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Some Extensions of Berliocchi–Lasry Theorem and Extremum Principles for Classes of Mathematical Programming Problems*

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Abstract. This paper gives some extended versions of a theorem of H. Berliocchi and J.M. Lasry in [3]. It is proved that these versions are equivalent to several extremum principles for classes of mathematical programming problems. An example is given in the last section of the paper to show the ability of using the above results to approach the existence of nonconvex variational and optimal control problems.

1. Introduction

A well-known characterization of the extreme points of an affinely constrained subset of a locally convex Hausdorff space had been invented by Berliocchi and Lasry in [3]. Later, Winkler [14] gave a generalization for extremes points of moment sets oriented towards optimizing affine functionals. Based on these results, the first author established a so-called "fundamental extremum principle", and with this as one of main tools, the first author obtained in [1] new results on the existence for optimal control problems without convexity assumptions. These results subsumed the well-known ones in the literature such as those of Cellina and Colombo [5], Raymond [11], Cesari [6], etc. In this paper, other versions of the extension of the Berliocchi–Lasry theorem are given. Moreover, in Sec. 3, we will prove that these versions are equivalent to several principles for classes of mathematical programming problems. The last section is left for an application of one

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of the mentioned results in deriving the existence of a simple class of nonconvex optimal control problems.

Let X be a locally convex Hausdorff space and A a subset of X (not necessarily convex). A subset E of A is called an *extremal subset* of A if $x_1, x_2 \in A$ and $\lambda x_1 + (1-\lambda)x_2 \in E$ for some $\lambda \in (0, 1)$ entails $x_1, x_2 \in E$. If an extremal subset E is a singleton subset of A, say $E = \{x_0\}$, then x_0 is said to be an *extreme point* of A. The set of all extreme points of A is denoted by $\partial_e A$ (see [8, 9, 12]).

We conclude this section by recalling the theorem of Berliocchi and Lasry [3].

Theorem 1. [3, II.2, Proposition 2, p. 145] *Let K be a convex, compact subset of a locally convex Hausdorff space and let $\Phi_1, \ldots, \Phi_n$ be n affine functions from K to $(-\infty, \infty]$*[a]*. Then each extreme point of the set*

$$G := \{x \in K \mid \Phi_i(x) \leq 0,\ i = 1, 2, \ldots, n\}$$

is a convex combination of at most $n+1$ extreme points of K.

2. Extensions of Berliocchi–Lasry Theorem

We are now in a position to introduce some extensions of Theorem 1. In this and the next section, K is a convex, compact subset of a locally convex Hausdorff space X. Besides, the symbol $\partial f(x)$ always stands for the subdifferential of the convex function f at the point x.

Theorem 2. *Let $\Phi_1, ..., \Phi_m$ be lower semicontinuous (l.s.c.) concave functions from K to $(-\infty, \infty]$. Let G be a subset of K defined by*

$$G := \{x \in K \mid \Phi_i(x) \leq 0,\ i = 1, 2, ..., m\}.$$

Suppose $G \neq \emptyset$ and $\partial(-\Phi_i)(x) \neq \emptyset$ for all $x \in G$ and for all $i = 1, 2, ..., m$. Then each extreme point of G is a convex combination of at most $(m+1)$ extreme points of K.

Proof. By the lower semicontinuity of $\Phi_i(.)$, $i = 1, 2, ..., m$, G is (nonempty) compact. It follows from Lemma 8.1 in [8, p. 21] or [9, 13A, p. 74][b] that $\partial_e G \neq \emptyset$. Take $y \in \partial_e G$, $\xi_i \in \partial(-\Phi_i)(y)$, $i = 1, 2, ..., m$. One obtains

$$\Phi_i(x) \leq \Phi_i(y) + \langle x - y, -\xi_i \rangle =: \bar{\Phi}_i(.), \text{ for all } x \in K, i = 1, 2, ..., m.$$

Set

$$G_y := \{x \in K \mid \bar{\Phi}_i(x) \leq \Phi_i(y),\ i = 1, 2, \ldots, m\}.$$

[a]For an example of such a kind of affine functions, one can take $K = \mathcal{M}^1_+(\mathbb{R})$ and $\mu \to \int_{\mathbb{R}} x^2 \mu(dx)$ (see also Sec. 4).

[b]Although, in [9], the definitions of extremal subsets and extreme points were just given for convex subsets of a locally convex space X, the lemma in [9, 13A p.74] is still valid for nonconvex subsets (and X is Hausdorff) as claimed by the author (without any changes in its proof given therein), provided that the extremal subsets and extreme points are understood as in Sec. 1.

Then $G_y \subset G$, and $y \in G_y$ (note that $\bar{\Phi}_i(y) = \Phi_i(y)$). Since $= y \in \partial_e G$, one obtains $y \in \partial_e G_y$. Theorem 1 now applies to the set G_y and guarantees the existence of $x_i \in \partial_e K$, $\lambda_i \in [0, 1]$ with $\sum_{i=1}^{m+1} \lambda_i 1$, $i = 1, 2, ..., m+1$ and $y = \sum_{i=1}^{m+1} \lambda_i x_i$. The proof is complete. ■

A further extension of Theorem 2 is as follows.

Theorem 3. *Let $\phi_0, \phi_1, \ldots, \phi_m : K \longrightarrow (-\infty, \infty]$ be l.s.c. concave functions. Let further ψ_i be functions from K to $(-\infty, \infty]$ defined by*

$$\psi_i(x) := \phi_i(x) + g_i(L_i(x)),\ x \in K,$$

where $L_i : K \longrightarrow \mathbb{R}^q$ are operators which are continuous and affine on K and $g_i : L_i(K) \longrightarrow (-\infty, +\infty]$ are l.s.c on $L_i(K)$, $i = 0, 2, \ldots, m$. Set

$$\mathcal{G} := \{x \in K \mid \psi_i(x) \leq 0,\ i = 0, 1, 2, ..., m\}.$$

Suppose $\mathcal{G} \neq \emptyset$ and $\partial[-\phi_i](x) \neq \emptyset$ for all $x \in K$. Then there exists a point $x_* \in \mathcal{G}$ such that x_* can be expressed as a convex combination of at most $(m+1)(2q+1)+1$ extreme points of K.

Proof. Take $\bar{x} \in \mathcal{G}$ and define $\mathcal{G}_{\bar{x}}$ to be the set of all $x \in K$ satisfying, for all $i = 0, 1, \ldots m, k = 1, 2, \ldots q,$

$$\phi_i(x) + g_i(L_i(\bar{x})) \leq 0, L_i^k(x) - L_i^k(\bar{x}) \leq 0, L_i^k(\bar{x}) - L_i^k(x) \leq 0,$$

where $L_i = (L_i^1, L_i^2, \ldots, L_i^q)$, $i = 0, 1, \ldots, m$. Then $\mathcal{G}_{\bar{x}}$ has the form of G in Theorem 2 where the roles of $\Phi_i(.)$ are played by either $\phi_i(.) + g_i(L_i(\bar{x}))$ or $L_i^k(.) - L_i^k(\bar{x})$, or $L_i^k(\bar{x}) - L_i^k(.)$, $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, q$.

Note that $\mathcal{G}_{\bar{x}}$ is not empty ($\bar{x} \in \mathcal{G}_{\bar{x}}$) and compact. Hence, $\partial_e \mathcal{G}_{\bar{x}} \neq \emptyset$. Take $x_* \in \partial_e \mathcal{G}_{\bar{x}}$. By definition, $x_* \in \mathcal{G}$. The conclusion now follows from Theorem 2. □

3. Extremum Principles

Let K, $\phi_i(.)$, $g_i(.)$, and $L_i(.)$, $i = 0, 1, \ldots, m$ be as in Theorem 3. Consider the optimization problem $(P1)$:

$$(P1):\ \text{minimize}\ \psi_0(x) := \phi_0(x) + g_0(L_0(x))$$

over all $x \in K$ satisfying

$$\psi_i(x) := \phi_i(x) + g_i(L_i(x)) \leq 0,\ i = 1, 2, \ldots, m.$$

As a consequence of Theorem 3, we obtain the following extremum principle for $(P1)$.

Theorem 4. *Suppose $\inf(P1) < +\infty$. Then problem $(P1)$ has an optimal solution which is a convex combination of at most $(m+1)(2q+1)+1$ extreme points of K.*

Proof. The existence of optimal solutions of $(P1)$ follows from the compactness of K, the lower semi-continuity of $\psi_i(.)$, $i = 0, 1, \ldots, m$, and Weierstrass theorem. For the last assertion, set

$$i_* := \inf(P1),$$

$$\bar{\mathcal{G}} := \{x \in K \mid \psi_0(x) \leq i_*, \ \psi_i(x) \leq 0, \ i = 1, 2, \ldots, m\}.$$

In other words, $\bar{\mathcal{G}}$ is the set of all optimal solutions of $(P1)$. By the previous argument, it is nonempty. Therefore, it follows from Theorem 3 that there exists $x_* \in \mathcal{G}$ which is a convex combination of at most $(m+1)(2q+1)+1$ extreme points of K. By the definition of $\bar{\mathcal{G}}$, x_* is an optimal solution of $(P1)$. The proof is complete. ■

Remark 1. Theorem 4 is equivalent to Theorem 3.

Obviously, Theorem 4 is a consequence of Theorem 3. Conversely, K and the functions $\phi_i(.)$, $g_i(.)$, $L_i(.)$, $\psi_i(.)$, $i = 0, 1, \ldots, m$ are as in Theorem 3. Suppose the set $\mathcal{G}$ defined in Theorem 3 is nonempty. Construct the problem $(P1)$ as above.

Note that $\inf(P1) < +\infty$ follows from the fact that $\mathcal{G} \neq \emptyset$. Hence, by Theorem 4, there exists an optimal solution x_* of $(P1)$ which is a convex combination of at most $(m+1)(2q+1)+1$ extreme points of K. It remains to prove that $x_* \in \mathcal{G}$. Since $\mathcal{G} \neq \emptyset$, we obtain $i_* := \inf(P1) \leq 0$. This gives $\psi(x_*) = i_* \leq 0$ which proves that $x_* \in \mathcal{G}$. ■

It is clear from the above proof that the element x_* that exists in Theorem 3 can be chosen in such a way that it is a minimizer of one of the functions $\psi_i(.)$, say ψ_{i_0}, over the set defined by $\psi_j(x) \leq 0$, $j \neq i_0$, $x \in K$.

As a direct consequence of Theorem 4, we obtain the following:

Corollary 1. (Extremum principle for concave programming problems with reverse convex constraints) *Let D be an open set containing a convex, compact subset K of a locally convex Hausdorff space X, and let $\phi_i(.)$, $i = 0, 1, 2, \ldots, m$ be l.s.c. concave functions from D to $(-\infty, +\infty]$. Suppose these functions are locally bounded from below on D. Consider the concave programming problem $(P2)$:*

$$(P2) : \textit{minimize } \phi_0(.)$$

over all $x \in K$ satisfying

$$\phi_i(x) \leq 0, \ i = 1, 2, \ldots, m \tag{$*$}$$

(constraints of the form $(*)$ are called reverse convex constraints (see [10])). Suppose the admissible set of $(P2)$ is nonempty. Then $(P2)$ has an optimal solution that is a convex combination of at most $m+2$ extreme points of K.

Remark 2. In Theorem 2, if all the functions $\Phi_i(.)$ except one are affine, then the hypotheses stated therein may be weakened but the price we have to pay for this is a weaker conclusion. As a matter of fact, this is a combination of Bauer's extremum principle (see [9, 13A, Corollary 2, p. 75], [7, Theorem 25.9, p. 102]), Theorem 1, and the previous argument. Concretely, we have

Corollary 2. *Let $\Phi_0 : K \longrightarrow (-\infty, \infty]$ be l.s.c. concave, and let $\Phi_1, \ldots, \Phi_m$ be l.s.c. affine functions from K to $(-\infty, \infty]$. Further, let G be a subset of K defined by*

$$G := \{x \in K \mid \Phi_i(x) \leq 0,\ i = 0, 1, 2, ..., m\}.$$

Suppose $G \neq \emptyset$. Then there exists a point $x_* \in G$ such that x_* is a convex combination of at most $(m+2)$ extreme points of K. Moreover, x_* can be chosen to be an extreme point of the set

$$\tilde{G} := \{x \in K \mid \Phi_i(x) \leq 0,\ i = 1, 2, ..., m\}.$$

Note that the assumption "$\partial[-\Phi_i](x) \neq \emptyset,\ \forall x \in G$" which played a very important role in the proof of Theorem 2 is absent.

Proof. Consider the problem $(P3)$:

$$(P3): \text{ minimize the function } \Phi_0(x) \text{ over } \tilde{G}.$$

Since $G \neq \emptyset$, $\inf(P3) < +\infty$. By Bauer's extremum principle, there exists $x_* \in \partial_e \tilde{G}$ which is an optimal solution for $(P3)$. It is evident that $x_* \in G$. The rest of the conclusion follows from Theorem 1. ■

The same argument as in the proof of Theorem 3 leads to the following, which is equivalent to the Fundamental Extremum Principle in [1] (see also Theorem 4 and Remark 1 above).

Corollary 3. *Let $\phi_0 : K \longrightarrow (-\infty, \infty]$ be l.s.c. concave, and let $\phi_1, ..., \phi_m : K \longrightarrow (-\infty, \infty]$ be affine l.s.c. functions. Further, let ψ_i be functions from K to $(-\infty, +\infty]$ defined by*

$$\psi_i(x) := \phi_i(x) + g_i(L_i(x)),\ x \in K,\ i = 0, 1, 2, \ldots, m$$

where $L_i : K \longrightarrow \mathbb{R}^q$ are operators which are continuous and affine on K and $g_i : L_i(K) \longrightarrow (-\infty, +\infty]$ are l.s.c on $L_i(K)$, $i = 0, 2, \ldots, m$. Suppose $\mathcal{G}$ is a subset of K defined by

$$\mathcal{G} := \{x \in K \mid \psi_i(x) \leq 0,\ i = 0, 1, 2, \ldots, m\}.$$

If $\mathcal{G} \neq \emptyset$, then there exists a point $x_* \in \mathcal{G}$ such that x_* is a convex combination of at most $(m+1)(2q+1)$ extreme points of K. ■

4. Application: The Existence for Nonconvex Optimal Control Problems

As an application of the above results, an existence theorem for a simple nonconvex optimal control problem will be proved in this section. In the course, Corollary 2 will play a crucial role. Despite of the simplicity of the problem in consideration, the existence is not trivial because of the lack of convexity assumptions.

4.1. Statement of the Problem and Assumptions

Consider problem (P) of minimizing the functional

$$J(u) := \int_0^1 g(t, u(t))dt + \int_0^1 \bar{g}(t, y(t))dt + \bar{c}(y) \qquad (P)$$

over all absolutely continuous mappings $y(.) : I \longrightarrow \mathbb{R}^n$ and over all measurable $u(.) : I \longrightarrow \mathbb{R}^q$ $(I := [0, 1])$ satisfying the following conditions:

$$\dot{y}(t) = B(t)y(t) + b(t, u(t)), \; y(0) = 0, \qquad (1)$$

$$u(t) \in U, \; t \in I, \qquad (2)$$

$$\int_0^1 g_i(t, u(t)) \leq \alpha_i, \; i = 1, 2, ..., m. \qquad (3)$$

Here, U is a Borel measurable, compact subset of $\mathbb{R}^q$, $B(.)$ is a integrable mapping from I with $n \times n$-matrix values, $b : I \times \mathbb{R}^q \longrightarrow \mathbb{R}^n$ is a map such that $b(., v)$ is measurable for all $v \in U$ and $b(t, .)$ is continuous for all $t \in I$. Moreover, $\bar{c} : C(I, \mathbb{R}^n) \longrightarrow (-\infty, \infty]$ is l.s.c., concave, and bounded from below on $C(I, \mathbb{R}^n)$ (for instance, $\bar{c}(y) = -\exp(-\|y\|)$). For the measurable functions $g, g_i : I \times \mathbb{R}^q \longrightarrow (-\infty, +\infty]$, $i = 1, 2, 3, ..., m$ and $\bar{g} : I \times \mathbb{R}^n \longrightarrow (-\infty, +\infty]$, the following hypotheses are imposed.

(H1) There exist a nonnegative function $\phi \in \mathcal{L}^1_+(I)$ and a function $\chi' : \mathbb{R}_+ \to [0, +\infty]$, nondecreasing, convex, l.s.c. with $\chi'(\xi)/\xi \to +\infty$ when $\xi \to \infty$, such that

$$g(t, v) \geq \chi'(|b(t, v)|) - \phi(t) \text{ on } I \times \mathbb{R}^q,$$
$$\chi'(|b(t, \cdot)|) \text{ is inf-compact on } \mathbb{R}^q \text{ for all } t \in I.$$
$$\forall \epsilon > 0 \, \exists \phi_\epsilon \in \mathcal{L}^1_+(I) : \; g_i(t, v) + \epsilon\chi'(|b(t, v)|)$$
$$\geq -\phi_\epsilon(t), i = 1, 2, ..., m, (t, v) \in I \times \mathbb{R}^q.$$

(H2) There exist nondecreasing functions $\bar{m}_1, \ldots, \bar{m}_d : \mathbb{R}_+ \to \mathbb{R}_+$ and $\bar{\psi}, \bar{\psi}_1, \ldots, \bar{\psi}_d \in \mathcal{L}^1_+(I)$ satisfying

$$\bar{g}(t, \eta) + \sum_{i=1}^d \bar{m}_i(|\eta|)\bar{\psi}_i(t) \geq -\bar{\psi}(t) \text{ on } I \times \mathbb{R}^n,$$
$$\lim_{\xi \to \infty} [\chi'(\xi) - \bar{\chi}(c_1\xi)] = +\infty,$$

where

$$c_1 := 1 + \sup_{(t,\tau) \in I^2} |E(t, \tau)|(1 + \int_0^1 |B(t)|dt)$$

and

$$\bar{\chi} : \xi \longrightarrow \sum_{i=1}^d \bar{m}_i(\xi) \int_0^1 \bar{\psi}_i(t) \, dt.$$

We shall prove that under the above assumptions together with $\inf(P) < +\infty$, problem (P) possesses at least one optimal solution, i.e., there exists an admissible control $u^*(.)$ (satisfying (1)–(3)) such that $J(u^*) = \inf(P)$. For this, we first extend the class of all controls to the class of relaxed controls (alias Young measures) and formulate the relaxed problem (P_{rel}) associated to (P). Secondly, we prove that the relaxed problem (P_{rel}) has an optimal relaxed solution. In this step, Corollary 2 will be used to guaranteed that (P_{rel}) has also an optimal relaxed control which is of a special form (exactly, the Minkowski form). Lastly, following the method described in Phase 3 in [1], from this special relaxed solution, we can construct the optimal control $u^*(.)$ for (P).

For further investigation, let us set

$$A(u)(t) := \int_0^t b(\tau, u(\tau))d\tau \text{ and } \Phi(z)(t) := z(t) + \int_0^t E(t,\tau)B(\tau)z(\tau)d\tau.$$

Then the solution of (1), $y_u(.)$, can be represented as

$$y_u(t) = \Phi(A(u))(t).$$

Here, $E_\tau := E(.,\tau)$ is the $n \times n$-matrix solution of the equation $\dot{E}_\tau = BE_\tau$ with the initial condition $E_\tau(\tau) = I$.

4.2. Relaxed Controls and Relaxed Problem $(P_{\rm rel})$

A *relaxed control* (or *Young measure*) from I to U is a function $\delta : I \longrightarrow \mathcal{M}_1^+(U)$ (the set of all probability measures on U) such that $\delta(.)(B) : t \longrightarrow \delta(t)(B)$ is a measurable function for each Borel set B in U. The set of all relaxed controls from I to U will be denoted by $\mathcal{R}$. A sequence $(\delta_n)_n \subset \mathcal{R}$ is said to *converge narrowly* to $\delta_0 \in \mathcal{R}$, denoted by $\delta_n \Longrightarrow \delta_0$, if for all measurable subsets $A \subset I$ and for all $c \in \mathcal{C}(U)$,

$$\lim_n \int_A [\int_U c(x)\delta_n(t)(dx)]dt = \int_A [\int_U c(x)\delta_0(t)(dx)]dt.$$

Remark 3. We will denote by ϵ_v the Dirac measure at v. Each measurable mapping $u(.) : I \longrightarrow U$ can be identified with a relaxed control ϵ_u defined by $\epsilon_u(t) := \epsilon_{u(t)}, t \in I$.

Remark 4. Note that $\mathcal{R}$ can be identified with a subset in the closed unit ball $\hat{\mathcal{R}}$ of the space $L^\infty(I, \mathcal{M}(U))$. $\mathcal{R}$ is convex while $\hat{\mathcal{R}}$ is compact for the weak star topology on $L^\infty(I, \mathcal{M}(U))$ by the Alaoglu–Bourbaki theorem ($\mathcal{M}(U)$ denotes the set of all bounded signed Borel measures on U). Note also that the narrow convergence mentioned above is nothing but the convergence with respect to the weak star topology on $L^\infty(I, \mathcal{M}(U))$. Moreover, each extreme point of the set $\mathcal{R}$ is of the form ϵ_u (see Remark 3) where $u(.)$ is a measurable mapping from I to U (see [1,2,4,13] for more details).

Let $h : I \times \mathbb{R}^q \longrightarrow \overline{\mathbb{R}}$, $\delta \in \mathcal{R}$. Denote

$$I_h(\delta) := \int_I [\int_U h(t,v)\delta(t)(dv)]dt$$

(note that $I_h(.)$ is linear in δ). If $\delta = \epsilon_u$, then

$$I_h(\epsilon_u) = \int_I h(t, u(t))dt.$$

Let also $r : \mathcal{R} \longrightarrow (-\infty, +\infty], \bar{r} : C(I, \mathbb{R}^n) \longrightarrow (-\infty, \infty]$ be the functions defined by

$$r(\delta) := \int_I [\int_U g(t, v)\delta(t)(dv)]dt, \quad \bar{r}(y) := \int_I \bar{g}(t, y(t))dt + \bar{c}(y).$$

With the assumption

$$a_{m+1}(\delta) := I_b(\delta) = \int_I [\int_U = |b(t, v)|\delta(t)(dv)]dt < +\infty, \tag{4}$$

the *relaxed trajectory* of the problem is defined by

$$y_\delta := \Phi(A(\delta)) \in C(I, \mathbb{R}^n) \text{ where } A(\delta)(t) := \int_0^t [\int_U b(\tau, v)\delta(\tau)(dv)]d\tau.$$

A relaxed control $\delta \in \mathcal{R}$ is called *admissible* if it satisfies (4) and

$$a_i(\delta) := I_{g_i}(\delta) \leq \alpha_i, \quad \text{for all } i = 1, 2, ..., m.$$

We are now able to state the relaxed problem (P_{rel}) of (P) which is as follows:

$$\text{minimize } \mathcal{J}(\delta) := r(\delta) + \bar{r}(y_\delta)$$

over all admissible relaxed controls. It is obvious that Inf $P_{rel} \leq$ inf P.

If we set $a_0(\delta) := \int_I [\int_U \chi'(|b(t, v)|)\delta(t)(dv)]\, dt$, then $a_0(.)$ is an inf-compact function on $\mathcal{R}$ (see [1, Lemma 4.1]).

Theorem 5. *Suppose* inf $P < \infty$. *Then problem* (P_{rel}) *possesses at least one relaxed solution.*

Theorem 6. *There exists an admissible relaxed control* $\delta_* \in \mathcal{R}$ *satisfying*

(i) $\mathcal{J}(\delta_*) = i_* :=$ inf P_{rel}*(i.e.,* δ_* *is an optimal relaxed solution for* (P_{rel})*);*
(ii) *there exist controls* $u_i(.)$, $\lambda_i \geq 0$, $i = 1, 2, ..., (m + 4)$, $\Sigma_{i=1}^{m+4} \lambda_i = 1$ *such that* $\delta_* = \Sigma_{i=1}^{m+4} \lambda_i \epsilon_{u_i}$.

Theorem 5 is a consequence of Lemma 4.4 in [1] while a proof (sketch) of Theorem 6 is given below.

Proof of Theorem 6 (sketch). Set

$$\mathcal{R}_0 := \{\delta \in \mathcal{R} \mid J(\delta) \leq i_*,\ a_i(\delta) \leq \alpha_i,\ i = 0, 1, 2, ..., m + 1\}.$$

It follows from Theorem 5 that $\mathcal{R}_0 \neq \emptyset$. On the other hand, the functions $\mathcal{J}(.)$ $a_i(.)$, $i = 0, 1, 2, \ldots, m + 1$ are l.s.c. by Lemma 4.1 in [1]. Moreover, $\mathcal{J}(.)$ is concave in δ. Therefore, these functions satisfy all the conditions from Corollary 2. By this corollary, there exists $\delta_* \in \mathcal{R}_0$ such that δ_* can be represented as a convex combination of at most $m + 4$ extreme points of $\mathcal{R}$. Taking Remark 4 into account, there exist controls $u_1(.), u_2(.), \ldots, u_{m+4}(.)$, such that

$$\delta_*(t) = \sum_{k=1}^{m+4} \lambda_k \epsilon_{u_k}(t),\ t \in I,\ \sum_k \lambda_k = 1,\ \lambda_k \geq 0,\ k = 1, 2, \ldots, m + 4$$

which is desired. ■

4.3. The Existence of Optimal Solutions for (P)

Theorem 7. *Suppose* $\inf P < +\infty$. *Then problem* (P) *possesses at least one optimal solution.* ■

Theorem 7 is a consequence of Phase 3 in [1] (including Lemmas 4.7 and 4.8 and Proposition 4.9 in [1]). There, the concavity of $\mathcal{J}(.)$ (in δ) and the extended Lyapunov theorem on the range of vector measures [2, Corollary A.11] (see also [3, Theorem 3, p. 153], [6], [8, Corollary 8.1, p. 24], etc.) were used to guarantee the existence of a measurable partition $C_1, C_2, \ldots, C_{m+4}$ of I so that the optimal control $u_*(.)$ of (P) can be constructed as follows:

$$u_*(t) := u_k(t) \text{ if } t \in C_k, k = 1, 2, ..., m + 4, \quad t \in I,$$

where $u_k(.)$, $k = 1, 2, \ldots, m + 4$ are the controls appearing in Theorem 6.

Remark 5. Problem (P) is a special case of a very abstract problem considered in [1]. The above application partly shows the ability of applying results obtained in Secs. 2 and 3 in approaching the existence for nonconvex optimal control (and also variational) problems.

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