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Borel's Lemma in the p-adic Case*

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Abstract. By using the p-adic Nevanlinna–Cartan theory, we prove a version of p-adic Borel's lemma.

Recent studies suggest that the hyperbolicity of the complex variety X is related to the finiteness of the number of rational or integral points of X (see [2, 5]). It is well known that complex Borel's lemma is one of the main tools in the construction of hyperbolic hypersurfaces. Let us recall it.

Borel's Lemma. *Let $f_1, \ldots, f_n$ $(n \geq 3)$ be holomorphic functions without zeros on C such that*

$$f_1 + f_2 + \cdots + f_n = 0.$$

Then the functions $f_1, \ldots, f_{n-1}$ are linearly dependent over C.

The most obvious analog would be to call a p-adic projective variety X *p-adic Brody hyperbolic* if the only p-adic holomorphic maps $f : C_p \to X$ are the constant maps, where C_p is the completion of the algebraic closure of Q_p, the field of p-adic numbers. In the p-adic case, since the holomorphic functions without zeros are constant (p-adic Picard's theorem [3]), p-adic Borel's lemma on C_p, if formulated as in the complex case, would be trivial. In this paper, we prove a version of p-adic Borel's lemma. The key of the proof is the p-adic Nevanlinna–Cartan theorem [4].

Theorem. (p-adic Borel's lemma) *Let $f_1, f_2, \ldots, f_n$ $(n \geq 3)$ be p-adic holomorphic functions without common zeros on C_p such that*

$$f_1 + f_2 + \cdots + f_n = 0.$$

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Then the functions $f_1, \ldots, f_{n-1}$ are linearly dependent if, for $j = 1, \ldots, n$, every zero of f_j is of multiplicity at least d_j and the following condition holds:

$$\sum_{j=1}^{n} \frac{1}{d_j} \leq \frac{1}{n-2}. \tag{1}$$

Proof. Following [4], we use the notations $h^+(f,t)$ for the height function of a meromorphic function f. The necessary properties of the height function can be found in [4].

We first claim that f_j, $1 \leq j \leq n-1$ are linearly dependent over C_p. Assume f_j, $1 \leq j \leq n-1$ are linearly independent over C_p. We define a holomorphic curve g in $P^{n-2}(C_p)$ by

$$g : z \mapsto (f_1(z), \ldots, f_{n-1}(z)).$$

Then g is linearly non-degenerated. Take the following hyperplanes in a general position,

$$H_1 = \{z_1 = 0\}, \ldots, H_{n-1} = \{z_{n-1} = 0\},$$
$$H_n = \{z_1 + \cdots + z_{n-1} = 0\}.$$

Then by p-adic Cartan–Nevanlinna theorem [4], we have

$$h^+(g,t) \leq \sum_{j=1}^{n} N_{n-2}(g \circ H_j, t) + \frac{(n-1)(n-2)}{2} t + 0(1).$$

On the other hand, we have

$$\begin{aligned} N(f_j,t) &\geq d_j N_1(f_j,t), \quad (j = 1, \ldots, n) \\ \Rightarrow N_{n-2}(g \circ H_j, t) &= N_{n-2}(f_j,t) \\ &\leq (n-2) N_1(f_j,t) \leq \frac{n-2}{d_j} N(f_j,t) \\ &\leq \frac{n-2}{d_j} \max_{1 \leq j \leq n-1} N(f_j,t), \quad (j = 1, 2, \ldots, n-1). \end{aligned}$$

For $j = n$, we still have

$$\begin{aligned} N_{n-2}(g \circ H_n, t) &= N_{n-2}(f_1 + \cdots + f_{n-1}, t) = N_{n-2}(f_n, t) \\ &\leq (n-2) N_1(f_n,t) \leq \frac{n-2}{d_n} N(f_n,t) \\ &= \frac{n-2}{d_n} N(f_1 + \cdots + f_{n-1}, t) \\ &\leq \frac{n-2}{d_n} \max_{1 \leq i \leq n-1} N(f_i,t) + 0(1). \end{aligned}$$

By Lemma 3.8 in [4], we obtain

$$h^+(g,t) = \max_{1\le j\le n-1} h^+(f_j,t) = \max_{1\le j\le n-1} N(f_j,t) + 0(1)$$

$$\Rightarrow \max_{1\le j\le n-1} N(f_j,t) \le \sum_{j=1}^{n} \frac{n-2}{d_j} \max_{1\le j\le n-1} N(f_j,t) + \frac{(n-1)(n-2)}{2} t + 0(1)$$

$$= (n-2)\left(\sum_{j=1}^{n} \frac{1}{d_j}\right) \max_{1\le j\le n-1} N(f_j,t) + \frac{(n-1)(n-2)}{2} t + 0(1)$$

$$\Rightarrow \left(\frac{1}{n-2} - \sum_{j=1}^{n} \frac{1}{d_j}\right) \max_{1\le j\le n-1} N(f_j,t) \le \frac{(n-1)}{2} t + 0(1).$$

By hypothesis

$$\sum_{j=1}^{n} \frac{1}{d_j} \le \frac{1}{n-2},$$

we obtain a contradiction as $t \to -\infty$.

We have the following corollary, which is another statement of p-adic Borel's lemma.

Corollary. *Let $f_1, \ldots, f_n$ be p-adic holomorphic functions satisfying*

$$f_1 + f_2 + \cdots + f_n = 1 \quad (n \ge 2).$$

Then $f_1, f_2, \ldots, f_n$ are linearly dependent if every zero of f_j is of multiplicity at least d_j and the following condition holds:

$$\sum_{j=1}^{n} \frac{1}{d_j} \le \frac{1}{n-1}.$$

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References

1. A. Boutabaa, Sur les courbes holomorphes p-adiques, *Ann. Fac. Sci. Toulouse Math.* **5**(6) (1996) 29–52.
2. W. A. Cherry, Hyperbolic p-adic analytic spaces, Ph.D. Thesis, Yale University, 1993.
3. H. H. Khoai, On p-adic meromorphic functions, *Duke Math. J.* **50** (1983) 695–711.
4. H. H. Khoai and M. V. Tu, p-adic Nevanlinna–Cartan theorem, *Inter. J. Math.* **6**(5) (1995) 719–731.
5. S. Lang, Hyperbolic and Diophantine analysis, *Bull. Amer. Math. Soc.* **14** (1986) 159–205.