present in the boundary condition(s) are called in [1] the eigenvalue problem with singularity and have been considered there in some particular cases for second order ordinary differential equations. In this paper we are dealing with such a problem for fourth order ordinary differential equation. We shall prove the existence of a countable set of eigenvalues and eigenfunctions and the high smoothness of the first eigenfunction.

2. The Problem in Strong Form

Let $p(x),\ q(x),\ r(x)$ be given functions on [0, 1] and s a given constant satisfying

$$0 < c_0 \leq p(x) \leq c_1\,, \quad q(x) \geq 0\,, \quad 0 < c_2 \leq r(x) \leq c_3\,, \quad s > 0\,, \tag{1}$$

where $c_0,\ c_1,\ c_2,\ c_3$ are positive constants.

In addition, assume

$$p(x),\ p'(x),\ p''(x),\ q(x),\ r(x) \in C^{\mu}[0,1]\,, \quad \mu = \text{integer} \geq 0\,. \tag{2}$$

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The eigenvalue problem in the strong form is stated as: *Find a scalar λ called eigenvalue and a function $w(x) \in C^4[0,1]$ not identically equal to zero called eigenfunction satisfying*

$$Lw := (pw'')'' + qw = \lambda rw\,, \quad 0 < x < 1\,, \tag{3}$$

$$lw := (pw'')'(0) = \lambda sw(0)\,, \tag{4}$$

$$w''(0) = 0\,, \;\; w(1) = 0\,, \;\; w'(1) = 0\,. \tag{5}$$

This problem arises from mechanics [1, p. 447]. Note that the parameter λ is also present in the boundary condition and that is why the problem is called in [1, p. 446, 447] eigenvalue problem with singularity.

3. The Problem in the Weak Form

Instead of (2), assume

$$p(x),\; q(x),\; r(x) \in C[0,1]\,. \tag{6}$$

Denote by $H^k(0,1)$ the Sobolev spaces $W_2^{(k)}(0,1)$ [3]. According to the theorem on the equivalence of norms in Sobolev spaces [3, p. 360, 361], we have

Lemma 1. *In $H^2(0,1)$, the following norms*

$$\|u\|_{H^2(0,1)} := \|u''\|_{L_2(0,1)} + \|u'\|_{L_2(0,1)} + \|u\|_{L_2(0,1)}\,,$$

$$\|u\|_1 := \|u''\|_{L_2(0,1)} + |u(0)|$$

$$\|u\|_2 := \{\|u''\|^2_{L_2(0,1)} + \|u'\|^2_{L_2(0,1)} + \|u\|^2_{L_2(0,1)}\}^{\frac{1}{2}}$$

are equivalent.

Let

$$V := \{v | v \in H^2(0,1),\; v(1) = 0,\; v'(1) = 0\}.$$

Let H be the closure of the space $C[0,1]$ with respect to the norm

$$\|u\|_H := (u,u)_H^{\frac{1}{2}}\,, \;\; (u,v)_H := \int_0^1 r(x)u(x)v(x)dx + su(0)v(0)\,.$$

It is obvious that V and H are Hilbert spaces (on R).

Let $u \in V \subset H^2(0,1)$. Then according to Lemma 1, $u \in L_2(0,1)$ and $u(0) < \infty$. So $V \subset H$.

Now, let $u \in V \subset H^2(0,1)$. Then we have

$$\|u\|_{L_2(0,1)} \leq \|u\|_{H^2(0,1)}\,, \quad |u(0)| \leq \|u\|_1 < c_1\|u\|_{H^2(0,1)}, \;\; c_1 = \text{const} > 0\,.$$

Then by addition we see that the embedding $V \to H$ is continuous.

Now, by the embedding theorem [2], the embedding $V \to L_2(0,1)$ is compact. Besides, if $\{u\}$ is a bounded set in V, then the set $\{u(0)\}$ is bounded. Hence, the embedding $V \mapsto H$ is compact. Consider in V the bilinear form

$$a(u,v) = \int_0^1 [p(x)u''(x)v''(x) + q(x)u(x)v(x)]dx\,, \quad u,v \in V.$$

We see that $a(u, v)$ can be considered as an inner product in V denoted by V_a. It yields another norm $\|u\|_a := \{a(u, u)\}^{1/2}$ equivalent to the norm in V.

The weak eigenvalue problem takes the following form: Find $u \in V$ such that

$$\int_0^1 [p(x)u''(x)v''(x) + q(x)u(x)v(x)]dx = \lambda\Big[\int_0^1 r(x)u(x)v(x)dx + su(0)v(0)\Big], \quad \forall v \in V. \tag{7}$$

It is obvious that $a(u, v)$ is symmetric. Moreover,

$$|a(u, v)| \le c_5\{\int_0^1 (|u''|.|v''| + |u|.|v|)dx\} \le c_5\{\|u\|_V\|v\|_V\}, \quad c_5 = \text{constant} > 0.$$

Then $a(u, v)$ is continuous on V.

Let $u \in V$. Then $u \in H^2(0, 1)$ and

$$a(u, u) \ge \int_0^1 p(x)[u''(t)]^2 dt \ge c_0\{\|u''\|^2_{L_2(0,1)}\}.$$

According to Lemma 1

$$|u(0)|^2 \le \|u\|_1^2 \le c_6 a(u, u), \quad c_6 = \text{constant} > 0\,.$$

So

$$a(u, u) \ge c_7\{\|u''\|^2_{L_2(0,1)} + |u(0)|^2\} \ge c_8\|u\|^2_{H^2(0,1)}, \quad c_7, c_8 = \text{constant} > 0\,.$$

Therefore, $a(.,.)$ is V-elliptic.

So all assumptions of Lemma 1 in [4] are verified.

Then under assumptions (1) and (6), we have

Theorem 1.

(1) *Problem (7) has countably many eigenvalues which are real with no finite limit points and can be arranged as*

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \cdots \lambda_m \to +\infty.$$

(2) *The corresponding eigenfunctions* $\{u_m(x) \in V \subset H^2(0, 1)\}$ *are orthonormal in* H *and the functions* $\{\lambda_m^{-1/2} u_m\}$ *form an orthonormal base in* V_a.

Now, we turn to the strong problem.

4. Solution of the Strong Problem

Under assumptions (1) and (2), we have

Theorem 2.

(1) *Each solution* $(\lambda, w(x))$ *of the strong problem (3)–(5) is a solution of the weak problem (7).*

(2) *To the solution* $(\lambda_1, u_1(x))$ *of the weak problem (7), there corresponds a solution* $(\lambda_1, v_1(x))$ *of the strong problem (3)–(5) such that*

$$v_1 \in C^{\mu+4}[0,1],\ \|v_1 - u_1\|_{H^2(0,1)} = 0,\ \|v_1\|_{H^{\mu+4}(0,1)} \le c_9 \lambda_1^{[(\mu+3)/4]+1}, \quad (8)$$

where c_9 *is a positive constant independent upon* u_1 *and* v_1.

For the proof, we consider the following boundary problem:

$$Lz := (p(x)z''(x))'' + q(x)z(x) = f(x),\ 0 < x < 1, \quad (9)$$

$$lz := (p(0)z''(0))' = g, \quad (10)$$

$$z''(0) = 0, \quad z(1) = 0,\ z'(1) = 0, \quad (11)$$

where $p(x),\ p'(x),\ p''(x), q(x)$ satisfy (1) and (2), and

$$f(x) \in C^{\nu}[0,1], \quad 0 \le \nu = \text{ integer } \le \mu, \quad g \in R. \quad (12)$$

Then we have

Lemma 2. *The boundary problem (9)–(12) has a unique solution*

$$z \in C^{\nu+4}[0,1], \quad (13)$$

such that

$$\|z\|_{H^p(0,1)} \le c_{10}\{\|f\|_{L_2(0,1)} + |g|\}, \quad p = 1,2,3,4, \quad (14)$$

$$\|z\|_{H^{p+4}(0,1)} \le c_{11}\{\|f\|_{L_2(0,1)} + |g| + \sum_{k=1}^{p} \|f^{(k)}\|_{L_2(0,1)}\}, \quad p = 1,2,\dots,\nu, \quad (15)$$

where c_{10} *and* c_{11} *are positive constants independent upon* z.

Proof. First, by taking the inner product in $L_2(0,1)$ of Lz by z, we can prove that the corresponding homogeneous problem has only a trivial solution. From that, the existence and uniqueness of Green function $G(x,\xi)$ and of the solution $z(x)$ satisfying (9)–(11) (see [1]) follow. Then we have

$$z(x) = \int_0^1 G(x,\xi) f(\xi) d\xi + G(x,0)g.$$

Now, to prove that the solution $z(x)$ satisfies (13), we must prove it analogously to [2, p. 74,75]. After that, taking the inner product in $L_2(0,1)$ of (9) by $z(x)$ and taking into account boundary conditions (10) and (11), we have

$$a(z,z) = (F,z)_H,$$

where

$$F = F(x) = \begin{cases} f(x)/r(x), & \text{if } 0 < x < 1, \\ g/s, & \text{if } x = 0. \end{cases}$$

Then we have (13) for $p = 1,2$. Since z verifies (14), we can differentiate (9) successively, and with the help of (14), for $p = 1,2$, we obtain (14) for $p = 3,4$ and (15) step by step. ■

Proof of Theorem 2. The first part follows from the inner product in $L_2(0,1)$ of (3) by $v(x) \in V$ and boundary conditions (4) and (5).

For the second part, let $(\lambda_1, u_1(x))$ be the solution of the weak problem (6). Then

$$u_1 \in V \subset H^2(0,1), \quad a(u_1,u_1) = \lambda_1\,, \quad \|u_1\|_H = 1. \tag{16}$$

Thus, $u_1 \in H^2(0,1)$. So by the imbedding theorem [2, p. 372], $u_1(0)$ is well defined and u_1 is equal almost everywhere (a.e.) on $[0,1]$ to a function $\tilde{u}_1 \in C^1[0,1]$.

Consider the auxiliary boundary value problem:

$$Lw_1 := (p(x)w_1''(x))'' + q(x)w_1(x) = \lambda r(x)\tilde{u}_1(x),\ 0 < x < 1, \tag{17}$$

$$lw_1 := (p(0)w_1''(0))' = \lambda s u_1(0), \tag{18}$$

$$w_1''(0) = 0, \quad w_1(1) = 0,\ w_1'(1) = 0. \tag{19}$$

According to Lemma 2, this problem has a unique solution:

$$w_1 \in C^{\min\{\mu,1\}+4}[0,1] = C^{\min\{\mu+4,5\}}, \tag{20}$$

such that

$$\begin{aligned}\|w_1\|_{H^p(0,1)} &\le c_{12}\lambda_1\{\|\tilde{u}_1\|_{L_2(0,1)} + |u_1(0)|\} = c_{12}\lambda\{\|u\|_{L_2(0,1)} + |u(0)|\}\\ &= c_{12}\lambda_1\|u\|_H = c_{12}\lambda_1,\ p = 1,2,\ldots,\min\{\mu+4,5\},\ c_{12} = \text{constant} > 0.\end{aligned} \tag{21}$$

We shall prove

$$\|w_1 - u_1\|_{V=H^2(0,1)} = 0. \tag{22}$$

By taking the inner product in $L_2(0,1)$ of (17) by $v(x) \in V \subset H^2(0,1)$, we have after integration by parts

$$\begin{aligned}\int_0^1 [p(x)w_1''(x)v''(x) + q(x)w_1(x)v(x)]dx - (p(0)w_1''(0))'v(0) + p(0)w_1''(0)v'(0)\\ = \lambda_1\int_0^1 r(x)\tilde{u}_1(x)v(x)dx.\end{aligned} \tag{23}$$

Then taking into account boundary conditions (18) and (19), Eq. (23) yields

$$\int_0^1 [p(x)w_1''(x)v''(x) + q(x)w_1(x)v(x)]dx = \lambda_1[\int_0^1 r(x)\tilde{u}_1(x)v(x)dx + su_1(0)v(0)]. \tag{24}$$

From (7) and (24), where $\tilde{u}_1 = u_1$ (a.e.) on $[0,1]$, we obtain

$$a(u_1,v) = a(w_1,v), \quad \forall v \in V.$$

Therefore, we have $\|w_1 - u_1\|_a = 0$ and hence (22) follows.

Thus, $u_1 = w$ (a.e.) and $u_1 = \tilde{u}_1$ (a.e.) on $[0,1]$. Then $\tilde{u}_1 = w_1$ (a.e.) on $[0,1]$. So $\tilde{u}_1 = w$ everywhere on $[0,1]$ because they are both continuous on $[0,1]$. So the problem (17)–(19) coincides with the problem (3)–(5). Therefore, λ_1 and w_1 satisfy the problem (3)–(5), and by (20)–(22), property (8) is verified for $\mu = 0$.

Since w_1 satisfies (3), (5) and (20), we successively have

$$w_1 \in C^{\min\{\mu+4,5\}} \Rightarrow w_1 \in C^{\min\{\mu+4,9\}} \Rightarrow \cdots \Rightarrow w_1 \in C^{\mu+4}$$

with the help of Lemma 2.

The last inequality of (8) is proved by applying Lemma 2 to w_1 step by step. So Theorem 2 is proved with $v_1 = w_1$. ■

Remark 1. By taking into account the previous results, we can approximate the first eigenvalue λ_1 and the first eigenfunction $u_1(x)$ by finite element method.

5. Another Problem

Let there be given a function $p(x) \geq c_0 = \text{constant} > 0$ and a constant $c \geq 0$.

The corresponding strong eigenvalue problem is [1, p. 447]: Find u such that

$$(pu'')'' = -\lambda u'' , \ 0 < x < 1 , \tag{24}$$

$$u''(0) = 0, \ (pu'')'(0) + cu(0) = -\lambda u'(0), \ c \geq 0, \ \ u(1) = u'(1) = 0 . \tag{25}$$

Let

$$V = \{v | v \in H^2(0,1) , \ v(1) = v'(1) = 0\} ,$$
$$H = \{v | v \in H^1(0,1) , \ v(1) = 0\} ,$$
$$(u, v)_H = \int_0^1 u'v' dx \ , \ \ \|u\|_H = (u, u)_H{}^{\frac{1}{2}} ,$$

and

$$a(u, v) = \int_0^1 pu''v'' dx + cu(0)v(0), \ \ u, v \in V .$$

The weak eigenvalue problem can be stated as follows: Find $u \in V$ such that

$$\int_0^1 pu''v'' dx + cu(0)v(0) = \lambda \int_0^1 u'v' dx \, , \quad v \in V . \tag{26}$$

For problems (24)–(26), there are results analogous to Theorems 1 and 2.

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