Vietnam Journal of Mathematics **26**:4 (1998) 363–366

Vietnam Journal
of
MATHEMATICS

Short Communication

Note on the Kolmogorov–Stein Inequality*

Ha Huy Bang

Institute of Mathematics, P. O. Box 631 Bo Ho, Hanoi, Vietnam

Hoang Mai Le

Department of Natural Science, Training Teachers College of Thai Nguyen
Thai Nguyen, Vietnam

Dedicated to Professor Ju. A. Dubinskii on the occasion of his 60th birthday

Received May 25, 1998

A. N. Kolmogorov has given the following result [4]: Let $f(x), f'(x), \ldots, f^{(n)}(x)$ be continuous and bounded on $\mathbb{R}$. Then

$$\|f^{(k)}\|_\infty^n \leq C_{k,n}\|f\|_\infty^{n-k}\|f^{(n)}\|_\infty^k,$$

where $0 < k < n$, $C_{k,n} = K_{n-k}^n / K_n^{(n-k)}$, $K_i = \frac{4}{\pi}\sum_{j=0}^\infty (-1)^j/(2j+1)^{i+1}$ for even i, while $K_i = \frac{4}{\pi}\sum_{j=0}^\infty 1/(2j+1)^{i+1}$ for odd i. Moreover, the constants are best possible.

This result has been extended by E. M. Stein to L_p-norm [7] and by Ha Huy Bang to any Orlicz norm [1]. The Kolmogorov–Stein inequality and its variants are an interest for many mathematicians and have various applications (see, for example, [2, 8] and their references).

In this paper, modifying the methods of [1, 7], we prove this inequality for another norm generated by concave functions. Note that the Orlicz norm is generated by convex functions and here we must overcome some essential difficulties because of the difference between convex and concave functions.

*This work is supported by the National Basic Research Program in Natural Science and by the NCST "Applied Mathematics".

Let $\mathcal{L}$ denote the family of all non-zero concave functions $\Phi(t) : [0, \infty) \to [0, \infty]$, which are non-decreasing and satisfy $\Phi(0) = 0$. Denote by $N_\Phi = N_\Phi(\mathbb{R})$ the space of measurable functions $f(x)$ such that $\|f\|_{N_\Phi} = \int_0^\infty \Phi\big(\lambda_f(y)\big)dy < \infty$, where $\lambda_f(y) = \text{mes}\{x : |f(x)| > y\}$, $(y \geq 0)$, and by $M_\Phi = M_\Phi(\mathbb{R})$, the space of measurable functions $g(x)$ such that

$$\|g\|_{M_\Phi} = \sup\Big\{\frac{1}{\Phi(\text{mes }\Delta)}\int_\Delta |g(x)|dx : \ \Delta \subset \mathbb{R}, \ 0 < \text{mes }\Delta < \infty\Big\} < \infty.$$

Then N_Φ and M_Φ are Banach spaces [5, 6].

The main result of this paper is the following:

Theorem 1. *Let $\Phi \in \mathcal{L}$, $f(x)$ and its generalized derivative $f^{(n)}(x)$ be in N_Φ. Then $f^{(k)}(x) \in N_\Phi$ for all $0 < k < n$ and*

$$\|f^{(k)}\|^n_{N_\Phi} \leq C_{k,n}\|f\|^{n-k}_{N_\Phi}\|f^{(n)}\|^k_{N_\Phi}. \tag{1}$$

Proof. The sketch of proof is as follows. We begin to prove (1) with the assumption that $f^{(k)}(x) \in N_\Phi, \ 0 \leq k \leq n.$

By Theorem 4.3 in [6] and [9, p. 113], we have

$$\|f^{(k)}\|_{N_\Phi} = \sup_{\|g\|_{M_\Phi}=1}\Big|\int_{-\infty}^{\infty} f^{(k)}(x)g(x)dx\Big|. \tag{2}$$

Let $\varepsilon > 0$. We choose a function $h(x) \in M_\Phi$ such that $\|h\|_{M_\Phi} = 1$ and

$$\Big|\int_{-\infty}^{\infty} f^{(k)}(x)h(x)dx\Big| \geq \|f^{(k)}\|_{N_\Phi} - \varepsilon. \tag{3}$$

Put

$$F(x) = \int_{-\infty}^{\infty} f(x+y)h(y)dy.$$

By Theorem 4.4 in [6], we obtain $F(x) \in L_\infty(\mathbb{R})$. Arguing as in [1], we have

$$F^{(r)}(x) = \int_{-\infty}^{\infty} f^{(r)}(x+y)h(y)dy, \quad 0 \leq r \leq n \tag{4}$$

in the distribution sense. It is easy to check that $|F^{(r)}(x)| \leq \|f^{(r)}\|_{N_\Phi}, \ \forall x \in \mathbb{R}$.

Now, we prove the continuity of $F^{(r)}(x)$ on $\mathbb{R}$ $(0 \leq r \leq n)$. We show this for $r = 0$ by contradiction: Assume that for some $\varepsilon > 0$, point x^0 and subsequence $|t_k| \to 0$,

$$\Big|\int_{-\infty}^{\infty}\Big(f(x^0+t_k+y) - f(x^0+y)\Big)h(y)dy\Big| \geq \varepsilon, \ k \geq 1. \tag{5}$$

Since $f \in N_\Phi$, we easily obtain $f \in L_{1,loc}(\mathbb{R})$. Then for any $m = 1, 2, \ldots$, $f(t_k + y) \to f(y)$ in $L_1(-m, m)$. Therefore, there exists a subsequence, denoted again by $\{t_k\}$, such that $f(t_k + y) \to f(y)$ a.e. in $(-m, m)$. Therefore, there exists a subsequence (for simplicity of notation, we assume it is coincident with $\{t_k\}$) such that $f(x^0 + t_k + y) \to f(x^0 + y)$ a.e. in $(-\infty, \infty)$.

On the other hand, $\{f(x^0+t_k+y)\}$ is bounded in N_Φ. So $\{f(x^0+t_k+y)\}$ is a weak precompact sequence. Therefore, there exist a subsequence denoted by $\{f(x^0+t_k+y)\}$ and a function $f_*(y) \in N_\Phi$ such that

$$\int_{-\infty}^{\infty} f(x^0+t_k+y)v(y)dy \to \int_{-\infty}^{\infty} f_*(y)v(y)d(y),\ \forall\, v(y) \in M_\Phi. \tag{6}$$

Let $u(x)$ be an arbitrary function in $C_0^\infty(\mathbb{R})$, then $u(x) \in M_\Phi$. It follows from $f(x^0+t_k+y) \to f(x^0+y)$ a.e. that

$$\int_{-\infty}^{\infty} f(x^0+t_k+y)u(y)dy \to \int_{-\infty}^{\infty} f(x^0+y)u(y)dy,\ \forall\, u \in C_0^\infty(\mathbb{R}). \tag{7}$$

Combining (6) and (7) and [3, p. 15], we obtain

$$\int_{-\infty}^{\infty} f(x^0+t_k+y)h(y)dy \to \int_{-\infty}^{\infty} f(x^0+y)h(y)dy$$

which contradicts (5). The cases $1 \le r \le n$ are proved similarly.

Therefore, it follows from the Kolmogorov inequality and (3) and (4) that

$$\left(\|f^{(k)}\|_{N_\Phi} - \varepsilon\right)^n \le |F^{(k)}(0)|^n \le C_{k,n}\|F\|_\infty^{n-k}\|F^{(n)}\|_\infty^k. \tag{8}$$

Because of $\|F\|_\infty \le \|f\|_{N_\Phi}$, $\|F^{(n)}\|_\infty \le \|f^{(n)}\|_{N_\Phi}$ and by letting $\varepsilon \to 0$, we have (1).

It remains to show that $f^{(k)} \in N_\Phi$, $0 < k < n$ if $f, f^{(n)} \in N_\Phi$.

Let $\psi_\lambda(x) \in C_0^\infty(\mathbb{R})$, $\psi_\lambda(x) \ge 0$, $\psi_\lambda(x) = 0$ for $|x| \ge \lambda$ and $\int \psi_\lambda(x)dx = 1$. We put $f_\lambda = f * \psi_\lambda$. Then $f_\lambda \in C^\infty(\mathbb{R})$. It is easy to check that $f_\lambda^{(k)} = f * \psi_\lambda^{(k)} \in N_\Phi$, $k \ge 0$ and $f_\lambda^{(n)} = f^{(n)} * \psi_\lambda$. Therefore,

$$\|f_\lambda^{(k)}\|_{N_\Phi}^n \le C_{k,n}\|f_\lambda\|_{N_\Phi}^{n-k}\|f_\lambda^{(n)}\|_{N_\Phi}^k,\ 0 < k < n.$$

Since $\|f_\lambda\|_{N_\Phi} \le \|f\|_{N_\Phi}$, $\|f_\lambda^{(n)}\|_{N_\Phi} \le \|f^{(n)}\|_{N_\Phi}$, we have that, for any $0 \le k \le n$, the sequence $\{f_\lambda^{(k)}\}$ is bounded in N_Φ.

By an argument similar to the previous one, we obtain $f_\lambda \to f$. Therefore, it follows that, for any $\varphi \in C_0^\infty(\mathbb{R})$,

$$\langle f_\lambda^{(k)}(x), \varphi(x)\rangle \to \langle f^{(k)}(x), \varphi(x)\rangle.$$

So $f^{(k)} \in N_\Phi$ for all $0 < k < n$ if $f, f^{(n)} \in N_\Phi$. The proof is complete. ■

Remark. For periodic functions, we have

Theorem 2. *Let $\Phi(t) \in \mathcal{L}$, $f(x)$ and its generalized derivative $f^{(n)}(x)$ be in $N_\Phi(\mathbf{T})$. Then $f^{(k)}(x) \in N_\Phi(\mathbf{T})$ for all $0 < k < n$ and*

$$|||f^{(k)}|||^n_{N_\Phi(\mathbf{T})} \le C_{k,n}|||f|||^{n-k}_{N_\Phi(\mathbf{T})}|||f^{(n)}|||^k_{N_\Phi(\mathbf{T})},$$

where $\mathbf{T}$ is the torus and $|||\,.\,|||_{N_\Phi(\mathbf{T})}$ is the corresponding norm.

Applying the obtained results, we can obtain imbedding theorems for spaces of infinite order. We give here, for example, one result.

Let $1 \le q \le \infty$, $\Phi \in \mathcal{L}$, and $a = \{a_k\}_{k\in P}$ be a sequence of non-negative real numbers, which contains an infinite subsequence of positive numbers, where P is the set of non-negative integers. We denote by $W^\infty_{a,\Phi,q}$ the space of functions f on the real line $\mathbf{R}$ whose following seminorms are finite;

$$|||f|||_{a,q} = \Big\{\sum_{n\in P}(a_n\|f^{(n)}\|_{N_\Phi})^q\Big\}^{1/q} \quad (q < \infty),$$

$$|||f|||_{a,\infty} = \sup_{n\in P}\{a_n\|f^{(n)}\|_{N_\Phi}\} \ (q = \infty).$$

The spaces $W^\infty_{a,\Phi,q}$ are called Sobolev spaces of infinite order. The space $W^\infty_{b,\Phi,q}$ is defined similarly.

Theorem 3. *If the following imbedding holds:*

$$W^\infty_{a,\Phi,q} \hookrightarrow W^\infty_{b,\Phi,q},$$

then there exists a constant M such that

$$F_{b,q}(t) \le MF_{a,q}(t) \quad \forall t \ge 0,$$

where

$$F_{a,q}(t) = \begin{cases} \sum_{n\in P} a_n^q t^n & (q < \infty) \\ \sup_{n\in P}\{a_n t^n\} & (q = \infty). \end{cases}$$

References

1. Ha Huy Bang, A remark on the Kolmogorov–Stein inequality, *J. Math. Anal. Appl.* **203** (1996) 861–867.
2. M. W. Certain and T. G. Kurtz, Landau–Kolmogorov inequalities for semigroups and groups, *Proc. Amer. Math. Soc.* **63** (1977) 226–230.
3. L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.
4. A. N. Kolmogorov, On inequalities between upper bounds of the successive derivatives of an arbitrary function on an infinite interval, *Amer. Math. Soc. Trans., Ser. 1* **2** (1962) 233–243.
5. M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, Inc., New York, 1995.
6. M. S. Steigerwalt and A. J. White, Some function spaces related to L_p, *Proc. London Math. Soc.* **22** (1971) 137–163.
7. E. M. Stein, Functions of exponential type, *Ann. Math.* **65** (1957) 582–592.
8. V. M. Tikhomirov and G. G. Magaril–Il'jaev, Inequalities for derivatives, in: *Kolmogorov A. N. Selected Papers*, Nauka, 1985, pp. 387–390.
9. K. Yosida, *Functional Analysis*, 4th ed., Springer-Verlag, New York, 1974.