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A Nonlinear Integral Equation of Gravimetry: Uniqueness and Approximation by Linear Moments

Dang Dinh Ang[1], Nguyen Van Nhan[2] and Dinh Ngoc Thanh[1]

[1] *School of Natural Sciences, Ho Chi Minh City National University 227 Nguyen Van Cu Str., 5 Distr., Ho Chi Minh City, Vietnam*

[2] *School of Economics, Ho Chi Minh City National University 59C Nguyen Dinh Chieu Str., 3 Distr., Ho Chi Minh City, Vietnam*

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Abstract. The authors consider the problem of determining by gravimetric methods the shape of an object in the interior of the Earth, the density of which differs from that of the surrounding medium. Assuming a flat earth model, the problem is that of finding a domain in the half-plane $z \leq H$, $H > 0$, represented by

$$0 \leq \sigma(x) < H, \quad 0 \leq x \leq 1,$$

where σ satisfies a nonlinear integral equation of the first kind. Uniqueness is proved and the integral equation is approximated by a linear moment problem.

1. Introduction

The determination of the shape of an object Ω in the interior of the Earth and the density, of which differs from that of the surrounding medium, is a fundamental problem of applied geophysics. Gravimetric methods are used for this purpose. They consist of measuring the gravity anomalies created on the surface by the difference in densities. A mathematical formulation of the problem can be found in [8] which contains several references to the literature. A uniqueness proof for the 3-D case was given in [1], and more recently, a general uniqueness theorem was proved for the 3-D case in [3]. In this paper, we consider the 2-D case and furthermore, we assume that we are given gravity gradient [9] instead of the anomaly in the gravity itself. The uniqueness proof for the 2-D case has certain peculiarities not present in the 3-D case. The gravity gradient approach presents advantages as shown in [9].

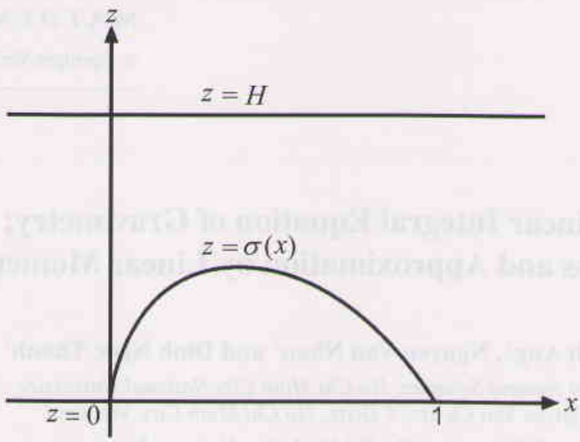

Fig. 1.

For our problem, we consider a flat earth model. Let the Earth be represented by the half plane (x, z), $-\infty < z \leq H$ where $H > 0$ and let the body Ω be represented by $0 \leq z \leq \sigma(x)$ (Fig. 1).

Let ρ be the relative density of Ω, i.e., the difference between the density of Ω and that of the surrounding medium. We assume ρ to be a constant.

Denote by $U = U(x, z)$ the gravity potential created by ρ:

$$U(x,z) = \frac{1}{2\pi} \int_\Omega \rho \ln \left((x-\xi)^2 + (z-\zeta)^2 \right) dv. \tag{1}$$

Then the gravity anomaly created by ρ is

$$-\frac{\partial U}{\partial z} = \frac{\rho}{2\pi} \int_\Omega \frac{z-\zeta}{(x-\xi)^2 + (z-\zeta)^2} dv \tag{2}$$

and the gravity gradient created on the surface $z = H$ is

$$\begin{aligned} -\left.\frac{\partial^2 U}{\partial z^2}\right|_{z=H} &= -\frac{\rho}{2\pi} \int_\Omega \frac{\partial}{\partial \zeta} \left(\frac{H-\zeta}{(x-\xi)^2 + (H-\zeta)^2} \right) dv \\ &= \frac{\rho}{2\pi} \left\{ \int_0^1 \frac{H-\sigma(\xi)}{(x-\xi)^2 + (H-\sigma(\xi))^2} d\xi - \int_0^1 \frac{H\,d\xi}{(x-\xi)^2 + H^2} \right\}. \end{aligned}$$

From now on, for convenience, we set

$$\rho \equiv 1.$$

Let $f_0 = f_0(x)$ be the gravity gradient on the surface $z = H$. Then we have

$$\frac{1}{2\pi} \int_0^1 \frac{H-\sigma(\xi)}{(x-\xi)^2 + (H-\sigma(\xi))^2} d\xi = f(x), \tag{3}$$

where we have set

$$f(x) = -f_0(x) - \frac{H}{2\pi} \int_0^1 \frac{d\xi}{(x-\xi)^2 + H^2}. \tag{4}$$

The problem is to determine $\sigma(x)$ from Eq. (3). The main results are a uniqueness result for Eq. (3) (Theorem 1) and a uniqueness theorem for the approximating linear moment problem (Theorem 2).

Theorem 1. (Uniqueness theorem) *Equation (3) admits at most one continuous solution* $\sigma = \sigma(x)$, $0 \leq x \leq 1$, *vanishing at* 0 *and* 1, *and such that* $0 < \sigma(x) \leq \alpha < H$ *for* $0 < x < 1$.

Proof. Let σ_1, σ_2 be two solutions of (4). Then

$$\int_0^1 \frac{H - \sigma_1(\xi)}{(x-\xi)^2 + (H - \sigma_1(\xi))^2} d\xi - \int_0^1 \frac{H - \sigma_2(\xi)}{(x-\xi)^2 + (H - \sigma_2(\xi))^2} d\xi = 0. \tag{5}$$

Put

$$F(x,z) = \int_{\mathbf{R}^2} (1_{\Omega_1} - 1_{\Omega_2}) \ln\left((x-\xi)^2 + (z-\zeta)^2\right) dv, \tag{6}$$

where $1_{\Omega_1}, 1_{\Omega_2}$ are the characteristic functions of Ω_1, Ω_2, respectively. It can be shown that

$$\frac{\partial F}{\partial z}(x,z) = \int_0^1 \ln\left(\frac{(x-\xi)^2 + (z-\sigma_1(\xi))^2}{(x-\xi)^2 + (z-\sigma_2(\xi))^2}\right) d\xi. \tag{7}$$

The function $\frac{\partial F}{\partial z}(x,z)$ is harmonic in $\mathbf{R}^2 \setminus (S_1 \cup S_2)$ where S_1, S_2 are the nonhorizontal portions of the boundaries of Ω_1, Ω_2, corresponding to σ_1, σ_2, respectively (Fig. 2).

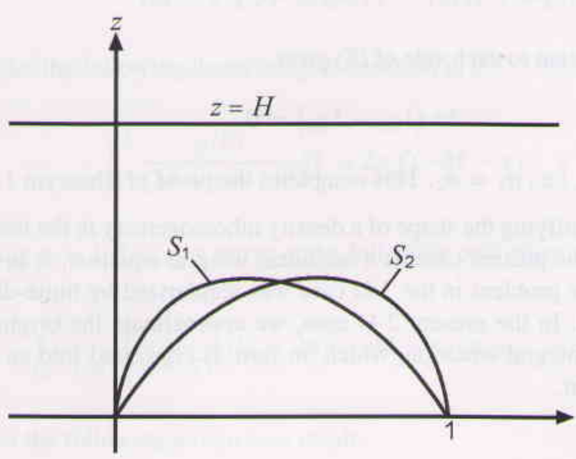

Fig. 2.

We shall show that $\frac{\partial F}{\partial z} \equiv 0$ in $z \geq H$. Indeed, $\frac{\partial^2 F}{\partial z^2}$ is harmonic in $\mathbf{R}^2 \setminus (S_1 \cup S_2)$, $\frac{\partial^2 F}{\partial z^2}(x,z) \to 0$ for $z \to \infty$ and $\frac{\partial^2 F}{\partial z^2}(x,H) = 0$, $\forall x$. Therefore, $\frac{\partial^2 F}{\partial z^2}(x,z) = 0$ for $z \geq H$ and hence, $\frac{\partial F}{\partial z}(x,z) = \lambda(x)$ for $z \geq H$. It follows from the harmonicity of F on $z \geq H$ that $\frac{\partial^3 F}{\partial z \partial x^2}(x,z) = \lambda''(x) = 0$ for $z \geq H$ so that λ is linear in x. Since $\lim_{x\to\infty} \lambda(x) = \lim_{x\to\infty} \frac{\partial F}{\partial z}(x,z) = 0$, we obtain $\lambda \equiv 0$ and thus, $\frac{\partial F}{\partial z}(x,z) \equiv 0$ in $z \geq H$ as desired. Consequently, $F(x,z) = \gamma(x)$ in $z \geq H$ and since F is harmonic for $z \geq H$, $\frac{\partial^2 F}{\partial x^2} = \gamma''(x) = 0$ for $z \geq H$ so that γ is linear in x. Since $\frac{\gamma(x)}{\ln(x^2+z^2)} = \frac{F(x,z)}{\ln(x^2+z^2)}$ has a finite limit as x tends to infinity, $\gamma(x)$ must be a constant.

Note that since $\frac{\partial F}{\partial z}$ is a harmonic function on $\mathbf{R}^2 \setminus (S_1 \cup S_2)$, by harmonic continuation, it follows that $\frac{\partial F}{\partial z}(x,z) = 0$ in the unbounded connected component K of $\mathbf{R}^2 \setminus (S_1 \cup S_2)$. Since $\frac{\partial F}{\partial z}$ is continuous everywhere in $\mathbf{R}^2$, and in particular on $S_1 \cup S_2$, and since $S_1 \cup S_2 \subset \partial K$, we have by continuity that

$$\frac{\partial F}{\partial z}(x,z) = 0 \quad \text{on } S_1 \cup S_2.$$

Let ω be a bounded connected component of $\mathbf{R}^2 \setminus (S_1 \cup S_2)$. Then $\partial\omega \subset S_1 \cup S_2$ and hence $\frac{\partial F}{\partial z} = 0$ on $\partial\omega$. By the maximum principle for harmonic functions, $\frac{\partial F}{\partial z}(x,z) = 0$ on ω. It follows that $\frac{\partial F}{\partial z}(x,z) = 0$ on $\mathbf{R}^2$ and that $F(x,z) = \delta(x)$ on $\mathbf{R}^2$ and hence from what precedes $\gamma(x) = \delta(x)$ on $z \geq H$ and thus $\delta(x)$ is a constant, i.e.,

$$\int_{\mathbf{R}^2} \left(1_{\Omega_1} - 1_{\Omega_2}\right) \ln\left((x-\xi)^2 + (z-\eta)^2\right) dv = \text{const. on } \mathbf{R}^2. \tag{8}$$

Applying the Laplacian to each side of (8) gives

$$2\pi \left(1_{\Omega_1} - 1_{\Omega_2}\right) = 0, \tag{9}$$

which gives $\Omega_1 = \Omega_2$, i.e., $\sigma_1 = \sigma_2$. This completes the proof of Theorem 1. ■

The problem of identifying the shape of a density inhomogeneity in the interior of the Earth, formulated in the present case as a nonlinear integral equation, is in general an ill-posed problem. The problem in the 3-D case was regularized by finite-dimensional approximations in [2]. In the present 2-D case, we approximate the original integral equation by a linear integral equation, which, in turn, is converted into an equivalent linear moment problem.

Consider Eq. (3)

$$\frac{1}{2\pi} \int_0^1 \frac{H - \sigma(\xi)}{(x-\xi)^2 + (H - \sigma(\xi))^2} d\xi = f(x).$$

As in Theorem 1, we shall assume that

$$0 < \sigma(x) \leq \alpha < H, \quad 0 < x < 1. \tag{10}$$

Setting $\varphi(x) \equiv H - \sigma(x)$, $x \in (0, 1)$, we see that the function

$$h(x) = \int_0^1 \frac{\varphi(\xi)d\xi}{(x-\xi)^2 + \varphi^2(\xi)}, \tag{11}$$

in view of (10), can be extended to complex analytic function on a strip of width $< H - \alpha$ around the real axis of the complex plane. Hence, h is completely defined by its values on an interval $(-\infty, -M]$, for any $M > 0$, i.e., (3) is equivalent to the following equation:

$$\int_0^1 \frac{\varphi(\xi)d\xi}{(x-\xi)^2 + \varphi^2(\xi)} = 2\pi f(x), \quad x \leq -M, \tag{12}$$

where φ is a continuous function on $[0, 1]$, $\varphi(0) = \varphi(1) = H$ and $H - \alpha \leq \varphi(x) < H$ for all $x \in (0, 1)$. Now, for large M and $x \geq 0$, we have the expansion

$$\frac{\varphi(\xi)}{(M+x+\xi)^2 + \varphi^2(\xi))} = \varphi(\xi) \Big/ (M+x+\xi)^2 \left(1 + \left(\frac{\varphi(\xi)}{M+x+\xi}\right)^2\right)$$

$$= \frac{\varphi(\xi)}{(M+x+\xi)^2} - \frac{\varphi^3(\xi)}{(M+x+\xi)^4} + \cdots .$$

As a first approximation, we take

$$\frac{\varphi(\xi)}{(M+x+\xi)^2 + \varphi^2(\xi)} \approx \frac{\varphi(\xi)}{(M+x+\xi)^2},$$

and consider the following linear integral equation in φ:

$$\int_0^1 \frac{\varphi(\xi)}{(M+x+\xi)^2} d\xi = 2\pi f(-M-x), \quad x > 0. \tag{13}$$

By taking $x = 1, 2, \ldots$, we arrive at the following equivalent moment problem:

$$\int_0^1 \frac{\varphi(\xi)}{(M+n+\xi)^2} d\xi = 2\pi f(-M-n) \equiv \mu_n, \quad n = 1, 2, \ldots . \tag{14}$$

We have the following uniqueness result:

Theorem 2. *Equation (14) admits at most one continuous solution $\varphi = \varphi(x)$, $0 \leq x \leq 1$.*

Proof. It is sufficient to prove that if

$$\int_0^1 \frac{\varphi(\xi)}{(M+n+\xi)^2} d\xi = 0, \quad n = 1, 2, \ldots, \tag{15}$$

then $\varphi \equiv 0$.

Now, the latter integral can be written as

$$\int_0^\infty \frac{\varphi(\xi)}{(M+n+\xi)^2} d\xi,$$

where $\varphi(\xi)$ is extended to be 0 for $x \geq 1$. We have

$$\int_0^1 \frac{\varphi(\xi)}{(x+\xi)^2} d\xi = \int_0^\infty e^{-xt} \left(\int_0^\infty t\varphi(\xi) e^{-t\xi} .d\xi \right) dt. \tag{16}$$

We put

$$\psi(t) \equiv t \int_0^\infty \varphi(\xi) e^{-t\xi} \, d\xi. \tag{17}$$

Then, by (15) and (16)

$$\int_0^\infty \left(\psi(t) \, e^{-Mt} \right) e^{-nt} dt = 0, \quad n = 1, 2, \ldots . \tag{18}$$

Since the set of continuous functions with compact supports on $\mathbf{R}^+$ is dense in $L^2(\mathbf{R}^+)$ (see [7]), an application of the Stone–Weierstrass theorem shows that the algebra generated by the sequence (e^{-nt}) is dense in $L^2(\mathbf{R}^+)$. It follows that $\psi(t)e^{-Mt} = 0 \, a.e.$ and hence that $\psi(t) = 0 \, a.e.$ In view of (17), it follows that

$$\int_0^\infty \varphi(\xi) e^{-t\xi} d\xi = 0, \quad t > 0. \tag{19}$$

Thus, the Laplace transform of $\varphi(\xi)$ is the null function. Hence, by uniqueness of the inverse Laplace transform (cf. [10, p. 243]) (or by the previous argument), we have

$$\varphi(\xi) = 0 \quad a.e. \tag{20}$$

This completes the proof of Theorem 2. ■

Remark. In proving that (18) implies $\psi(t)e^{-Mt} = 0$, we have actually proved (the known fact) that if the Laplace transform $\hat{g}$ of a function g vanishes at $n = 1, 2, \ldots$, then $g = 0$ (from which follows the uniqueness of the inverse Laplace transform). Since we have not been able to find a handy reference, we sketch a proof.

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