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Short Communication

First Holomorphic Cohomology Group and Linear Topological Properties

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1. Introduction

Let E be a Fréchet space with a fundamental system of semi-norms $\{\|.\|_k\}$. For each subset B of E, we define $\| \,.\, \|_B^* : E^* \to [0, +\infty)$ by

$$\|u\|_B^* = \sup\{|u(x)| : x \in B\},$$

where $u \in E^*$, the topological dual space of E.

Instead of $\| \,.\, \|_{U_k}^*$, we write $\| \,.\, \|_k^*$, where

$$U_k = \{x \in E : \|x\|_k \leq 1\}.$$

Using this notation, we say that E has the properties

$$\text{(DN)} \quad \text{if } \exists p \ \forall q, d > 0 \ \exists k, c > 0 : \ \|x\|_q^{1+d} \leq C\|x\|_k\|x\|_p^d \ \text{ for } x \in E,$$

$$(\Omega) \quad \text{if } \forall p \ \exists q \ \forall k \ \exists d, c > 0 \ \forall y \in E^* : \ \|y\|_q^{*1+d} \leq C\|y\|_k^*\|y\|_p^{*d}.$$

The properties (DN) and (Ω) have been introduced and investigated by Vogt (see [6, 7]).

The aim of this paper is to establish that

$$H^1(F^*, \mathcal{O}_{F^*}^{E^*}) = 0 \tag{1}$$

in the relation with linear topological invariants (DN) and (Ω).

Remark that Dineen proved that $H^1(\Omega, \mathcal{O}) = 0$ for every pseudoconvex domain Ω in a vector space equipped with the finest topology. After that, in the first case where $E = C$ and F is a Fréchet nuclear space, (1) has been established by Colombeau–Perrot in [2], and in the second case, where E is a Fréchet space with property (DN) and $F = C$ has been established by Vogt in [5]. Here, by using the linear topological properties (DN) and (Ω), we extend the results of Colombeau–Perrot and Vogt to infinite-dimensional cases.

2. Holomorphic Cohomology Group $H^1(F^*, \mathcal{O}_{F^*}^{E^*})$

Let E, F be locally convex spaces. For $q \in N$, we denote by $\Lambda_q(F, E)$ the vector space of all continuous skew symmetric q-antilinear forms on F^q with values in E. The space $\Lambda_q(F, E)$ is endowed with the topology of uniform convergence on bounded subset of F. By $C^{(0,q)}(\Omega, E)$, where Ω is an open subset in F, we denote the vector space of all C^∞-functions $\omega : \Omega \to \Lambda_q(F, E)$. $C^{(0,q)}(\Omega, E)$ is equipped with the topology of uniform convergence on compact subsets of Ω for functions, together with all their respective derivatives.

For each $q \in N$, we define a linear operator $\bar\partial$ from $C^{(0,q)}(\Omega, E)$ into $C^{(0,q+1)}(\Omega, E)$ by the formula

$$(\bar\partial\omega(x))(y_1, y_2, \ldots, y_{q+1}) = \frac{1}{q+1}\sum_{k=1}^{q}(-1)^{k+1}\frac{1}{2}(d\omega(x)[y_k] + i d\omega(x)[iy_k])(y_1, y_2, \ldots, \hat{y}_k, \ldots, y_{q+1}),$$

where $x \in \Omega$, $y_i \in F$ if $1 \le i \le q+1$, $d\omega(x)$ denotes the real differential of ω and the hat sign on y_k means that y_k is omitted.

In the case $q = 0$, we set $C^{(0,0)}(\Omega, E) = C^\infty(\Omega, E)$ as the space of E-valued C^∞-functions on Ω.

By definition, an element ω of $C^{(0,q)}(\Omega, E)$ is said to be $\bar\partial$-closed if $\bar\partial\omega = 0$ and $\bar\partial$-exact if it can be written as $\omega = \bar\partial f$.

Let E, F be locally convex spaces. By $H^1(F, \mathcal{O}_F^E)$, we denote the quotient space of the space of $\bar\partial$-closed C^∞-forms ω of type (0, 1) on F with values in E by the space of those which are $\bar\partial$-exact, where $\mathcal{O}_F^E$ denotes the sheaf of germs of E-valued holomorphic functions on F.

The main result of the note is the following:

Theorem 1. *Let E be a Fréchet nuclear space having the property (DN) and F a Fréchet nuclear space with property* (Ω). *Then*

$$H^1(F^*, \mathcal{O}_{F^*}^{E^*}) = 0.$$

We need the following:

Lemma 2. *Let E be a Fréchet nuclear space having the property (DN) and F a Fréchet nuclear space with property* (Ω). *Let W be a balanced convex neighborhood of* $0 \in F^*$. *Then for every continuous linear map*

$$T : E \to Z^1(W), \text{ where } Z^1(W) = \{\omega \in C^{(0,1)}(W) : \bar\partial\omega = 0\},$$

there exists a neighborhood V of 0 in W such that $T : E \to Z^1(V)$ can be lifted to a continuous linear map $S : E \to C^\infty(V)$.

Proof. Following [6], F^* is isomorphic to a subspace of s^*, where s is the space of rapidly decreasing sequences. Choose an open polydisc D in s^* such that $D \cap F^* \subset W$ and the image of the restriction map $\mathcal{O}(W) \to \mathcal{O}(D \cap F^*)$ is contained in $H = R(\mathcal{O}(D))$, where $R : \mathcal{O}(D) \to \mathcal{O}(D \cap F^*)$ is the restriction map. Such a map exists because the family of all open polydiscs in s^* forms a neighborhood basis of $0 \in s^*$ [4] and by the nuclearity of s^*. Note that $\mathcal{O}(D) \in (\Omega)$ [4], and hence, $H \in (\Omega)$, where H is equipped with the quotient topology.

The following argument is a modification of [6]. Put $H^1 = \mathcal{O}(W)$. It is a nuclear Fréchet space. For each $k \geq 1$, $H^1_k = \hat{(}H^1/\ker\|\,.\,\|_k)$ is the Banach space associated to the kth semi-norm $\rho_k : H^1 \to H^1_k$ and $\rho_{n,k} : H^1_n \to H^1_k$, $n > k$ are the canonical maps.

Following [2], there exists an exact sequence

$$0 \to H^1 \to C^\infty(W) \overset{\overline{\partial}}{\to} Z^1(W) \to 0.$$

Consider the fiber product

$$P = \{(x, y) \in C^\infty(W) \times E : \overline{\partial} x = Ty\}$$

and the canonical projections $\alpha : P \to E$, $\beta : P \to C^\infty(W)$.

It also follows that the sequence

$$0 \to H^1 \to P \overset{\alpha}{\to} E \to 0$$

is exact.

For each $k \geq 1$, since $\rho_k : H^1 \to H^1_k$ is nuclear, mapping ρ_k can be extended to $\Phi \in L(P, H^1_k)$.

Let $\tilde{\Psi}_k = \rho_{k+1,k} \circ \Phi_{k+1} - \Phi_k \in L(P, H^1_k)$. Since $P/H^1 \cong E$ and $\tilde{\Psi}_k|_{H^1} = 0$, hence, $\tilde{\Psi}$ induces a continuous linear map $\overline{\Psi}_k : E \to H^1_k$ which is nuclear. Since $H^1_k \subset H_k = \hat{(}H/\ker\|\,.\,\|_k)$, we can consider $\Phi_k \in L(P, H_k)$, $\overline{\Psi}_k \in L(E, H_k)$. Because $H \in (\Omega)$, and hence, by [6], there exists a continuous linear map Q from s onto H. Moreover, we can assume that, for each k, we have an induced quotient map $Q_k : s_k \to H_k$, where

$$s_k = \{x = (x_1, \dots, x_n, \dots) : \|x\|_k = \sup_j |x_j| j^k < \infty\},$$

with the norm $\|\,.\,\|_k$.

Because of its nuclearity, $\overline{\Psi}_k$ can be lifted to $\Psi_k \in L(E, s_k)$. Write $\Psi_k = (\Psi^k_1, \Psi^k_2, \dots)$, where $\Psi^k_j \in E^*$ and $\{j^k \Psi^k_j : j = 1, 2, \dots\}$ is equi-continuous on E. Hence, by changing index, we can consider that

$$j^k \Psi^k_j \subset U^0_k,$$

where $\{U_k\}$ is a decreasing neighborhood basis of $0 \in E$ and U^0_k is the polar of U_k.

Since E has property (DN), we can find a neighborhood U of $0 \in E$ such that

$$U_k^0 \subset rU^0 + \frac{2^{-k-2}}{r} U_{k+1}^0,$$

for all $r > 0$, $k \geq 1$. Choosing $r = j2^{-k-1}$, we obtain, after multiplication by $2j^{-k}$,

$$2j^{-k} U_k^0 \subset j^{-k+1} 2^{-k} U^0 + j^{-(k+1)} U_{k+1}^0 \qquad (*)$$

for all $j, k \in N$.

For each j fixed, $j \geq 1$, we determine inductively a sequence $A_j^k \in E^*$ with $A_j^k \in j^{-k} U_k^0$. We start with $A_j^0 = 0$. If $A_j^k \in j^{-k} U_k^0$, we have $\Psi_j^k + A_j^k \in 2j^{-k} U_k^0$.

From (*), we can find $A_j^{k+1} \in j^{-(k+1)} U_{k+1}^0$ such that

$$\Psi_j^k + A_j^k - A_j^{k+1} \in j^{-k+1} 2^{-k} U^0.$$

Defining $A_k x = (A_1^k(x), A_2^k(x), \ldots)$, we obtain an $A_k \in L(E, s_k)$. Now, we define

$$\tilde{A}_k = Q_k \circ A_k \circ \alpha \in L(P, H_k),$$
$$\Pi_k = \Phi_k - \tilde{A}_k \in L(P, H_k).$$

For $x \in \alpha^{-1}(U)$, we have

$$\begin{aligned} \|\rho_{k-1,k}\Pi_{k+1}(x) - \Pi_k(x)\|_{k-1} &= \|\tilde{\Psi}_k(x) + \tilde{A}_k(x) - \tilde{A}_{k+1}(x)\|_{k-1} \\ &= \|(\Psi_k + A_k - A_{k+1})\alpha x\|_{k-1} \leq 2^{-k}. \end{aligned}$$

It follows that

$$\lim_{\substack{k\to\infty \\ k>n}} (\rho_{k,n} \circ \Pi_k)x$$

exists for every $n \geq 1$ and every $x \in P$. Put

$$\tilde{\Pi}_n(x) = \lim_{\substack{k\to\infty \\ k>n}} (\rho_{k,n} \circ \Pi_k)x \quad \text{for } x \in P.$$

We have $\tilde{\Pi}_n \in L(P, H_n)$. Since $\rho_{n+1,n}\tilde{\Pi}_{n+1} = \tilde{\Pi}_n$ and $H = \lim \operatorname{proj}(H_n, \rho_{n,n-1})$, there exists $\Pi \in L(P, H)$ with $\tilde{\Pi}_n = \rho_n \circ \Pi$ for $n \geq 1$. For $x \in H^1$, we have

$$\tilde{\Pi}_n(x) = \lim_{\substack{k\to\infty \\ k>n}} (\rho_{k,n} \circ \Phi_k(x)) = \rho_n(x) \quad \text{for } n \geq 1.$$

So $\Pi(x, 0) = x|_V$ for $x \in H^1 = \mathcal{O}(W)$, where $V = D \cap F^*$. This yields that $T : E \to Z^1(V)$ is lifted to $S \in L(E, C^\infty(V))$.

Proof of Theorem 1. By [2], we have the exact sequences

$$0 \to \mathcal{O}_{F^*} \to C^{\infty}_{F^*} \to \ker \overline{\partial}^1_{F^*} \to 0,$$

$$0 \to \mathcal{O}(F^*) \to C^{\infty}(F^*) \to \ker \hat{\overline{\partial}}^1_{F^*} \to 0.$$

Lemma 2 and the Vogt's splitting theorem [6] imply that the sequences

$$0 \to \mathcal{O}^{E^*}_{F^*} \to C^{\infty E^*}_{F^*} \to \ker \overline{\partial}^{1E^*}_{F^*} \to 0$$

$$0 \to \mathcal{O}(F^*)\hat{\otimes}_{\epsilon} E^* \to C^{\infty}(F^*)\hat{\otimes}_{\epsilon} E^* \to \ker \hat{\overline{\partial}}^1_{F^*}\hat{\otimes}_{\epsilon} E^* \to 0$$

are exact. This yields

$$H^1(F^*, \mathcal{O}^{E^*}_{F^*}) = 0,$$

and Theorem 1 is completely proved. ■

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