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Multipoint Boundary-value Problems for Transferable Differential-algebraic Equations II — Quasilinear Case*

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Abstract. This paper deals with the solvability and approximate solution of multipoint boundary-value problems (BVPs) for quasilinear differential-algebraic equations (DAEs). If the corresponding linear multipoint BVPs for transferable DAEs are regular, then under certain hypotheses, the Schauder fixed point principle ensures the solvability of multipoint BVPs for quasilinear DAEs. Otherwise, in irregular cases, a Tikhonov iterative regularization process can be implemented for finding approximate solutions of quasilinear multipoint BVPs.

1. Introduction

This paper is the second part of our work on multipoint *boundary value problems* (BVPs) for *differential-algebraic equations* (DAEs) [1]. As already mentioned in [1], the paper is motivated by a series of works by März and her colleagues on two-point BVPs for DAEs (see [6, 7] for an extensive bibliography) and Sweet's results on multipoint BVPs for ordinary differential equations (ODEs for short) [8]. It is also closely related to our early works on nonlinear BVPs [2–5].

Consider the following multipoint BVP for quasilinear differential algebraic system:

$$Lx := A(t)x' + B(t)x = f(x,t), \quad t \in J := [t_0, T], \tag{1.1}$$

$$\Gamma x := \int_{t_0}^{T} d\eta(t)x(t) = \gamma, \tag{1.2}$$

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where $A, B \in C(J, \mathbb{R}^{n\times n})$ are continuous matrix-valued functions, $\eta \in BV(J, \mathbb{R}^{n\times n})$ is a matrix-valued function of bounded variations, $\gamma \in \mathbb{R}^n$, and $f : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$ is a nonlinear vector-valued function. By the Riesz theorem, the left-hand side of (1.2) represents a general form of linear bounded operators from $C := C(J, \mathbb{R}^n)$ to $\mathbb{R}^n$. In what follows, we assume the pair of matrices $\{A, B\}$ satisfies the transferability conditions [2], i.e.,

(1) there exists a continuously differentiable projector-function $Q \in C^1(J, \mathbb{R}^{n\times n})$, i.e., $Q^2(t) = Q(t)$, such that $\operatorname{Im} Q(t) = \operatorname{Ker} A(t)$ for all $t \in J$;
(2) the matrix $G := A + BQ$ is non-singular for all $t \in J$.

Let $P := I - Q$, then $P \in C^1(J, \mathbb{R}^n)$ is also a projector-function and $PQ = QP = 0$. Since $Ax' = APx' = A(Px)' - AP'x$, we should ask for solutions of (1.1) belonging to the Banach space:

$$\mathcal{X} := \{x \in C : Px \in C^1(J, \mathbb{R}^n)\}$$

with the norm $\|x\| := \|x\|_\infty + \|(Px)'\|_\infty$. In what follows, we use the expression Ax' as an abbreviation of $A\{(Px)' - P'x\}$.

Denote $Q_s(t) := Q(t)G^{-1}(t)B(t)$; $P_s(t) := I - Q_s(t)$ and let Y be the fundamental solution matrix of the ordinary initial valued problem (IVP for short):

$$Y' = (P'P_s - PG^{-1}B)Y; \; Y(t_0) = I.$$

It has been shown [2] that $X(t) := P_s(t)Y(t)P(t_0)$ is the fundamental solution matrix, whose columns belong to $\mathcal{X}$, satisfying the relations:

$$A(t)X' + B(t)X(t) = 0; \; P(t_0)(X(t_0) - I) = 0.$$

Moreover, $\operatorname{Ker} X(t) = \operatorname{Ker} A(t_0)$ for every $t \in J$.

In the remainder of this section, we recall the main results of [1]. Denote by D the so-called shooting matrix $\int_{t_0}^T d\eta(t)X(t)$ and by $\mathcal{R}_0$ the (closed) subspace $\{\int_{t_0}^T d\eta(t)x(t) : x \in C\}$ of $\mathbb{R}^n$. Let $\mathcal{Y} := C \times \mathbb{R}^n$ be a Banach space with the norm

$$\left\| \begin{pmatrix} q \\ \gamma \end{pmatrix} \right\| := \|q\|_\infty + |\gamma|$$

and $\mathcal{Y}_0 := C \times \mathcal{R}_0$ a closed subspace of $\mathcal{Y}$. Here, $|\gamma|$ denotes an arbitrary norm of $\gamma \in \mathbb{R}^n$; however, for definiteness, we shall use the max-norm.

Theorem 1.1. *The operator*

$$\mathcal{L}x := \begin{pmatrix} Lx \\ \Gamma x \end{pmatrix}$$

acting from $\mathcal{X}$ into $\mathcal{Y}_0$ is continuously invertible if and only if the shooting matrix D satisfies conditions:

$$\operatorname{Ker} D = \operatorname{Ker} A(t_0); \operatorname{Im} D = \mathcal{R}_0. \tag{1.3}$$

Now, let us turn to the irregular case where at least one of the conditions (1.3) does not hold. Since $\operatorname{Ker} A(t_0) \subset \operatorname{Ker} D$, for the sake of simplicity, we suppose $\dim\operatorname{Ker} A(t_0) = \nu < \dim\operatorname{Ker} D = p$. Let $\{w_i^0\}_{i=1}^\nu$ be an orthonormal basis of $\operatorname{Ker} A(t_0)$, i.e., $w_i^{0T} w_j^0 = \delta_{ij}$, where, in what follows, the superscript T denotes transposition. An extension of $\{w_i^0\}_{i=1}^\nu$ to an orthonormal basis of $\operatorname{Ker} D$ will be denoted as $\{w_i^0\}_{i=1}^p$. Define a column matrix

$$\Phi(t) := (\varphi_{\nu+1}(t), \ldots, \varphi_p(t)),$$

where $\varphi_i(t) := X(t)w_i^o$ and put $M := \int_{t_0}^T \Phi^T(t)\Phi(t)dt$.

It is easy to see [1] that M is nonsingular and $\mathcal{X}$ can be decomposed into a direct sum of closed subspaces $\mathcal{X} = \mathrm{Ker}\mathcal{L} \oplus \mathrm{Ker}\mathcal{W}$, where

$$(\mathcal{W}x)(t) := \Phi(t)M^{-1}\int_{t_0}^T \Phi^T(s)x(s)ds$$

and

$$\mathrm{Ker}\mathcal{L} = \{x = \Phi(t)a : a \in \mathbb{R}^{p-\nu}\}.$$

Further, $\{w_i\}_{i=1}^p$ will denote an orthonormal basis of $\mathrm{Ker}D^T$ and $W_0 := (w_{\nu+1}^o, \ldots, w_p^o)$, $W := \begin{pmatrix} w_1^T \\ \vdots \\ w_p^T \end{pmatrix}$ are $n \times (p-\nu)$ and $p \times n$ matrices, respectively.

Theorem 1.2. *The following assertions hold:*

(1) *The operator*

$$\mathcal{L}x := \begin{pmatrix} Lx \\ \Gamma x \end{pmatrix}$$

mapping $\mathcal{X}$ into $\mathcal{Y}$ is a bounded linear Noëther operator, and

$$\mathrm{Ind}\mathcal{L} = \mathrm{dimKer}\mathcal{L} - \mathrm{codimIm}\,\mathcal{L} = -\nu.$$

(2) *The linear multipoint BVP:*

$$A(t)x' + B(t)x = q(t); \quad \int_{t_0}^T d\eta(t)x(t) = \gamma, \tag{1.4}$$

with the given data $q \in C$, $\gamma \in \mathbb{R}^n$ is solvable if and only if

$$W\left(\gamma - \int_{t_0}^T d\eta(t)F(t)\right) = 0,$$

where

$$F(t) := X(t)\int_{t_0}^t Y^{-1}(s)P(s)[I + P'(s)]G^{-1}(s)q(s)ds + Q(t)G^{-1}(t)q(t).$$

(3) *A general solution of (1.4) can be represented as:*

$$x(t) = X(t)(\overline{x}_0 + W_0\alpha) + F(t) + \Phi(t)a, \tag{1.5}$$

where

$$\overline{x}_0 = \widehat{D}^{-1}\left(\gamma - \int_{t_0}^T d\eta(t)F(t)\right),$$

$\widehat{D}$ is the restriction of D into $\mathrm{Im}\,D^T$, a is an arbitrary vector of $\mathbb{R}^{p-\nu}$, and

$$\alpha = -M^{-1}\int_{t_0}^T \Phi^T(s)\{X(s)\overline{x}_0 + F(s)\}ds.$$

2. Multipoint BVPs for Quasilinear DAEs with Linear Regular Parts

Consider multipoint BVP (1.1)–(1.2) with a transferable pair of matrices $\{A, B\}$. The triplet $\{A, B, \eta\}$ is said to be regular if the regularity condition (1.3) holds.

Theorem 2.1. *Suppose $\gamma \in \mathcal{R}_0$ and the triplet $\{A, B, \eta\}$ is regular. Further, assume the nonlinear function $f : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$ satisfies the following conditions:*

$$|f(x,t)| \le c_1|x|^\mu + c_2; \quad \forall x \in \mathbb{R}^n, \forall t \in J, \tag{2.1}$$

where μ, c_1, c_2 are constants and $0 \le \mu < 1$.

$$|f(x,t_1) - f(x,t_2)| \le \omega(r, |t_1 - t_2|); \ \forall t_1, t_2 \in J; \ \forall x \in \mathbb{R}^n, \ |x| \le r, \tag{2.2}$$

where $\omega(r,s) : \mathbb{R}^+ \times [0, T - t_0] \to \mathbb{R}^+$ is continuous in s for every fixed $r \in \mathbb{R}^+ := [0,\infty)$ and $\omega(r, 0) = 0$.

Finally, suppose the derivative $f'_x(x,t)$ is a continuous matrix-function; moreover,

$$\mathrm{Ker} A(s) \subset \mathrm{Ker} f'_x(x,t); \ \forall x \in \mathbb{R}^n; \quad \forall t, s \in J. \tag{2.3}$$

Then there exists at least one solution of (1.1) and (1.2).

Proof. Since the triplet $\{A, B, \eta\}$ is regular, from Theorem 1.1, it follows that $\mathcal{L}$ possesses a bounded inverse. Thus, problem (1.1)–(1.2) is reduced to a fixed-point equation:

$$x = \mathcal{L}^{-1}\mathcal{F}(x) \tag{2.4}$$

where $\mathcal{F} : C \to C \times \mathbb{R}^n$ is defined by

$$\mathcal{F}(x) := \begin{pmatrix} f(x(t), t) \\ \gamma \end{pmatrix}, \quad t \in J.$$

Using the growth condition (2.1), we have

$$\forall x \in \mathcal{X}, \forall t \in J \quad |f(x(t), t)| \le c_1|x(t)|^\mu + c_2 \le c_1\|x\|_\infty^\mu + c_2,$$

therefore,

$$\|\mathcal{L}^{-1}\mathcal{F}(x)\| \le \|\mathcal{L}^{-1}\| \max_{t\in J}(|f(x(t),t)| + |\gamma|) \le \|\mathcal{L}^{-1}\|(c_1\|x\|^\mu + c_2 + |\gamma|)$$

or equivalently,

$$\|\mathcal{L}^{-1}\mathcal{F}(x)\|/\|x\| \le \|\mathcal{L}^{-1}\|\{c_1\|x\|^{\mu-1} + (c_2 + |\gamma|)\|x\|^{-1}\}.$$

Since the right-hand side of the last inequality tends to zero as $\|x\| \to \infty$, this inequality implies $\mathcal{L}^{-1}\mathcal{F}$ maps a closed ball $\mathcal{B}_r := \{x \in \mathcal{X} : \|x\| \le r\}$ with a sufficiently large radius $r > 0$ into itself. Further, to show the relative compactness of the set $\mathcal{L}^{-1}\mathcal{F}(\mathcal{B}_r)$, it suffices to prove that the set $\mathcal{N}$ of all functions $t \mapsto f(x(t), t)$ for $x \in \mathcal{B}_r$ is relatively compact in $C(J, \mathbb{R}^n)$.

First, we observe that the uniform boundedness of $\mathcal{N}$ is immediately implied from the following estimates:

$$\forall x \in \mathcal{B}_r, \forall t \in J, \ |f(x(t), t)| \le c_1|x(t)|^\mu + c_2 \le c_1 r^\mu + c_2.$$

Further, as $f'_x(x,t)$ is continuous on the compact $\Delta := \{(x,t) : |x| \le r; t \in J\}$, there exists a constant $C_r > 0$ such that

$$|f'_x(x,t)| \le C_r; \;\; \forall (x,t) \in \Delta. \tag{2.5}$$

Using (2.3) and (2.5) and taking into account the fact that $\|(Px)'\|_\infty \le \|x\| \le r$ for $x \in \mathcal{B}_r$, we find

$$\begin{aligned}
&\forall x \in \mathcal{B}_r,\ \forall t, \bar{t} \in J,\ |f(x(t),t) - f(x(\bar{t}),\bar{t})| \le |f(x(t),t) - f(x(\bar{t}),t)| \\
&+ |f(x(\bar{t}),t) - f(x(\bar{t}),\bar{t})| \le \Big| \int_0^1 f'_x(x(\bar{t}) + s(x(t) - x(\bar{t})), t)(x(t) - x(\bar{t}))ds \Big| \\
&+ \omega(r, |t-\bar{t}|) = \Big| \int_0^1 f'_x(x(\bar{t}) + s(x(t) - x(\bar{t})), t)(P(t)x(t) - P(\bar{t})x(\bar{t}))ds \Big| \\
&+ \omega(r, |t-\bar{t}|) \le C_r |P(t)x(t) - P(\bar{t})x(\bar{t})| \\
&+ \omega(r, |t-\bar{t}|) \le rC_r|t-\bar{t}| + \omega(r, |t-\bar{t}|).
\end{aligned}$$

Thus, the equi-continuity of $\mathcal{N}$ is proved. Applying the well-known Schauder fixed point theorem, we come to the conclusion of Theorem 2.1. ■

Remark. Conditions (2.1)–(2.3) are trivially fulfilled for a special case, where $f(x,t) \equiv q(t)$, $(t \in J)$, and $q \in C(J, \mathbb{R}^n)$.

Example. We consider problem (1.1)–(1.2) with the following data:

$$A = \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -t \\ 0 & 0 & 0 \end{pmatrix};\ B = \begin{pmatrix} 1 & -(1+t) & t^2+2t \\ 0 & -1 & t-1 \\ 0 & 0 & 1 \end{pmatrix};\ J := [0,1] \tag{2.6}$$

$$f(x,t) = (a(t)(x_1^2+1)^{\mu/2} + q_1(t), q_2(t), q_3(t))^T, \tag{2.7}$$

where $\mu \in (0,1)$; $a, q_i \in C(J, \mathbb{R})$ and $a(t) > 0$ for every $t \in J$. The boundary condition is given as

$$\int_0^1 x_i(s)ds = \gamma_i \ \ (i = 1,\ 2)\,.$$

Obviously,

$$d\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} dt\,. \tag{2.8}$$

It has been verified [1] that the pair $\{A, B\}$ is transferable and the triplet $\{A, B, \eta\}$ is regular. It is easy to see that conditions (2.1) and (2.2) are fulfilled. Further, since

$$f'_x(x,t) = \begin{pmatrix} \mu a(t) x_1 (1+x_1^2)^{\mu/2-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it follows that for any $x \in \mathbb{R}^3$ and $t, s \in J$

$$\operatorname{Ker} f'_x(x,t) = \begin{cases} \operatorname{Span}\{(0,1,0)^T;\ (0,0,1)^T\} & x_1 \ne 0 \\ \mathbb{R}^3 & x_1 = 0\,. \end{cases}$$

Therefore, $\mathrm{Ker}A(s) = \mathrm{Span}\{(0, s, 1)^T\} \subset \mathrm{Ker} f'_x(x, t), \forall x \in \mathbb{R}^3, \forall t, s \in J$ and condition (2.3) holds.

By Theorem 2.1, problem (1.1) and (1.2) with the data given by (2.6)–(2.8) is solvable for any $\gamma_1, \gamma_2 \in \mathbb{R}$.

3. Multipoint BVPs for Quasilinear DAEs with Linear Irregular Parts

In this section, we deal with problem (1.1)–(1.2), where the triplet $\{A, B, \eta\}$ is irregular. The reader is referred to previous sections for some notations and results.

Suppose the function $f(x, t)$ and its derivative $f'_x(x, t)$ are continuous on the set $\Delta := \{(x, t) : x \in \mathbb{R}^n;\ |x| \le R;\ t \in J\}$. Let $H : C \to C$ be an operator of Hammerstein–Nemyski type defined on the set $\mathcal{B}_0 := \{x \in C : \|x\|_\infty \le R\}$ by

$$H(x)(t) := X(t) \int_{t_0}^{t} Y^{-1}(s)P(s)(I + P'(s))G^{-1}(s) f(x(s), s)ds$$
$$+ Q(t)G^{-1}(t) f(x(t), t); \quad t \in J.$$

For $x \in \mathcal{B}_0$, we denote by $E(x)$ the matrix

$$E(x) = -W \int_{t_0}^{T} d\eta(t) \Big\{ X(t) \int_{t_0}^{t} Y^{-1}(s)P(s)[I + P'(s)]G^{-1}(s) f'_x(x(s), s)\Phi(s)\, ds$$
$$+ Q(t)G^{-1}(t) f'_x(x(t), t)\Phi(t) \Big\}.$$

Further, assume the initial approximation $x_0(t)$ is given and the m-approximation $x_m(t)$ $(m \ge 0)$ has been found. We decompose $x_m(t)$ into two components $x_m(t) = u_m(t) + \Phi(t)a_m$, where $u_m \in \mathrm{Ker}\mathcal{W}$, $a_m \in \mathbb{R}^{p-\nu}$. Suppose $x_m \in \mathcal{X}$ and $(x_m(t), t) \in \Delta$ for every $t \in J$, we define the following quantities:

$$f_m(t) := f(x_m(t); t); \quad F_m(t) := H(x_m)(t);$$
$$\gamma_m := \gamma - \sum_{i=1}^{p} w_i^T \left(\gamma - \int_{t_0}^{T} d\eta(t) F_m(t) \right) w_i;$$
$$E_m := E(x_m); \quad N_m := E_m^T E_m + \alpha_m I;$$
$$\Psi_m := W \left(\gamma - \int_{t_0}^{T} d\eta(t) F_m(t) \right).$$

Here, for the sake of brevity, we denote the $n \times n$ and $(p - \nu) \times (p - \nu)$ identity matrices by the same symbol I. They will be easily recognized from the context.

Then the first component of the next approximation can be determined from the equation

$$\mathcal{L}u_{m+1} = \begin{pmatrix} f_m \\ \gamma_m \end{pmatrix}. \tag{3.1}$$

From Theorem 1.2, it follows that problem (3.1) is solvable and its general solution can be represented as

$$u_{m+1}(t) = X(t)(\overline{x}_{0,m} + W_0\omega_m) + F_m(t)$$

where

$$\bar{x}_{0,m} = \widehat{D}^{-1}\left(\gamma_m - \int_{t_0}^{T} d\eta(t)F_m(t)\right)$$

and

$$\omega_m = -M^{-1}\int_{t_0}^{T} \Phi^T(t)\{X(t)\bar{x}_{0,m} + F_m(t)\}dt.$$

Following [5], we find the second component by an iterative regularization process:

$$a_{m+1} = a_m - N_m^{-1}\{E_m^T\Psi_m + \alpha_m(a_m - a^{(0)})\} \tag{3.2}$$

where $a^{(0)} \in \mathbb{R}^{p-\nu}$ is a specially chosen element and $\{\alpha_m\}_1^\infty$ are regularization parameters.

Finally, the $(m+1)$-approximation can be computed as

$$x_{m+1}(t) = u_{m+1}(t) + \Phi(t)a_{m+1} \quad (m \geq 0). \tag{3.3}$$

It should be emphasized that process (3.1)–(3.3) is always well defined, however, its convergence can be established only in particular cases. For example, we have the following result:

Theorem 3.1. *Assume problem (1.1)–(1.2) possesses a solution* $x^* \in \mathcal{X}$ *such that* $\|x^*\|_\infty < R$ *and* $f_x'(x^*(t), t) = 0$ *for every* $t \in J$. *Further, let*

$$|f_x'(x,t) - f_x'(\bar{x},t)| \leq l|x - \bar{x}| \quad \forall (x,t), (\bar{x},t) \in \Delta.$$

Then process (3.1)–(3.3) with a specially chosen sequence $\{\alpha_m\}$ *and*

$$a^{(0)} = M^{-1}\int_{t_0}^{T} \Phi^T(s)x^*(s)ds$$

will be locally convergent.

Proof. Let $\varepsilon > 0$ be a fixed small number, such that $\varepsilon < l(R - \|x^*\|_\infty)$. Define two sets $\Delta_0 := \{(x,t) : t \in J;\ |x - x^*(t)| \leq r\}$ and $\mathcal{B} := \{x \in \mathcal{X} : \|x - x^*\|_\infty \leq r\}$, where $r := \varepsilon/l$. Obviously, $\Delta_0 \subset \Delta$ and $\mathcal{B} \subset \mathcal{B}_0$; moreover, for an arbitrary fixed $x \in \mathcal{B}$ and for any $t \in J$, $(x(t), t) \in \Delta_0$.

Observing that for every $(x,t) \in \Delta_0$, $|f_x'(x,t)| = |f_x'(x,t) - f_x'(x^*(t),t)| \leq l|x - x^*(t)| \leq \varepsilon$, we come to the inequality:

$$|f_x'(x,t)| \leq \varepsilon, \quad \forall (x,t) \in \Delta_0. \tag{3.4}$$

Further, if $x, \bar{x} \in \mathcal{B}$, then for any $t \in J$ and $s \in [0,1]$, $(\bar{x}(t) + s(x(t) - \bar{x}(t)), t) \in \Delta_0$. By virtue of (3.4) and the relation $|f(x(t),t) - f(\bar{x}(t),t)| = |\int_0^1 f_x'(\bar{x}(t) + s(x(t) - \bar{x}(t)), t)(x(t) - \bar{x}(t))ds|$ we obtain

$$|f(x(t),t) - f(\bar{x}(t),t)| \leq \varepsilon|x(t) - \bar{x}(t)|. \tag{3.5}$$

Suppose the m-approximation $x_m \in \mathcal{B}$ has been found ($x_0 \in \mathcal{B}$ given). The next $(m+1)$-approximation will be constructed by formulae (3.1)–(3.3). Since $x_m, x^* \in \mathcal{B}$, relation (3.5) implies

$$\|f_m - f^*\|_\infty \le \varepsilon\|x_m - x^*\|_\infty \le \varepsilon\|x_m - x^*\|, \tag{3.6}$$

where $f^*(t) := f(x^*(t), t)$.

Further, let $F^*(t) := H(x^*)(t)$; $\Psi^* := W(\gamma - \int_{t_0}^T d\eta(t)F^*(t))$ and $x^*(t) = u^*(t) + \Phi(t)a^*$, where $u^* \in \mathrm{Ker}\mathcal{W}$, $a^* \in \mathbb{R}^{p-\nu}$. From Theorem 1.2, it is easy to conclude that $\Psi^* = 0$.

Using (3.1) and taking into account the fact that $\mathcal{L}u^* = \begin{pmatrix} f^* \\ \gamma \end{pmatrix}$, we find

$$\|u_{m+1} - u^*\| \le \mathcal{H}(\|f_m - f^*\|_\infty + |\gamma_m - \gamma|). \tag{3.7}$$

Here, $\mathcal{H}$ denotes a bound of $\|\widehat{\mathcal{L}}^{-1}\|$, where $\widehat{\mathcal{L}}$ is the restriction of $\mathcal{L}$ into $\mathrm{Ker}\mathcal{W}$. It is not difficult to express $\mathcal{H}$ in terms of the initial data $\{A, B, \eta\}$ and their related quantities $X(t), Y(t), P(t), Q(t), G(t), D, \ldots$ Before continuing our calculation, we note that

$$|w| \le |w|_2 \le \sqrt{p}|w| \qquad \forall w \in \mathbb{R}^p, \tag{3.8}$$

$$|E^T| \le p|E| \qquad \forall E \in \mathbb{R}^{p\times(p-\nu)} \tag{3.9}$$

$$|M| \le \sqrt{p-\nu}|M|_2 \quad \forall M \in \mathbb{R}^{(p-\nu)\times(p-\nu)}, \tag{3.10}$$

where, as usual, $|.|$ and $|.|_2$ stand for the max-norm and Euclidean norm of vectors or matrices, respectively.

Using (3.8) and noting that $|w_i|_2 = 1$ $(i = 1, \ldots, p)$, we obtain

$$\begin{aligned}
|\gamma_m - \gamma| &= \Big|\sum_{i=1}^p w_i^T \int_{t_0}^T d\eta(t)(F_m(t) - F^*(t))w_i\Big| \le \sum_{i=1}^p \Big|w_i^T \int_{t_0}^T d\eta(F_m - F^*)\Big| \\
&\le \sum_{i=1}^p \Big|w_i^T \int_{t_0}^T d\eta(F_m - F^*)\Big|_2 \\
&\le \sum_{i=1}^p \Big|\int_{t_0}^T d\eta(F_m - F^*)\Big|_2 \le p\sqrt{p}\Big|\int_{t_0}^T d\eta(F_m - F^*)\Big| \\
&\le \left(p\sqrt{p} \vee_{t_0}^T \eta\right)\|F_m - F^*\|_\infty,
\end{aligned}$$

where $\vee_{t_0}^T \eta$ is the total variation of η on J.

On the other hand, it follows from (3.5) that

$$\begin{aligned}
|F_m(t) - F^*(t)| &= \Big|X(t)\int_{t_0}^t Y^{-1}P(I+P')G^{-1}[f(x_m(s), s) - f(x^*(s), s)]ds \\
&\qquad + QG^{-1}[f(x_m(t), t) - f(x^*(t), t)]\Big| \\
&\le \varepsilon\{\|X\|_\infty \int_{t_0}^T |Y^{-1}P(I+P')G^{-1}|ds + \|QG^{-1}\|_\infty\}\|x_m - x^*\|,
\end{aligned}$$

therefore,

$$|\gamma_m - \gamma| \leq \varepsilon c_1 \|x_m - x^*\|, \tag{3.11}$$

where $c_1 := c_2 \left(p\sqrt{p} \vee_{t_0}^T \eta\right)$ and

$$c_2 := \|X\|_\infty \int_{t_0}^T |Y^{-1}P(I+P')G^{-1}|ds + \|QG^{-1}\|_\infty.$$

Combining (3.6) and (3.11) together with (3.7), we obtain

$$\|u_{m+1} - u^*\| \leq \varepsilon \mathcal{H}(1+c_1)\|x_m - x^*\|. \tag{3.12}$$

Now, let us deal with the second component a_{m+1}. First, note that

$$\Psi_m = \Psi_m - \Psi^* = W \int_{t_0}^T d\eta(t)[H(x^*)(t) - H(x_m)(t)]. \tag{3.13}$$

It is easy to prove that $H : \mathcal{B}_0 \to C$ is continuously differentiable and for any $x \in \mathcal{B}_0,\ h \in C$,

$$\begin{aligned}\left(H_x'(x)h\right)(t) = X(t) \int_{t_0}^t Y^{-1}(s)P(s)\left[I + P'(s)\right]G^{-1}(s) f_x'(x(s),s)h(s)\,ds \\ + Q(t)G^{-1}(t) f_x'(x(t),t)h(t), \quad t \in J.\end{aligned} \tag{3.14}$$

Further, from (3.14), it follows that

$$\|H_x'(x)\| \leq c_2\,\varepsilon, \quad \forall x \in \mathcal{B}.$$

Finally, it is obvious that

$$\|H_x'(x) - H_x'(\bar{x})\| \leq c_2 l \|x - \bar{x}\|_\infty, \quad \forall x, \bar{x} \in \mathcal{B}_0. \tag{3.15}$$

Using the Lipschitz continuity of $f_x'(x,t)$ on Δ, we have

$$f(x^*(t),t) - f(x_m(t),t) = f_x'(x_m(t),t)\big(x^*(t) - x_m(t)\big) + g_m(t), \quad \forall t \in J,$$

where

$$g_m(t) = \int_0^1 \left[f_x'(x_m(t) + \tau(x^*(t) - x_m(t)), t) - f_x'(x_m(t),t) \right](x^*(t) - x_m(t))\,d\tau$$

and

$$|g_m(t)| \leq l|x_m(t) - x^*(t)|^2 \int_0^1 \tau\,d\tau \leq \frac{l}{2}\|x_m - x^*\|_\infty^2.$$

Taking into account the expression of $f(x^*(t),t) - f(x_m(t),t)$ and the estimation of $g_m(t)$, we find

$$\begin{aligned}&H(x^*)(t) - H(x_m)(t)\\ &= X(t) \int_{t_0}^t Y^{-1}(s)P(s)\left[I + P'(s)\right]G^{-1}(s) f_x'(x_m(s),s)(x^*(s) - x_m(s))\,ds\\ &\quad + Q(t)G^{-1}(t) f_x'(x_m(t),t)(x^*(t) - x_m(t)) + \bar{g}_m(t),\end{aligned}$$

where

$$\bar{g}_m(t) := X(t)\int_{t_0}^{t} Y^{-1}(s)P(s)[I + P'(s)]G^{-1}(s)g_m(s)\,ds + Q(t)G^{-1}(t)g_m(t),$$

and

$$\|\bar{g}_m\|_\infty \le c_2\|g_m\|_\infty \le \frac{c_2}{2}l\|x_m - x^*\|^2_\infty. \tag{3.16}$$

Using the decomposition $x_m - x^* = u_m - u^* + \Phi(t)(a_m - a^*)$, we can rewrite (3.13) as

$$\begin{aligned}\Psi_m - \Psi^* &= W\int_{t_0}^{T} d\eta[H'_x(x_m)(t)(x^* - x_m) + \overline{g}_m]\\ &= W\Big\{\int_{t_0}^{T} d\eta H'_x(x_m)(t)\Phi(t)(a^* - a_m)\\ &\quad + \int_{t_0}^{T} d\eta H'_x(x_m)(t)(u^* - u_m) + \int_{t_0}^{T} d\eta\overline{g}_m\Big\}.\end{aligned}$$

Putting $K_m = W\int_{t_0}^{T} d\eta H'_x(x_m)(t)(u^* - u_m)$ and $\widetilde{g}_m = W\int_{t_0}^{T} d\eta\overline{g}_m$ and observing that

$$\begin{aligned}W\int_{t_0}^{T} d\eta H'_x(x_m)\Phi(a^* - a_m) &= \Big\{W\int_{t_0}^{T} d\eta(t)\Big[X(t)\int_{t_0}^{t} Y^{-1}(s)P(s)[I + P'(s)]\\ &\quad G^{-1}(s)f'_x(x_m(s), s)\Phi(s)\,ds + Q(t)G^{-1}(t)f'_x(x_m(t), t)\Phi(t)\Big]\Big\}(a^* - a_m)\\ &= -E_m(a^* - a_m),\end{aligned}$$

we can reduce (3.2) to the form:

$$a_{m+1} = a_m - N_m^{-1}\{E_m^T[E_m(a_m - a^*) + K_m + \widetilde{g}_m] + \alpha_m(a_m - a^{(0)})\}. \tag{3.17}$$

Recalling the hypothesis of Theorem 3.1 that $a^{(0)} = a^*$, we can rewrite (3.17) as

$$a_{m+1} - a^* = -N_m^{-1}E_m^T K_m - N_m^{-1}E_m^T\widetilde{g}_m. \tag{3.18}$$

By virtue of (3.9), we have $|E_m^T| \le p|E_m| \le c_3\varepsilon$, where $c_3 := pc_2|W|\|\Phi\|_\infty \vee_{t_0}^{T}\eta$. On the other hand using (3.10), we obtain

$$\begin{aligned}|N_m^{-1}| = |(E_m^T E_m + \alpha_m I)^{-1}| &\le \sqrt{p - \nu}\,|(E_m^T E_m + \alpha_m I)^{-1}|_2\\ &\le \sqrt{p - \nu}\max_{\lambda \ge 0}\frac{1}{\lambda + \alpha_m}\\ &= \sqrt{p - \nu}.\alpha_m^{-1}.\end{aligned}$$

From (3.16), it follows that

$$|\widetilde{g}_m| \le \left(|W| \vee_{t_0}^{T}\eta\right)\|\bar{g}_m\|_\infty \le \left(\frac{c_2 l}{2}|W| \vee_{t_0}^{T}\eta\right)\|x_m - x^*\|^2,$$

therefore,

$$|N_m^{-1}E_m^T\widetilde{g}_m| \le c_4\varepsilon\alpha_m^{-1}\|x_m - x^*\|^2,$$

where

$$c_4 = \frac{c_2 c_3 l}{2}\sqrt{p-\nu}|W| \vee_{t_0}^{T} \eta.$$

Observing that $f_x'(x^*(t), t) = 0$ for every $t \in J$, hence, $\big(H_x'(x^*)(u^* - u_m)\big)(t) \equiv 0$ and taking into account inequality (3.15), we have

$$|K_m| = |W \int_{t_0}^{T} d\eta\{H_x'(x_m)(t) - H_x'(x^*)(t)\}(u^* - u_m)|$$
$$\leq \big(c_2 l |W| \vee_{t_0}^{T} \eta\big) \|x_m - x^*\| \|u_m - u^*\|.$$

Thus,

$$|N_m^{-1} E_m^T K_m| \leq \frac{c_5 \varepsilon}{\alpha_m} \|x_m - x^*\| \|u_m - u^*\|,$$

where $c_5 := c_2 c_3 l \sqrt{p-\nu}|W| \vee_{t_0}^{T} \eta$. From (3.18) and the last estimate, it follows that

$$|a_{m+1} - a^*| \leq \frac{c_4 \varepsilon}{\alpha_m} \|x_m - x^*\|^2 + \frac{c_5 \varepsilon}{\alpha_m} \|x_m - x^*\| \|u_m - u^*\|. \tag{3.19}$$

Now, let $\varepsilon > 0$ be sufficiently small, such that

$$\varepsilon < \frac{q}{2} \min\left\{\frac{1}{(c_4 + c_5/2)c_6}; \frac{1}{\mathcal{H}(1 + c_1)}\right\}, \tag{3.20}$$

where $q \in (0, 1)$ is an arbitrary fixed number and $c_6 := \|\Phi\|_\infty + \|(P\Phi)'\|_\infty$. Recalling that $\Phi(t) = (\varphi_{\nu+1}(t), \ldots, \varphi_p(t))$ with $\varphi_i(t) = X(t) w_i^0$, we find $(P\Phi)' = (PX)' W_0$. Let

$$\widetilde{\mathcal{B}} := \{x \in \mathcal{X} : \|u - u^*\| \leq r/2;\ |a - a^*| \leq r(2c_6)^{-1}\}$$

and $\mathcal{B}_1 := \{x \in \mathcal{X} : \|x - x^*\| \leq r\}$.

Clearly, $\widetilde{\mathcal{B}} \subset \mathcal{B}_1 \subset \mathcal{B}$. Let $x_0 \in \widetilde{\mathcal{B}}$ be an arbitrary initial approximation. We choose the regularization parameters $\alpha_m = rq^m$. The following relations will be proved by induction:

$$x_m \in \mathcal{B}_1 \quad (m \geq 0); \tag{3.21}$$

$$\|u_m - u^*\| \leq rq^m/2; \tag{3.22}$$

$$|a_m - a^*| \leq rq^m (2c_6)^{-1}. \tag{3.23}$$

For $m = 0$, relations (3.21)–(3.23) automatically hold. Assume (3.21)–(3.23) are satisfied for $k \leq m$. From (3.12), (3.20) and (3.22), it follows that $\|u_{m+1} - u^*\| \leq \varepsilon \mathcal{H}(1 + c_1) rq^m \leq rq^{m+1}/2$. Further, using (3.19) and taking into account (3.22) together with the fact that

$$\|x_m - x^*\| \leq \|u_m - u^*\| + c_6 |a_m - a^*| \leq rq^m, \tag{3.24}$$

we find

$$|a_{m+1} - a^*| \leq \frac{c_4 \varepsilon}{rq^m}(rq^m)^2 + \frac{c_5 \varepsilon}{rq^m} rq^m \frac{rq^m}{2} = \varepsilon(c_4 + c_5/2) rq^m \leq rq^{m+1}(2c_6)^{-1}.$$

Finally, $\|x_{m+1} - x^*\| \leq \|u_{m+1} - u^*\| + c_6 |a_{m+1} - a^*| \leq rq^{m+1} \leq r$, therefore, $x_{m+1} \in \mathcal{B}_1$. Thus, relations (3.21)–(3.23) hold for every $m \geq 0$. Since $\|x_m - x^*\| \leq rq^m$, the local convergence of process (3.1)–(3.3) is proved. ∎

Example. We are concerned with the problem (1.1)–(1.2) with the same pair of matrices $\{A, B\}$ as in (2.6) and a three-point boundary condition $D_1x(0)+D_2x(\frac{1}{2})+D_3x(1) = 0$, where

$$D_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{e} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; D_3 = \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $f(x, t) = (x_1 - \sin x_1, x_2^2, x_3(e^{x_3} - 1))^T$. It is easy to show that

$$\eta(t) = \begin{cases} -D_1 & t = 0 \\ 0 & 0 < t < \frac{1}{2} \\ D_2 & \frac{1}{2} \le t < 1 \\ D_2 + D_3 & t = 1. \end{cases} \tag{3.25}$$

In this case, the triplet $\{A, B, \eta\}$ is irregular [1]. Clearly, $x^*(t) \equiv 0$ is an exact solution of our problem. Further,

$$f'_x(x, t) = \begin{pmatrix} 1 - \cos x_1 & 0 & 0 \\ 0 & 2x_2 & 0 \\ 0 & 0 & e^{x_3} - 1 + x_3e^{x_3} \end{pmatrix},$$

therefore, $f'_x(x^*, t) = 0$ for every $t \in J := [0, 1]$. Obviously, $f'_x(x, t)$ satisfies the Lipschitz condition in x on each domain $\Delta := \{(x, t) : |x| \le R;\ t \in J\}$. Theorem 3.1 ensures the local convergence of process (3.1)–(3.3) with $a^{(0)} = 0$ and an initial approximation $x_0(t)$ sufficiently close to 0.

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Corrigendum

The author has noticed some computational errors concerning matrices $PG^{-1}B$, Y, X, D on p. 355 in [1]. Fortunately, the conclusions on the solvability of BVP (1), (2) with given data (19), (20) or (19), (21) remain true. Further, the matrice D_1 in three-point boundary condition (22) on p. 357 in [1] should be changed as in Example 3.1 of the present paper. The necessary and sufficient conditions for the solvability of the above-mentioned three-point BVP should read:

$$\gamma_2 = e^{\frac{1}{2}} q_3 \left(\frac{1}{2}\right) /2 + e \int_0^{\frac{1}{2}} se^{-s} q_2(s) ds; \; \gamma_s = q_3 \left(\frac{1}{2}\right) + q_3(1).$$

The author apologies to the reader for the errors.