Vietnam Journal of Mathematics **26**:4 (1998) 323–335

Vietnam Journal of MATHEMATICS

Convergence in Pettis Norm Under Extreme Point Condition

Allal Amrani [1], Charles Castaing [2], and Michel Valadier [2]

[1] *University of Mohammed V, Rabat, Morocco*
[2] *Department of Mathematics, University of Montpellier II*
F-34095 Montpellier, Cedex 5, France

Received November 3, 1997

Abstract. Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space, E a separable Banach space, and E' the topological dual of E. Under an extreme point condition, we show that a sequence of Pettis μ-integrable E-valued functions defined on Ω, which converges in the topology of pointwise convergence on $L^\infty_{\mathbf{R}}(\mu) \otimes E'$, converges in Pettis norm and in φ-measure respectively. We also give a version of Olech's lemma in Pettis integration.

1. Introduction

Throughout this paper, E will denote a separable Banach space, E' the topological dual of E, $\overline{B}_{E'}$ the closed unit ball in E' and $(\Omega, \mathcal{F}, \mu)$ a complete probability space. The symbol $P^1_E(\Omega, \mathcal{F}, \mu)$ ($P^1_E(\mu)$ for short) denotes the space of E-valued Pettis μ-integrable functions $f : \Omega \to E$ endowed with the Pettis norm $\|f\|_{Pe} = \sup_{x' \in \overline{B}_{E'}} \int_\Omega |\langle x', f\rangle| \, d\mu$ (see, for example, [10, 13, 16–19, 21, 22]). Strong convergence results in the space $L^1_E(\mu)$ of Lebesgue–Bochner functions related to extreme point condition and strict convexity have been studied extensively (see, for example, [1, 3–6, 8, 9, 11, 20, 24–27]). The purpose of this paper is to characterize convergence in Pettis norm and in φ-measure for Pettis integrable functions under extreme point condition (resp. denting point condition) and to give a version of Olech's lemma in Pettis integration.

We will need the following definitions and notations. A subset $\mathcal{H} \subset P^1_E(\mu)$ is *Pettis uniformly integrable* (PUI for short) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mu(A) \leq \delta \Longrightarrow \sup_{u \in \mathcal{H}} \|1_A u\|_{Pe} \leq \varepsilon.$$

If $f \in P^1_E(\mu)$, the singleton $\{f\}$ is PUI since the set $\{\langle x', f\rangle : \|x'\| \leq 1\}$ is uniformly integrable [18, p. 82]. Let us mention a more general fact. If $\mathcal{H}$ is a subset of $P^1_E(\mu)$

satisfying

$$\lim_{a\to\infty}\sup_{f\in\mathcal{H}}\sup_{x'\in\overline{B}_{E'}}\int_{\{|\langle x',f\rangle|>a\}}|\langle x',f\rangle|\,d\mu=0,$$

then $\mathcal{H}$ is Pettis uniformly integrable. Recall $\|1_A u\|_{Pe}=\sup_{x'\in\overline{B}_{E'}}\int_A|\langle x',u\rangle|\,d\mu$. For any $x'\in\overline{B}_{E'}$, one has

$$\int_A|\langle x',u\rangle|\,d\mu=\int_{A\cap\{|\langle x',u\rangle|\le a\}}|\langle x',u\rangle|\,d\mu+\int_{A\cap\{|\langle x',u\rangle|>a\}}|\langle x',u\rangle|\,d\mu. \qquad (*)$$

Let a be large enough in order to ensure

$$\forall x'\in\overline{B}_{E'},\ \forall u\in\mathcal{H},\quad \int_{\{|\langle x',u\rangle|>a\}}|\langle x',u\rangle|\,d\mu\le\varepsilon/2.$$

Thus, the last term of $(*)$ is $\le\varepsilon/2$. Now, if δ is small enough in order to ensure $a\delta\le\varepsilon/2$, we obtain

$$\int_{A\cap\{|\langle x',u\rangle|\le a\}}|\langle x',u\rangle|\,d\mu\le a\mu(A)\le\varepsilon/2$$

as soon as $\mu(A)\le\delta$.

By $\mathcal{L}wc(E)$ (resp. $cwk(E)$ and $ck(E)$), we will denote the set of all nonempty closed convex locally weakly compact containing no line [14] (resp. weakly compact and norm compact) subsets of E. If K is a convex subset of E, $\partial_{ent}(K)$ (resp. $\partial_{ext}(K)$) is the set of denting (resp. extremal) points of K. A multifunction $\Gamma:\Omega\to cwk(E)$ is scalarly integrable if, for every $x'\in E'$, the scalar function

$$\omega\mapsto\delta^*(x',\Gamma(\omega)):=\sup\{\langle x',x\rangle : x\in\Gamma(\omega)\}$$

is $\mathcal{F}$-measurable and μ-integrable. We denote by $\mathcal{S}_\Gamma^{Pe}$ the set of all Pettis integrable selections of Γ. If $\mathcal{S}_\Gamma^{Pe}$ is nonempty, the *integral of* Γ *over a* $\mathcal{F}$*-measurable set* A is defined by

$$\int_A\Gamma\,d\mu:=\left\{\int_A f\,d\mu : f\in\mathcal{S}_\Gamma^{Pe}\right\},$$

where $\int_A f\,d\mu$ is the Pettis integral of f over A.

A scalarly integrable multifunction: $\Gamma:\Omega\to cwk(E)$ is *Pettis-integrable* if the set

$$\{\delta^*(x',\Gamma) : \|x'\|\le 1\}$$

is uniformly integrable in $L^1_{\mathbf{R}}(\mu)$.

A sequence (u_n) in $P^1_E(\mu)$ *weakly* converges to $u\in P^1_E(\mu)$ if $u_n\to u$ in the topology of pointwise convergence on $L^\infty_{\mathbf{R}}(\mu)\otimes E'$.

A subset $\mathcal{H}$ in $P^1_E(\mu)$ is $cwk(E)$-tight (resp. $ck(E)$-tight) if, for every $\varepsilon>0$, there exists a $cwk(E)$-valued (resp. $ck(E)$-valued) Pettis-integrable multifunction Γ_ε satisfying

$$\sup_{u\in\mathcal{H}}\mu(\{\omega\in\Omega : u(\omega)\notin\Gamma_\varepsilon(\omega)\})\le\varepsilon.$$

We will summarize the following basic results.

The following result is a *sequentially compact* version of an analogous compactness result given in Theorem V.13 in [13].

Theorem 1.1. *Suppose E is a separable Banach space and $\Gamma : \Omega \to cwk(E)$ is a Pettis integrable multifunction, then the set $\mathcal{S}_\Gamma^{Pe}$ is nonempty and sequentially compact for the topology of pointwise convergence on $L^\infty_{\mathbf{R}}(\mu) \otimes E'$. In particular, if Γ is $ck(E)$-valued, then the integral*

$$\int_\Omega \Gamma \, d\mu := \left\{ \int_\Omega f \, d\mu : f \in \mathcal{S}_\Gamma^{Pe} \right\}$$

is convex and compact in E.

Proof. (See [12, Theorem 4.2, Remark 1 of Theorem 4.4] and [13]).
Let us mention a useful fact.

Fact 1.2. Suppose $L : \Omega \to \mathcal{L}wc(E)$ is a measurable multifunction, $(u_n)_{n\in\mathbf{N}}$ a sequence of scalarly integrable E-valued selections of L, and $u : \Omega \to E$ a scalarly integrable function such that

$$\lim_{n\to\infty} \int_A \langle x', u_n \rangle \, d\mu = \int_A \langle x', u \rangle \, d\mu$$

for every $A \in \mathcal{F}$ and every $x' \in E'$, then $u(\omega) \in L(\omega)$ μ a.e.

Proof. Suppose the conclusion is not true. Then by Lemma III. 34 in [14], there exist $x' \in E'$, $A \in \mathcal{F}$ with $\mu(A) > 0$ such that

$$\langle x', u(\omega) \rangle > \delta^*(x', L(\omega)) := \sup\{\langle x', x \rangle : x \in L(\omega)\}$$

for all $\omega \in A$. By integrating the above inequality, we obtain

$$\int_A \delta^*(x', L) \, d\mu < \int_A \langle x', u \rangle \, d\mu. \tag{1.1}$$

Since $\langle x' u_n \rangle$ converges $\sigma(L^1, L^\infty)$ to $\langle x', u \rangle$ and the u_n are Pettis selections of L, we deduce that

$$\int_A \delta^*(x', L) \, d\mu \geq \lim_{n\to\infty} \int_A \langle x', u_n \rangle \, d\mu = \int_A \langle x', u \rangle \, d\mu, \tag{1.2}$$

which contradicts (1.1). ■

Theorem 1.3. *Suppose $\Gamma : \Omega \to cwk(E)$ is a scalarly integrable multifunction such that $\mathcal{S}_\Gamma^{Pe}$ is nonempty, then*

$$\partial_{ext}(\mathcal{S}_\Gamma^{Pe}) = \mathcal{S}^{Pe}_{\partial_{ext}(\Gamma)}.$$

Proof. Since the inclusion $\mathcal{S}^{Pe}_{\partial_{ext}(\Gamma)} \subset \partial_{ext}(\mathcal{S}_\Gamma^{Pe})$ is obvious, it remains to prove the inverse inclusion. Assume by contradiction that there is $f \in \partial_{ext}(\mathcal{S}_\Gamma^{Pe}) \setminus \mathcal{S}^{Pe}_{\partial_{ext}(\Gamma)}$. Since the graph of the multifunction $\partial_{ext}(\Gamma)$ belongs to $\mathcal{F} \otimes \mathcal{B}(E)$ because E is separable [14, Corollary IV.5], the set

$$A = \{\omega \in \Omega : f(\omega) \notin \partial_{ext}(\Gamma)(\omega)\}$$

is $\mathcal{F}$-measurable with $\mu(A) > 0$. By the same construction as in Theorem IV.14 in [13], there are two $A \cap \mathcal{F}$-measurable selections g and h of Γ such that

$$g(\omega) \neq h(\omega) \text{ and } f(\omega) = \frac{1}{2}[g(\omega) + h(\omega)]$$

for all $\omega \in A$. Since $\|g\|$ and $\|h\|$ are $\mathcal{F}$-measurable, we find $a > 0$ and $B \in A \cap \mathcal{F}$ with $\mu(B) > 0$ such that, $\forall \omega \in B$, $\sup\{\|g(\omega)\|, \|h(\omega)\|\} \leq a$. Set

$$g_1 = 1_B g + 1_{\Omega \setminus B} f \text{ and } g_2 = 1_B h + 1_{\Omega \setminus B} f.$$

Then $g_i \in \mathcal{S}_\Gamma^{Pe}$ $(i = 1, 2)$, $g_1 \neq g_2$ and $f = \frac{1}{2}(g_1 + g_2)$. This proves that $f \notin \partial_{ext}(\mathcal{S}_\Gamma^{Pe})$, contradicting the extreme nature of f. ■

We will need the following lemma which is borrowed from Lemma 3 in [3 p. 172].

Lemma 1.4. *Let $(u_n)_{n \in \mathbf{N}}$ be a sequence of real $\mathcal{F}$-measurable functions converging in measure to a $\mathcal{F}$-measurable function u on a $\mathcal{F}$-measurable set A. If $u(\omega) < 0$ a.e. on A, then, for every $\eta > 0$, there exist $a < 0$ and $k_1 \in \mathbf{N}$ such that*

$$k \geq k_1 \Longrightarrow \mu(\{\omega \in A : u_k(\omega) > a\}) < \eta.$$

2. Main Results

We will first present a version of the Lebesgue–Vitali theorem in $P_E^1(\mu)$.

Proposition 2.1. *Suppose $(u_n)_{n \in \mathbf{N}}$ is a PUI sequence in $P_E^1(\mu)$ converging in measure to $u_\infty \in P_E^1(\mu)$, then $\|u_n - u_\infty\|_{Pe} \to 0$.*

Proof. Suppose the conclusion does not hold. There exists $\varepsilon > 0$ such that, for a subsequence still denoted by $(u_n)_n$, $\forall n$, $\|u_n - u_\infty\|_{Pe} > \varepsilon$. Since $(u_n)_{n \in \mathbf{N}}$ is PUI, there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $\forall n \in \mathbf{N} \cup \{\infty\}$, $\|1_A u_n\|_{Pe} \leq \varepsilon/3$. There exists a subsequence still not relabeled such that $\|u_n(.) - u_\infty(.)\| \to 0$ almost everywhere (a.e. for short). By virtue of Egorov's theorem, there exists $B \in \mathcal{F}$ such that $\mu(\Omega \setminus B) < \delta$ and $\|u_n(.) - u_\infty(.)\| \to 0$ uniformly on B. Let n_0 be such that $\forall n \geq n_0$, $\|u_n(\omega) - u_\infty(\omega)\| \leq \varepsilon/3$ on B. Then $\forall n \geq n_0$, and we have

$$\begin{aligned}\|u_n - u_\infty\|_{Pe} &\leq \|1_B(u_n - u_\infty)\|_{Pe} + \|1_{\Omega\setminus B}(u_n - u_\infty)\|_{Pe} \\ &\leq \varepsilon/3 + \|1_{\Omega\setminus B} u_n\|_{Pe} + \|1_{\Omega\setminus B} u_\infty\|_{Pe} \leq \varepsilon.\end{aligned}$$

This contradicts the initial assumption. ■

We begin with the following lemma formulated for simplicity in a special case.

Lemma 2.2. *Let E be a separable Banach space, $\Gamma : \Omega \to ck(E)$ a Pettis integrable multifunction, and (u_n) a sequence in $\mathcal{S}_\Gamma^{Pe}$ which weakly converges to $u \in \mathcal{S}_\mathcal{F}^{Pe}$. Suppose $u \in \partial_{ext}(\mathcal{S}_\Gamma^{Pe})$, then $\|u_n - u\|_{Pe} \to 0$.*

Proof. Without loss of generality, we may suppose $u \equiv 0$ so that, by Theorem 1.3, $0 \in \partial_{ext}(\Gamma(\omega))$ a.e. Since the sequence (u_n) is Pettis uniformly integrable, by Proposition 2.1, we need only to prove that $\|u_n(.)\| \to 0$ in measure. Suppose not. Then there exist $\varepsilon > 0$ and $\eta > 0$ such that

$$\mu(\{w \in \Omega : \|u_n(\omega)\| \geq \varepsilon\}) \geq \eta$$

for infinitely many n, namely, there exists an infinite subset $S \subset \mathbf{N}$ such that the preceding inequality holds for all $n \in S$. Let us consider the following Pettis integrable functions

$$v_n := 1_{\{\omega \in \Omega : u_n(\omega) \notin B(0,\varepsilon)\}} u_n$$

and $w_n := u_n - v_n$. By virtue of Theorem 1.1, the sequence (v_n) is relatively sequentially $\sigma(P^1_E(\mu), L^\infty_{\mathbf{R}}(\mu) \otimes E')$ compact. By extracting an appropriate subsequence, we may suppose v_n converges to $v \in P^1_E(\mu)$ for this topology. It follows that w_n weakly converges to $-v$. Since $0 \in \partial_{ext}(\Gamma(\omega))$ a.e. and u_n weakly converges to 0, we obtain $v = 0$. Since $\partial_{ext}(\Gamma(\omega)) = \partial_{ent}(\Gamma(\omega))$ because $\Gamma(\omega)$ is convex norm compact, $0 \in \partial_{ent}(\Gamma(\omega))$ a.e. Hence, $0 \notin \overline{\text{co}}(\Gamma(\omega) \setminus B(0, \varepsilon))$ a.e., where $B(0, \varepsilon)$ is the open ball of center 0 and radius ε. Now, we set $A = \{\omega \in \Omega : \Gamma(\omega) \setminus B(0, \varepsilon) \neq \emptyset\}$. Then the multifunction $\Sigma : \omega \to \overline{\text{co}}(\mathcal{F}(\omega) \setminus B(0, \varepsilon))$ from A to $ck(E)$ has its graph in $(A \cap \mathcal{F}) \otimes \mathcal{B}(E)$, hence, the multifunction Ψ defined on A with nonempty values in $\overline{B}_{E'}$ (thanks to the Hahn–Banach theorem):

$$\Psi(\omega) = \{x' \in \overline{B}_{E'} : \delta^*(x', \Sigma(\omega)) < 0\}$$

has its graph in $(A \cap \mathcal{F}) \otimes \mathcal{B}(\overline{B}_{E'})$, where $\overline{B}_{E'}$ is endowed with the topology of compact convergence on E'. Hence, Σ admits a $\mathcal{F}$ measurable selection $\sigma : A \mapsto \overline{B}_{E'}$ (see, for example, [14, Theorem III. 22]). Since $\overline{B}_{E'}$ is compact metrizable for the topology of compact convergence, there is a sequence (σ_k) of simple $\mathcal{F}$-measurable mappings from A to $\overline{B}_{E'}$ such that σ_k pointwise converges to σ for the compact convergence. It follows that

$$\lim_{k \to \infty} \delta^*(\sigma_k(\omega), \Sigma(\omega)) = \delta^*(\sigma(\omega), \Sigma(\omega)) < 0$$

for every $\omega \in A$. Then applying Lemma 1.4 to the sequence $(\delta^*(\sigma_k(.), \Sigma(.)))_k$ provides $a < 0$ and $k_1 \in \mathbf{N}$ such that

$$\forall k \geq k_1,\ \mu(\{\omega \in A : \delta^*(\sigma_k(\omega), \Sigma(\omega)) > a\}) < \frac{\eta}{2}.$$

Let $k \geq k_1$ be fixed and set

$$A_k = \{\omega \in A : \delta^*(\sigma_k(\omega), \Sigma(\omega)) \leq a\} \text{ and } B_k = A \setminus A_k\,.$$

Then we have

$$\limsup_{n \to \infty} \langle \sigma_k(\omega), v_n(\omega) \rangle \leq 0$$

for all $\omega \in A_k$. Since σ_k is a simple function with values in $\overline{B}_{E'}$ and v_n converges $\sigma(P^1_E(\mu), L^\infty_{\mathbf{R}}(\mu) \otimes E')$ to 0, $(\langle \sigma_k, v_n \rangle)_n$ converges $\sigma(L^1_{\mathbf{R}}(A_k), L^\infty_{\mathbf{R}}(A_k))$ to 0. From Théorème 26 in [15, p. 44], it follows that $(\langle \sigma_k, v_n \rangle)_n$ converges to 0 in measure on A_k. Consequently, there exists N_1 such that

$$\forall n \geq N_1 \Longrightarrow \mu(\{\omega \in A_k : \langle \sigma_k(\omega), v_n(\omega) \rangle \leq a\}) < \frac{\eta}{2}.$$

We have

$$\begin{aligned}\{\omega \in A : v_n(\omega) \neq 0\} &= \{\omega \in A_k : v_n(\omega) \neq 0\} \cup \{\omega \in B_k : v_n(\omega) \neq 0\}\\ &\subset \{\omega \in A_k : v_n(\omega) \neq 0\} \cup B_k\\ &\subset \{\omega \in A_k : \langle \sigma_k(\omega), v_n(\omega)\rangle \leq a\} \cup B_k .\end{aligned}$$

For $n \geq N_1$, we have

$$\begin{aligned}\mu(\{\omega \in A : v_n(\omega) \neq 0\}) &\leq \mu(\{\omega \in A_k : \langle \sigma_k(\omega), v_n(\omega)\rangle \leq a\}) + \mu(B_k)\\ &< \frac{\eta}{2} + \frac{\eta}{2} = \eta.\end{aligned}$$

Hence, for $n \geq N_1$, $n \in S$, we obtain the contradiction

$$\mu(\{\omega \in \Omega : u_n(\omega) \notin B(0,\varepsilon)\}) = \mu(\{\omega \in A : v_n(\omega) \neq 0\}) < \eta. \quad \blacksquare$$

Remark. It is worthy to mention an alternative proof of Lemma 2.2 via a new notion of K-convergence developed by Balder [6], which we include in the following for the sake of completeness.

Since the sequence (u_n) is Pettis uniformly integrable, by Proposition 2.1, we need only to prove that $\|u_n(.) - u(.)\| \to 0$ in measure. Balder captures Theorem 2.5 in [6, pp. 29–30] the case of scalarly integrable functions. We reproduce here his arguments.

Since E is separable, the functions u_n are strongly measurable and the associated Young measures ε_{u_n} are unambiguously defined. The set they form $\{\varepsilon_{u_n} : n \in \mathbf{N}\}$ is norm tight (E being equipped of its norm) because $\Gamma(\omega) \in ck(E)$ for every $\omega \in \Omega$. By Theorem A.5 in [6, p.41], there exists an infinite subset S of $\mathbf{N}$ and a Young measure δ_*, such that the subsequence $(\varepsilon_{u_m})_{m\in S}$ K-converges to δ_*, which is written as

$$\varepsilon_{u_m} \xrightarrow{K} \delta_* .$$

Because $\delta_*(\omega)$ is carried by $\Gamma(\omega)$, it has a barycenter bar $\delta_*(\omega)$. Let $(x'_j)_j$ be a $\sigma(E', E)$-dense sequence in $\overline{B}_{E'}$. Since $\{\delta^*(x', \Gamma(.)) : x' \in \overline{B}_{E'}\}$ is uniformly integrable, so is $\{\langle x'_j, u_m(.)\rangle : m \in S\}$. As in [6, p. 27], the number depending on m, which is defined for any $B \in \mathcal{F}$ and $j \in \mathbf{N}$ by

$$\int_B \Big[\int_E \langle x'_j, x\rangle \varepsilon_{u_m}(dx)\Big] \mu(d\omega) = \int_\Omega \langle 1_B \otimes x'_j, u_m\rangle d\mu = \langle 1_B \otimes x'_j, u_m\rangle,$$

converges both to $\langle 1_B \otimes x'_j, u\rangle = \int_B \langle x'_j, u\rangle d\mu$ and to

$$\int_B \Big[\int_E \langle x'_j, x\rangle \delta_*(dx)\Big] \mu(d\omega) = \int_B \langle x'_j, \text{bar}\,\delta_*(\omega)\rangle\, \mu(d\omega).$$

Hence, bar $\delta_*(\omega) = u(\omega)$ a.e. Now, the remainder of the proof is the same: The whole sequence $(u_n)_n$ converges in measure to u (E being equipped of its norm). $\blacksquare$

By combining Theorem 1.1 and Lemma 2.2, we obtain a version of Olech's lemma in Pettis integration.

Theorem 2.3. *Suppose E is a separable Banach space, $\Gamma : \Omega \to ck(E)$ is a Pettis integrable multifunction, and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{S}_\Gamma^{Pe}$ satisfying:*

(a) $\lim_{n\to\infty} \int_\Omega u_n(\omega)\, d\mu = e$ *for the weak topology* $\sigma(E, E')$;

(b) $e \in \partial_{ext}\Big(\int_\Omega \Gamma(\omega)\, \mu(d\omega)\Big)$.

Then there exists a unique Pettis integrable selection u of Γ such that

$$e = \int_\Omega u(\omega)\, \mu(d\omega) \ \text{ and } \ \|u_n - u\|_{Pe} \to 0.$$

Proof. We will proceed in three steps.

Step 1. There exists a unique $u \in \mathcal{S}_\Gamma^{Pe}$ such that $e = \int_\Omega u\, d\mu$. Suppose not. Then there exist $u_i \in \mathcal{S}_\Gamma^{Pe}$ $(i = 1, 2)$ such that

$$u_1 \neq u_2 \ \text{ and } \ e = \int_\Omega u_1\, d\mu = \int_\Omega u_2\, d\mu\,. \tag{2.1}$$

Hence, there exist $x' \in E'$ and $A \in \mathcal{F}$ with $\mu(A) > 0$ such that

$$\langle x', u_2(\omega) - u_1(\omega)\rangle > 0 \tag{2.2}$$

for all $\omega \in A$. Now, we write

$$\begin{aligned} e &= \frac{1}{2}\Big(\int_\Omega u_1\, d\mu + \int_\Omega u_2\, d\mu\Big) \\ &= \frac{1}{2}\Big(\int_\Omega (1_A u_1 + 1_{\Omega\setminus A} u_2) d\mu\Big) + \frac{1}{2}\Big(\int_\Omega (1_{\Omega\setminus A} u_1 + 1_A u_2) d\mu\Big). \end{aligned} \tag{2.3}$$

Note that the functions $1_A u_1 + 1_{\Omega\setminus A} u_2$ and $1_{\Omega\setminus A} u_1 + 1_A u_2$ are Pettis integrable selections of Γ. Then by (b) and (2.3), we deduce that

$$\int_A u_1\, d\mu + \int_{\Omega\setminus A} u_2\, d\mu = \int_{\Omega\setminus A} u_1\, d\mu + \int_A u_2\, d\mu\,. \tag{2.4}$$

From (2.1) and (2.4), it follows that

$$\int_A u_1\, d\mu = \int_A u_2\, d\mu \tag{2.5}$$

which contradicts (2.3).

Step 2. $u(\omega) \in \partial_{ext}(\Gamma(\omega))$ a.e. Suppose not. Since the graph of the multifunction $\partial_{ext}(\Gamma(.))$ belongs to $\mathcal{F} \otimes B(E)$ [14, Corollary IV.5], the set

$$A = \{\omega \in \Omega : u(\omega) \notin \partial_{ext}(\Gamma(\omega))\}$$

is $\mathcal{F}$-measurable with $\mu(A) > 0$. This allows us to repeat the arguments in Theorem 1.3 providing two Pettis integrable selections g and h of Γ such that

$$g \neq h \quad \text{and} \quad u = \frac{1}{2}(g+h).$$

Let us consider a $\mathcal{F}$-measurable set $B \subset A$ of positive measure such that

$$\int_B g\,d\mu \neq \int_B h\,d\mu$$

and set

$$g_1 = 1_B g + 1_{\Omega\setminus B} u \quad \text{and} \quad g_2 = 1_B h + 1_{\Omega\setminus B} u.$$

Then we have

$$e = \int_\Omega u\,d\mu = \frac{1}{2}\Big(\int_\Omega g_1\,d\mu + \int_\Omega g_2\,d\mu \Big)$$

with $\int_\Omega g_1\,d\mu \neq \int_\Omega g_2\,d\mu$, thus contradicting the extreme nature of e.

Step 3. $\|u_n - u\|_{Pe} \to 0$. By Theorem 1.1, there exists a subsequence (u_{n_k}) of (u_n) which weakly converges to an element $v \in S_\Gamma^{Pe}$. By (a), it follows that $e = \int_\Omega v\,d\mu$. Using Step 1, we deduce that $v = u$. By Step 2, we may apply Lemma 2.2 to the sequence (u_{n_k}) providing

$$\lim_{k\to\infty} \|u_{n_k} - u\|_{Pe} = 0.$$

From what has been proved, any subsequence (v_n) of (u_n) admits a subsequence (w_n) such that $\|w_n - u\|_{Pe} \to 0$. So we may conclude that $\|u_n - u\|_{Pe} \to 0$. ■

Now, we proceed to a generalize Lemma 2.2.

Theorem 2.4. *Suppose E is a separable Banach space, $\Phi : \Omega \to \mathcal{L}wc(E)$ an $\mathcal{L}wc(E)$-valued measurable multifunction, $(u_n)_{n\in\mathbf{N}}$ a Pettis uniformly integrable and $ck(E)$-tight sequence in $P_E^1(\mu)$ which weakly converges to $u \in P_E^1(\mu)$ with $u_n(\omega) \in \Phi(\omega)$ $(n \in \mathbf{N})$ and $u(\omega) \in \partial_{ext}(\Phi(\omega))$ a.e., then $\|u_n - u\|_{Pe} \to 0$.*

Proof. We may suppose $u \equiv 0$ and $0 \in \Phi(\omega)$ for all $\omega \in \Omega$ because $\overline{\text{co}}(\Phi(\omega) \cup \{0\})$ still belongs to $\mathcal{L}wc(E)$. So we have $0 \in \partial_{ext}(\Phi(\omega))$ for almost anywhere (a.a. for short) $\omega \in \Omega$. Let $\varepsilon > 0$. Since $(u_n)_{n\in\mathbf{N}}$ is Pettis uniformly integrable, there exists $\delta > 0$ such that

$$\mu(A) < \delta \Longrightarrow \sup_{n\in\mathbf{N}} \|1_A u_n\|_{Pe} \leq \varepsilon. \tag{2.6}$$

By the tightness hypothesis, we find a $ck(E)$-valued Pettis integrable multifunction Γ_δ such that

$$\sup_{n\in\mathbf{N}} \mu(\{\omega \in \Omega : u_n(\omega) \notin \Gamma_\delta(\omega)\}) \leq \delta. \tag{2.7}$$

For every $n \in \mathbf{N}$, we set

$$v_n := 1_{\{\omega\in\Omega : u_n(\omega)\in\Gamma_\delta(\omega)\}} u_n,$$

$$w_n := 1_{\{\omega\in\Omega : u_n(\omega)\notin\Gamma_\delta(\omega)\}} u_n,$$

and

$$\Delta(\omega) := \Phi(\omega) \cap \overline{co}(\Gamma_\delta(\omega) \cup \{0\})$$

for all $\omega \in \Omega$. Then it is clear that the multifunction Δ has nonempty convex compact values and is Pettis integrable because

$$\forall x' \in E', \ 0 \leq \delta^*(x', \Delta) \leq \delta^*(x', \Gamma_\delta \cup \{0\}),$$

so that $\{\delta^*(x', \Delta) : \|x'\| \leq 1\}$ is uniformly integrable. Thus, $v_n \in \mathcal{S}_\Delta^{Pe}$ where $\mathcal{S}_\Delta^{Pe}$ denotes the set of Pettis integrable selections of the $ck(E)$-valued Pettis integrable multifunction Δ. By Theorem 1.1, we may suppose v_n weakly converges to $v \in \mathcal{S}_\Delta^{Pe}$, by extracting a subsequence if necessary. Hence, we have

$$0 = \text{weak-} \lim_{n\to\infty} u_n = \text{weak-} \lim_{n\to\infty} [v_n + w_n] = v + w$$

with $w \in \mathcal{S}_\Phi^{Pe}$ using Fact 1.2. Since $0 \in \partial_{ext}(\Phi(\omega))$ a.e., it follows that $v = w = 0$ a.e. and $0 \in \partial_{ext}(\Delta(\omega))$ a.e. This allows us to apply Lemma 2.2 to the sequence $(v_n)_n$ showing that $\|v_n\|_{Pe} \to 0$. Since

$$\|u_n\|_{Pe} \leq \|v_n\|_{Pe} + \|w_n\|_{Pe} \leq \|v_n\|_{Pe} + \varepsilon$$

for all $n \in \mathbf{N}$, using (2.6) and (2.7), and ε is arbitrary > 0, $\|u_n\|_{Pe} \to 0$. ■

Remarks. (1) Assume the hypotheses of Theorem 2.4 are satisfied. Then $\Sigma(\omega) := \overline{\text{co}}(\{u_n(\omega) : n \in \mathbf{N}\}) \in \mathcal{L}wc(E)$ for all $\omega \in \Omega$, $u(\omega) \in \partial_{ext}(\Sigma(\omega))$ a.e., and thanks to Fact 1.2, $u \in \mathcal{S}_\Sigma^{Pe}$.

(2) If $\Phi : \Omega \to cwk(E)$ is a scalarly integrable multifunction, $(u_n)_{n\in\mathbf{N}}$ is a scalarly integrable sequence of selections of Φ which weakly converges to a scalarly integrable E-valued function u, thanks to the measurability of $\overline{co}w\text{-}Ls(u_n(.))$ and the weak compactness of $\Phi(\omega)$ (see, for example, [3, Lemma 2]), we obtain more

$$u(\omega) \in \overline{\text{co}}w\text{-}Ls(u_n(\omega)) \text{ a.e.}$$

So it is natural to pose the question of the validity of Theorem 2.4 under the sole extremal condition

$$u(\omega) \in \partial_{ext}(\overline{\text{co}}w\text{-}Ls(u_n(\omega))) \text{ a.e.}$$

This is an open problem (compare Theorem 14 in [26]).

It is worthy to mention two useful properties of Pettis uniformly integrable sequences in $P_E^1(\mu)$, namely, we have the following.

(3) If $(u_n)_{n\in\mathbf{N}}$ is Pettis uniformly integrable and $cwk(E)$-tight (resp. $ck(E)$-tight), then, for every $A \in \mathcal{F}$, the sequence $\left(\int_A u_n \, d\mu\right)_{n\in\mathbf{N}}$ is relatively weakly compact (resp. norm compact). It is enough to prove this fact when $A = \Omega$. Let $\varepsilon > 0$. An easy inspection of the proof of the preceding theorem shows that $u_n = v_n + w_n$, where $\left(\int_\Omega v_n \, d\mu\right)_{n\in\mathbf{N}}$ is relatively weakly compact (resp. norm compact) by Theorem 1.1 and $\|w_n\|_{Pe} \leq \varepsilon$ for all $n \in \mathbf{N}$. It follows that the sequence $\left(\int_\Omega u_n \, d\mu\right)_{n\in\mathbf{N}}$ is relatively weakly compact (resp. norm compact) too. Even in the Bochner integration, the norm compactness of $\left(\int_A u_n \, d\mu\right)_{n\in\mathbf{N}}$ is useful in several places. See, for example, [7] in which the authors state the Pettis-norm convergence via Bocce criteria and the preceding compactness result.

(4) If $(u_n)_{n\in\mathbf{N}}$ is a sequence in $P_E^1(\mu)$ satisfying

(i) $\{\langle x', u_n\rangle : \|x'\| \le 1;\ n \in \mathbf{N}\}$ is uniformly integrable in $L^1_{\mathbf{R}}(\mu)$;

(ii) $\bigcup_{n\in\mathbf{N}}\{\int_A u_n\, d\mu\}$ is relatively weakly compact, for every $A \in \mathcal{F}$,

then there is a subsequence $(u_{\alpha(n)})$ such that $\int_A \langle x', u_{\alpha(n)}\rangle d\mu$ converges in $\mathbf{R}$ for every $A \in \mathcal{F}$ and every $x' \in E'$. We refer to [2, 13] for proof.

So by combining these remarks, we obtain a "conditionally weakly compact" criteria in $P_E^1(\mu)$, namely, suppose $(u_n)_{n\in\mathbf{N}}$ is a $cwk(E)$-tight sequence in $P_E^1(\mu)$ such that $\{\langle x', u_n\rangle : \|x'\| \le 1,\ n \in \mathbf{N}\}$ is uniformly integrable in $L^1_{\mathbf{R}}(\mu)$, then there exists a subsequence $(u_{\varphi(n)})$ such that $\int_A \langle x', u_{\varphi(n)}\rangle d\mu$ converges in $\mathbf{R}$ for every $A \in \mathcal{F}$ and every $x' \in E'$. See [9, 17] for other related results.

The following result shows that Pettis-norm convergence is implied by strict convexity.

Theorem 2.5. *Let $\varphi : \Omega \times E \to]-\infty, \infty]$ be a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand such that $\varphi(\omega, .)$ is convex lower semicontinuous on E for every fixed $\omega \in \Omega$. Let $(u_n)_{n\in\mathbf{N}}$ be a Pettis uniformly integrable and $ck(E)$-tight sequence in $P_E^1(\mu)$. Suppose* $\operatorname{epi}\varphi_\omega \in \mathcal{L}wc(E)$ *for every $\omega \in \Omega$, u_n weakly converges to $u \in P_E^1(\mu)$, the functions $\varphi(., u_n(.))$ and $\varphi(., u(.))$ are integrable, and $\varphi(., u_n(.))$ converges $\sigma(L^1, L^\infty)$ to $\varphi(., u(.))$ with*

$$(u(\omega), \varphi(\omega, u(\omega))) \in \partial_{ext}(\operatorname{epi}\varphi(\omega, .))\ \mu\text{-a.e.},$$

then $\|u_n - u\|_{Pe} \to 0$.

Remark. If $\varphi(\omega, .)$ is strictly convex, $(x, \varphi(\omega, x))$ is always an extremal point of $\operatorname{epi}\varphi(\omega, .)$ for all $x \in E$.

Proof. Since $\varphi(., u_n(.))$ converges $\sigma(L^1, L^\infty)$ to $\varphi(., u(.))$, applying Theorem 2.4 to the Pettis uniformly integrable $E \times \mathbf{R}$-valued functions

$$(u_n, \varphi(., u_n(.)) \text{ and } (u, \varphi(., u(.))$$

and to the multifunction $\Phi(\omega, .) = \operatorname{epi}\varphi(\omega, .)$ yields the desired result; the details are left to the reader. ∎

It would be interesting to address the question: What happens if one replaces $ck(E)$ by $cwk(E)$ in Theorem 2.4? This leads to the following variant.

Theorem 2.6. *Suppose E is a Banach space with strongly separable dual, $\Gamma : \Omega \to cwk(E)$ a Pettis integrable multifunction, (u_n) a sequence in S_Γ^{Pe} which weakly converges to $u \in S_\Gamma^{Pe}$ with $u \in \partial_{ext}(S_\Gamma^{Pe})$, and $\varphi : \Omega \times E \to \mathbf{R}^+$ an integrand satisfying*

(i) *φ is $\mathcal{F} \otimes \mathcal{B}(E)$-measurable;*

(ii) *for every $\omega \in \Omega$, $\varphi(\omega, .)$ is (E', E) continuous on E;*

(iii) *for every $\omega \in \Omega$, $\varphi(\omega, 0) = 0$;*

then $\varphi(., u_n(.) - u(.)) \to 0$ in measure.

Proof. We may suppose $u = 0$ and we will repeat the arguments of the proof of Lemma 2.2 with appropriate modifications. Suppose the conclusion is not true. Then there are $\varepsilon > 0$ and $\eta > 0$ such that

$$\mu(\{\omega \in \Omega : \varphi(\omega, u_n(\omega)) \geq \varepsilon\}) \geq \eta \tag{2.8}$$

for infinitely many n, namely, there exists an infinite subset $S \subset \mathbf{N}$ such that (2.8) holds for all $n \in S$. For every $\omega \in \Omega$, set

$$W_\omega(0, \varepsilon) := \{x \in E : \varphi(\omega, x) < \varepsilon\}. \tag{2.9}$$

Then by (ii) and (iii), $W_\omega(0, \varepsilon)$ is a weakly open neighborhood of 0, and by weak compactness of $\Gamma(\omega)$, we have $0 \notin \overline{\text{co}}(\Gamma(\omega) \setminus W_\omega(0, \varepsilon))$ [3, Lemma 1′]. Moreover, by (i), the multifunction $\omega \mapsto W_\omega(0, \varepsilon)$ has its graph in $\mathcal{F} \otimes \mathcal{B}(E)$ and (2.8) is equivalent to

$$\forall n \in S, \ \mu(\{\omega \in \Omega : u_n(\omega) \notin W_\omega(0, \varepsilon)\}) \geq \eta. \tag{2.10}$$

Set

$$A = \{\omega \in \Omega : \Gamma(\omega) \setminus W_\omega(0, \varepsilon) \neq \emptyset\}.$$

As in Lemma 2.2, let us define v_n and w_n in $P_E^1(\mu)$ by

$$v_n := 1_{\{\omega \in \Omega : u_n(\omega) \notin W_\omega(0,\varepsilon)\}} u_n$$

and $w_n := u_n - v_n$. By Theorem 1.1, the sequence (v_n) is relatively sequentially weakly compact. Hence, we may suppose v_n weakly converges to $v \in P_E^1(\mu)$ for this topology. It follows that w_n weakly converges to $-v$. Since $0 \in \partial_{ext}(\Gamma(\omega))$ a.e. and u_n weakly converges to 0, we obtain $v = 0$. Now, we can repeat the arguments of the proof of Lemma 2.2 invoking the separability of the strong dual of E to finish the proof. ■

Remark. It is also possible to reproduce the arguments involving the K-convergence for Young measures in the second proof of Theorem 2.3 which provide a better result since the strong separability hypothesis on the dual can be relaxed. The details are left to the reader who is familiar with this concept.

3. Convergence in Pettis Norm Under Denting Point Condition

In light of the preceding results, it would be interesting to address the following question: What happens if one replaces "extreme point condition" by the denting point condition? This leads to following partial analog.

Theorem 3.1. *Let E be a Banach space, K a closed convex subset of E, $e \in \partial_{ent}(K)$, and $(u_n)_{n \in \mathbf{N}}$ a Pettis uniformly integrable sequence in $P_E^1(\mu)$ such that $\forall n \in \mathbf{N}$, $\forall \omega \in \Omega$, $u_n(\omega) \in K$. Suppose $\|\int_\Omega u_n \, d\mu - e\| \to 0$, then $\|u_n - e\|_{Pe} \to 0$.*

Proof. Without loss of generality, we may suppose $e = 0$. By virtue of Proposition 2.1, it is sufficient to prove that $\|u_n(.)\| \to 0$ in measure. We will follow the arguments in Theorem 15 in [25]. Suppose not. Then there are $\varepsilon > 0$ and $\eta > 0$ such that

$$\mu(\{\omega \in \Omega : \|u_n(\omega)\| \geq \varepsilon\}) \geq \eta \qquad (*)$$

for infinitely many n, namely, there exists an infinite subset $S \subset \mathbf{N}$ such that $(*)$ holds for all $n \in S$. Set

$$\Omega_n := \{\omega \in \Omega : \|u_n(\omega)\| \geq \varepsilon\}.$$

If $\mu(\Omega_n) = 1$ for infinitely many $n \in S$, there exists an infinite subset S_1 of S such that

$$\int_\Omega u_n \, d\mu \in \overline{\text{co}}(K \setminus B(0, \varepsilon))$$

for all $n \in S_1$, hence, $\int_\Omega u_n \, d\mu$ does not converge to 0 because $0 \notin \overline{\text{co}}(K \setminus B(0, \varepsilon))$. If $\{n \in S : \mu(\Omega_n) = 1\}$ is finite, there exists an infinite set $S_2 \subset S$ such that $\mu(\Omega_n) < 1$ for all $n \in S_2$, then $\int_\Omega u_n \, d\mu = \mu(\Omega_n)\psi_n + (1 - \mu(\Omega_n))\zeta_n$, where

$$\psi_n = \frac{1}{\mu(\Omega_n)} \int_{\Omega_n} u_n \, d\mu \quad \text{and} \quad \zeta_n = \frac{1}{1 - \mu(\Omega_n)} \int_{\Omega \setminus \Omega_n} u_n \, d\mu.$$

Note that $\psi_n \in \overline{\text{co}}(K \setminus B(0, \varepsilon))$. Then either $\zeta_n \to 0$ and $\int_\Omega u_n \, d\mu$ cannot converge to 0 or there exists $\alpha > 0$ such that $\|\zeta_n\| \geq \alpha$ for infinitely many $n \in S$. Since we may suppose $\alpha \leq \varepsilon$, we have

$$\psi_n \in \overline{\text{co}}(K \setminus B(0, \alpha)) \quad \text{and} \quad \zeta_n \in \overline{\text{co}}(K \setminus B(0, \alpha)).$$

Hence, we obtain again a contradiction

$$\int_\Omega u_n \, d\mu \in \overline{\text{co}}(K \setminus B(0, \alpha))$$

for infinitely many $n \in S$. ■

Comments.

(1) In Bochner integration, the results we present above are valid under fairly general conditions (see, for example, [3, 5, 6, 8, 24, 26, 27]).

(2) Theorem 3.1 is a partial analog of the results stated in [20, 24] in the Bochner integration.

In the context of this paper, it should be mentioned that the main results are original and focus on the difference between Bochner and Pettis integration.

References

1. A. Amrani, Entaillabilité et convergence dans l'espaces L^1, *Bull. Polish Acad. Sc.* **42** (1994) 229–236.

2. A. Amrani and C. Castaing, Weak compactness in Pettis integration, *Bull. Polish Acad. Sc.* **45** (1997) 139–150.
3. A. Amrani, C. Castaing, and M. Valadier, Méthodes de troncatures appliquées à des problèmes de convergence faible ou forte dans L^1, *Arch. Rational Mech. Anal.* **117** (1992) 167–191.
4. A. Amrani, C. Castaing, and M. Valadier, Convergence forte impliquée par la convergence faible. Méthodes de troncature, *C.R. Acad. Sci. Paris, Ser. 1*, **314** (1992) 91–94.
5. Z. Arstein and T. Rzezuchowski, A note on Olech's lemma, *Stud. Math.* **98** (1991) 91–94.
6. E.J. Balder, On equivalence of strong and weak convergence in L^1-spaces under extreme point condition, *Israel J. Math.* **75** (1991) 21–47.
7. E.J. Balder, M. Girardi, and V. Jalby, From weak to strong types of L^1_E-convergence by the Bocce-criterion, *Stud. Math.* **111**(3) (1994) 241–262.
8. H. Benabdellah, Extrémalité, stricte convexité et convergence in L^1_E, *Séminaire Anal. Convexe Montpellier* **21** (1991) 4.1–4.18.
9. J. M. Borwein and A. S. Lewis, Strong rotundity and optimization, *SIAM J. Optimization* **4**(1) (1994) 146–158.
10. J. K. Brooks and N. Dinculeanu, On weak compactness in the space of Pettis integrable functions, *Adv. in Math.* **45** (1982) 53–58.
11. C. Castaing, Convergence faible et sections extremales, *Séminaire Anal. Convexe Montpellier* (1988) 2.1–2.18.
12. C. Castaing, Weak compactness criteria in set-valued integration, Université Montpellier II, 1995 (preprint 1995/03).
13. C. Castaing, Weak compactness and convergence in Bochner and Pettis integration, *Vietnam J. Math.* **24**(3) (1996) 241–286.
14. C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, Berlin, 1977.
15. C. Dellacherie and P. Meyer, *Probabilités et Potentiel*, Hermann, Paris, 1975.
16. J. Diestel and J.J. Uhl, *Vector Measures*, American Mathematical Society, Providence, 1977.
17. G. Emmanuelle and K. Musial, Weak precompactness in the space of Pettis intergrable functions, *J. Math. Anal. Appl.* **148** (1990) 245–250.
18. R. Geitz, Pettis integration, *Proc. Amer. Math. Soc.* **82** (1981) 81–86.
19. R. Huff, Remarks on Pettis integration, *Proc. Amer. Math. Soc.* **96** (1986) 402–404.
20. B. L. Lin, P. K. Lin, and S. L. Troyanski, Characterizations of denting points, *Proc. Amer. Math. Soc.* **102** (1988) 526–528.
21. K. Musial, Vitali and Lebesgue theorems for Pettis integral in locally convex spaces, *Atti. Sem. Mat. Fis. Modena* **25** (1987) 159–166.
22. K. Musial, Topics in the theory of Pettis integration, Rendiconti dell'instituto di matematica dell' Universita di Trieste, School on Measure Theory and Real Analysis Grado (Italy), 14–15 October 1991, pp. 176–262 (lecture).
23. J. Neveu, *Bases Mathématiques du Calcul des Probabilités*, Masson, Paris, 1964.
24. C. Olech, Existence theory in optimal control, in: *Control Theory and Topics in Functional Analysis, IAEA, Vienna*, Vol. 1, 1976, pp. 191–228.
25. T. Rzezuchowski, Impact of dentability on weak convergence in L^1, *Boll. Un. Mat. Italy* **7** (1992) 71–80.
26. M. Valadier, Différents cas où grâce à une propriété d' extrémalité une suite de fonctions intégrables faiblement convergente, converge faiblement, *Séminaire Anal. Convexe Montpellier*, 1989, 20 pp.
27. M. Visitin, Strong convergence results related to strict convexity, *Comment. Partial Diff. Equation* **9** (1984) 439–466.